Mathematics 1132 SOLUTIONS
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1. Find a bound $B$ such that $|f''(x)| \leq B$ for $-2 \leq x \leq 3$, where $f(x) = x^2 e^{-x} \sin x$. *Make sure each step you take is clear. Do not use any information about the size of e other than* $2 < x < 3$ *and do not use a calculator more than once.*

   **Solution:** $f'(x) = 2xe^{-x}\sin x - x^2e^{-x}\sin x + x^2e^{-x}\cos x$.

   $f''(x) = (2e^{-x}\sin x - 2xe^{-x}\sin x + 2xe^{-x}\cos x) - (2xe^{-x}\sin x - x^2e^{-x}\sin x + x^2e^{-x}\cos x) + (2xe^{-x}\cos x - x^2e^{-x}\cos x - x^2e^{-x}\sin x)$
   $= 2e^{-x}\sin x - 4xe^{-x}\sin x + 4xe^{-x}\cos x - 2x^2e^{-x}\cos x$
   $f''(x) = 2e^{-x}(\sin x - 2x\sin x + 2x\cos x - x^2\cos x)$

   So, on $[-2,3]$, $|f''(x)| \leq 2e^{-x}(|\sin x| + 2|x\sin x| + |2x\cos x| + |x^2\cos x|)$

   Since $|\sin x| \leq 1$ and $|\cos x| \leq 1$, $|f''(x)| \leq 2e^{-x}(1+2|x|+2|x|+x^2) = 2e^{-x}(1+4|x|+x^2)$.

   On $[-2,3]$, $e^{-x} \leq e^2 =\leq 3^2 = 9$, $|x| \leq 3$ and $x^2 \leq 3^2 \leq 9$, we have $|f''(x)| \leq 2 \cdot 9(1 + 4 \cdot 3 + 9) = 396$.

   So we may take $B = 396$.

2. Assume you need to use Simpson's Rule to estimate $\int_{-2}^{3} f(x)\,dx$ with an error no greater than $5 \cdot 10^{-12}$ and you have found that $|f^{(4)}(x)| \leq 58$ on $[-2,3]$. Find a smallest possible value of $n$ which the error formula for Simpson's Rule $\left(|E_S| \leq \dfrac{K^*(b-a)^5}{180n^4}\right)$ guarantees can be used, where the notation used here is that generally used in class in this context. *Note: If you use a calculator prior to the very last step, you are misusing it.*

   **Solution:** Since we may take $K^* = 58$, $a = -2$ and $b = 3$, $|E_S|$ will be $\leq 5 \cdot 10^{-12}$ if $\dfrac{58[3-(-2)]^5}{180n^4} \leq 5 \cdot 10^{-12}$, $\dfrac{58 \cdot 5^5}{180n^4} \leq 5 \cdot 10^{-12}$, $n^4 \geq \dfrac{58 \cdot 5^5}{180 \cdot 5 \cdot 10^{-12}} = \dfrac{58 \cdot 5^4 \cdot 10^{12}}{180}$.

   $n \geq 5 \cdot 10^3 \sqrt[4]{\dfrac{58}{180}} \approx 3767.1149843$.

   So we may take $n = 3768$.

3. Without evaluating it, prove $\int_1^\infty \frac{1}{x^2+x}\,dx < \infty$.

**Solution:** $\frac{1}{x^2+x} < \frac{1}{x^2}$. Since, by the P-Test, $\int_1^\infty \frac{1}{x^2}\,dx < \infty$, it follows from the Comparison Test that $\int_1^\infty \frac{1}{x^2+x}\,dx < \infty$.

4. Evaluate $\int_1^\infty \frac{1}{x^2+x}\,dx$.

**Solution:** Using a Partial Fractions Expansion, $\frac{1}{x^2+x} = \frac{1}{x(1+x)} = \frac{a}{x} + \frac{b}{1+x} = \frac{a(1+x)+bx}{x(1+x)}$.

Equating numerators, $a(1+x)+bx = 1$.

Letting $x = 0$, we get $a = 1$.

Letting $x = -1$, we get $-b = 1$, $b = -1$.

Thus $\frac{1}{x^2+x} = \frac{1}{x} + \frac{-1}{1+x}$ and $\int \frac{1}{x^2+x}\,dx = \int \frac{1}{x} - \frac{1}{1+x}\,dx = \ln x - \ln(1+x) = \ln\left(\frac{x}{1+x}\right)$. *Since $x$ and $1+x$ are both positive, we don't need absolute value notation.*

$\int_1^\infty \frac{1}{x^2+x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+x}\,dx = \lim_{t\to\infty} \ln\left(\frac{x}{1+x}\right)\Big|_1^t = \lim_{t\to\infty}[\ln\left(\frac{t}{1+t}\right) - \ln\left(\frac{1}{2}\right)] = 0 - (-\ln 2) = \ln 2$.

