Different Types of Limits

Besides ordinary, two-sided limits, there are one-sided limits (left-hand limits and right-hand limits), infinite limits and limits at infinity.

One-Sided Limits

Consider $\lim_{x\to 5}\sqrt{x^2-4x-5}$.

One might think that since $x^2-4x-5\to 0$ as $x\to 5$, it would follow that $\lim_{x\to 5}\sqrt{x^2-4x-5}=0$.

But since $x^2-4x-5=(x-5)(x+1)<0$ when x is close to 5 but smaller than 5, $\sqrt{x^2-4x-5}$ is undefined for some values of x very close to 5 and the limit as $x\to 5$ doesn't exist.

But we would still like a way of saying $\sqrt{x^2-4x-5}$ is close to 0 when x is close to 5 and $x>5$, so we say the *Right-Hand Limit* exists, write $\lim_{x\to 5^+}\sqrt{x^2-4x-5}=0$ and say $\sqrt{x^2-4x-5}$ *approaches* 0 *as* x *approaches* 5 *from the right.*

Sometimes we have a Left-Hand Limit but not a Right-Hand Limit. Sometimes we have both Left-Hand and Right-Hand Limits and they're not the same. Sometimes we have both Left-Hand and Right-Hand Limits and they're equal, in which case the ordinary limit exists and is the same.

Example

$$f(x)=\begin{cases} x^2 & \text{if } x<1 \\ x^3 & \text{if } 1<x<2 \\ x^2 & \text{if } x>2. \end{cases}$$

$\lim_{x\to 1^-}f(x)=\lim_{x\to 1^+}f(x)=1$, so the left and right hand limits are equal and $\lim_{x\to 1}f(x)1$.

$\lim_{x\to 2^-}f(x)=8$ while $\lim_{x\to 2^+}f(x)=4$, so the left and right hand limits are different and $\lim_{x\to 2}f(x)$ doesn't exist.

Limits at Infinity

Suppose we're interested in estimating about how big $\dfrac{2x}{x+1}$ is when x is very big. It's easy to see that $\dfrac{2x}{x+1}=\dfrac{2x}{x(1+\frac{1}{x})}=\dfrac{2}{1+\frac{1}{x}}$ if $x\neq -1$ and thus $\dfrac{2x}{x+1}$ will be very close to 2 if x is very big. We write

$$\lim_{x\to\infty}\frac{2x}{x+1}=2$$

and say *the limit of* $\dfrac{2x}{x+1}$ *is* 2 *as* x *approaches* ∞.

Limits at Infinity

Similarly, $\dfrac{2x}{x+1}$ will be very close to 2 if x is very small and we write

$\lim_{x\to-\infty} \dfrac{2x}{x+1} = 2$

and say *the limit of* $\dfrac{2x}{x+1}$ *is* 2 *as* x *approaches* $-\infty$. Here, *small* does not mean close to 0, but it means that x is a negative number with a large magnitude (*absolute value*).

Calculating Limits at Infinity

A convenient way to find a limit of a quotient at infinity (or minus infinity) is to factor out the largest term in the numerator and the largest term in the denominator and cancel what one can.

$\lim_{x\to\infty} \dfrac{5x^2-3}{8x^2-2x+1} = \lim_{x\to\infty} \dfrac{x^2(5-\frac{3}{x^2})}{x^2(8-\frac{2}{x}+\frac{1}{x^2})} =$
$\lim_{x\to\infty} \dfrac{5-\frac{3}{x^2}}{8-\frac{2}{x}+\frac{1}{x^2}} = \dfrac{5}{8}$

Example

$\lim_{x\to\infty} \dfrac{5x-3}{8x^2-2x+1} = \lim_{x\to\infty} \dfrac{x(5-\frac{3}{x})}{x^2(8-\frac{2}{x}+\frac{1}{x^2})} =$
$\lim_{x\to\infty} \dfrac{5-\frac{3}{x}}{x(8-\frac{2}{x}+\frac{1}{x^2})} = 0$

Infinite Limits

If x is close to 1, it's obvious that $\dfrac{1}{(x-1)^2}$ is very big. We write

$\lim_{x\to1} \dfrac{1}{(x-1)^2} = \infty$

and say *the limit of* $\dfrac{1}{(x-1)^2}$ *is* ∞ *as* x *approaches 1.*

Similarly, $\lim_{x\to1} -\dfrac{1}{(x-1)^2} = -\infty$.

A Technicality

Technically, a function with an *infinite limit* doesn't actually have a limit. Saying a function has an *infinite limit* is a way of saying it doesn't have a limit in a very specific way.

Calculating Infinite Limits

Infinite limits are inferred fairly intuitively. If one has a quotient $\frac{f(x)}{g(x)}$, one may look at how big $f(x)$ and $g(x)$ are. For example:

If $f(x)$ is close to some positive number and $g(x)$ is close to 0 and positive, then the limit will be ∞.

If $f(x)$ is close to some positive number and $g(x)$ is close to 0 and negative, then the limit will be $-\infty$.

If $f(x)$ is close to some negative number and $g(x)$ is close to 0 and positive, then the limit will be $-\infty$.

If $f(x)$ is close to some negative number and $g(x)$ is close to 0 and negative, then the limit will be ∞.

Variations of Limits

One can also have one-sided infinite limits, or infinite limits at infinity.

$\lim_{x \to 1^+} \frac{1}{x-1} = \infty$

$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$

Asymptotes

If $\lim_{x \to \infty} f(x) = L$ then $y = L$ is a horizontal asymptote.

If $\lim_{x \to -\infty} f(x) = L$ then $y = L$ is a horizontal asymptote.

If $\lim_{x \to c^+} f(x) = \pm\infty$ then $x = c$ is a vertical asymptote.

If $\lim_{x \to c^-} f(x) = \pm\infty$ then $x = c$ is a vertical asymptote.