Continuity

Definition 1 (Continuity)**.** *A function f is said to be continuous at c if* $\lim_{x \to c} f(x) = f(c)$.

Goemetrically, this corresponds to the absence of any breaks in the graph of f at c.

When we've calculated limits, most of the time we started with a function that was not continuous at the limit point, simplified to get another function which was equal to the original function except at the limit point but was continuous at the limit point, and then it was easy to find the limit of the latter function.

Rule of Thumb: Most functions we run across will be continuous except at points where there is an obvious reason for them to fail to be continuous.

Examples of Continuous Functions

- Polynomial Functions
- Rational Functions (Quotients of Polynomial Functions) – except where the denominator is 0.
- The exponential function
- The natural logarithm function
- sin and cos
- tan – except at odd multiples of $\pi/2$, where it obviously isn't since $\tan = \frac{\sin}{\cos}$ and cos takes on the value 0 at odd multiples of $\pi/2$.

Properties of Continuous Functions

When we perform most algebraic manipulations involving continuous functions, we wind up with continuous functions. Again, the exception is if there's an obvious reason why the new function wouldn't be continuous somewhere.

- The sum of continuous functions is a continuous function.
- The difference of continuous functions is a continuous function.
- The product of continuous functions is a continuous function.
- The quotient of continuous functions is a continuous function – except where the denominator is 0.
- The composition of continuous functions is a continuous function.

Extreme Value Theorem

Theorem 1 (Extreme Value Theorem)**.** *If a function is continuous on a closed interval, it must attain both a maximum value and a minimum value on that interval.*

The necessity of the continuity on a closed interval may be seen from the example of the function $f(x) = x^2$ defined on the open interval $(0, 1)$.

f clearly has no minimum value on $(0, 1)$, since 0 is smaller than any value taken on while no number greater than 0 can be a minimum.

This also has no maximum value on $(0, 1)$, since 1 is larger than any value taken on while no number less than 1 can be a maximum.

Intermediate Value Theorem

Theorem 2 (Intermediate Value Theorem)**.** *If a function is continuous on a closed interval $[a, b]$, then the function must take on every value between $f(a)$ and $f(b)$.*

Corollary 3 (Zero Theorem)**.** *If a function is continuous on a closed interval $[a, b]$ and takes on values with opposite sign at a and at b, then it must take on the value 0 somewhere between a and b.*

The *Zero Theorem* leads to the *Bisection Method*, which is a foolproof way of estimate a zero of a continuous function to any desired precision provided we are able to find both positive and negative values of the function.

The Bisection Method

The Bisection Method works as follows.

We wish to estimate a zero for a continuous function f. We start by finding points a_0, b_0 at which the function has opposite sign. Without loss of generality (WLOG), let us assume
$a_0 < b_0$.

We evaluate the function at the midpoint
$m_0 = \dfrac{a_0 + b_0}{2}$. Unless $f(m_0) = 0$, in which case we have found a zero for f and can stop, $f(m_0)$ must have a different sign than either $f(a_0)$ or $f(b_0)$.

The Bisection Method

If $f(m_0)$ differs in sign from $f(a_0)$, then we know f has a zero on the interval $[a_0, m_0]$; otherwise, we know f has a zero on the interval $[m_0, b_0]$.

Either way, we now have an interval we may denote by $[a_1, b_1]$ which is half the width of the original interval but which also contains a zero of f.

We can repeat this process until we have an interval as small as we desire.

Number of Iterations

We can also determine, in advance, how many iterations we need in order to obtain an interval of width smaller than any predetermined number $\epsilon > 0$.

Clearly, after the first iteration we will have an interval of width $\frac{b_0 - a_0}{2}$.

After the next iteration, we will have an interval of width $\frac{b_0 - a_0}{4} = \frac{b_0 - a_0}{2^2}$.

After the third iteration, we will have an interval of width $\frac{b_0 - a_0}{8} = \frac{b_0 - a_0}{2^3}$.

Continuing, it should be clear that after n iterations we will have an interval of width $\frac{b_0 - a_0}{2^n}$.

$$\frac{b_0 - a_0}{2^n}$$

We can easily find a value of n for which this width is less than ϵ by solving as follows.

$\frac{b_0 - a_0}{2^n} < \epsilon$

$\frac{b_0 - a_0}{\epsilon} < 2^n$

$2^n > \frac{b_0 - a_0}{\epsilon}$

$n \ln 2 > \ln\left(\frac{b_0 - a_0}{\epsilon}\right)$

$n > \dfrac{\ln\left(\frac{b_0 - a_0}{\epsilon}\right)}{\ln 2}$.

$$n > \frac{\ln\left(\frac{b_0 - a_0}{\epsilon}\right)}{\ln 2}$$

The number of iterations we will need is thus the smallest integer greater than or equal to $\dfrac{\ln\left(\frac{b_0 - a_0}{\epsilon}\right)}{\ln 2}$.

One should not memorize this formula; the calculations are routine enough so that you should be able to carry them out fairly easily and quickly when you need to and doing so is good practice.