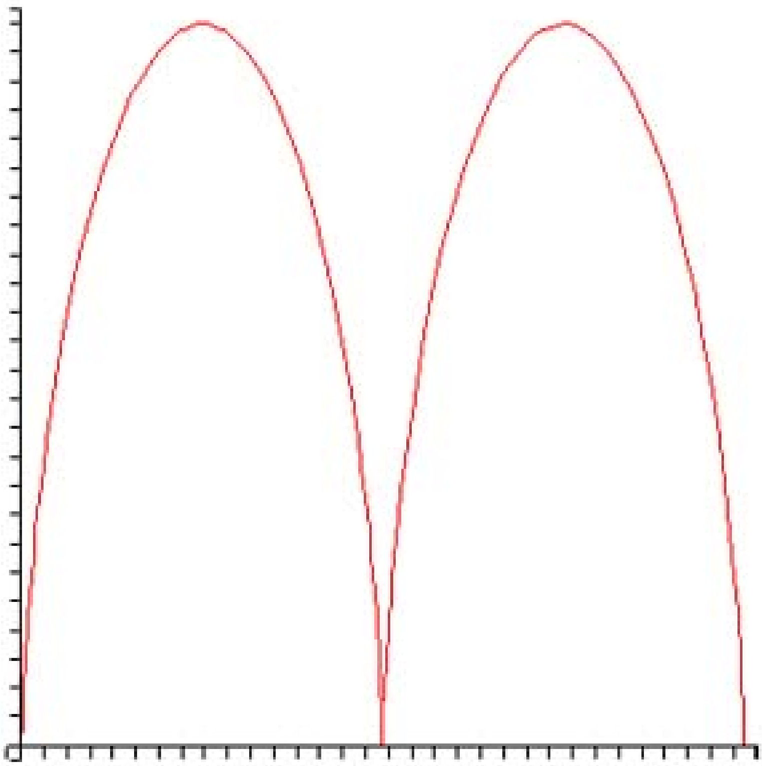
5. Determine whether $\int_0^\infty \frac{x^2}{e^x}\,dx$ is convergent. Justify your conclusion.

**Solution:** For large $x$, $e^x > x^4$, so $\frac{x^2}{e^x} < \frac{x^2}{x^4} = \frac{1}{x^2}$.

Since, by the P-Test, $\int_1^\infty \frac{1}{x^2}\,dx < \infty$, it follows by the Comparison Test that $\int_1^\infty \frac{x^2}{e^x}\,dx < \infty$ and thus $\int_0^\infty \frac{x^2}{e^x}\,dx < \infty$.

6. Sketch the parametric curve $x = 5(t - \sin t)$, $y = 5(1 - \cos t)$, $0 \le t \le 4\pi$ and represent its length by a definite integral.

**Solution:**



The length is $s = \int_0^{4\pi} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\, dt$.

$\frac{dx}{dt} = 5(1 - \cos t)$, $\frac{dy}{dt} = 5 \sin t$, so $(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 = 25(1 - 2\cos t + \cos^2 t) + 25 \sin^2 t = 50(1 - \cos t)$.

Thus $s = 5\sqrt{2} \int_0^{4\pi} \sqrt{1 - \cos t}\, dt$.

*Extra Credit: Evaluate the integral.*

**Solution:** We know $\sin^2 t = \frac{1 - \cos 2t}{2}$, so $1 - \cos 2t = 2\sin^2 t$ and $1 - \cos t = 2\sin^2(t/2)$.

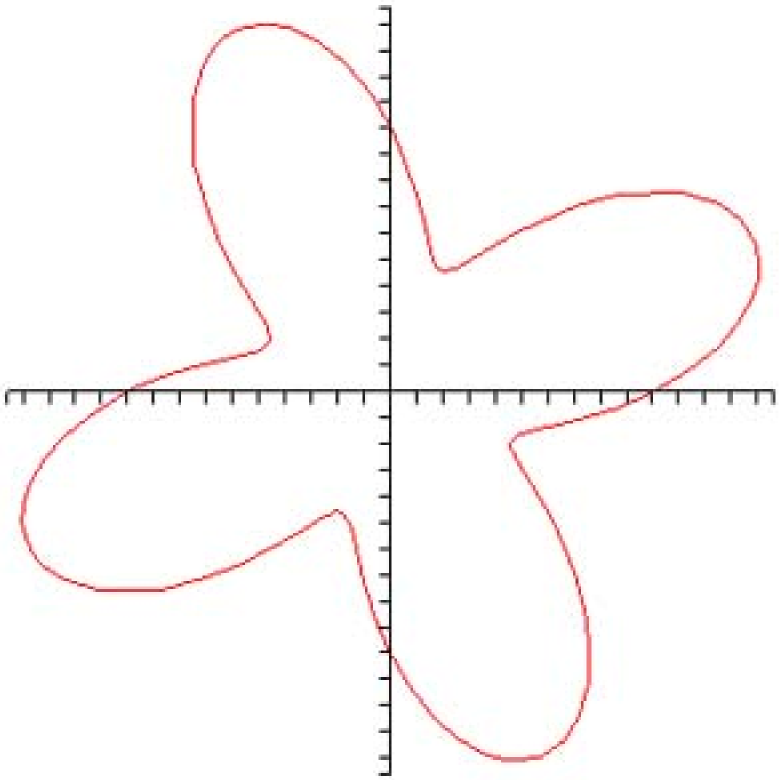
Between 0 and $2\pi$, $\sin(t/2) \ge 0$, so $\sqrt{1 - \cos t} = \sqrt{2\sin^2(t/2)} = \sqrt{2}\sin(t/2)$, so $\int \sqrt{1 - \cos t}\, dt = -2\sqrt{2}\cos(t/2)$.

Between $2\pi$ and $4\pi$, $\sin(t/2) \le 0$, so $\sqrt{1 - \cos t} = \sqrt{2\sin^2(t/2)} = -\sqrt{2}\sin(t/2)$, so $\int \sqrt{1 - \cos t}\, dt = 2\sqrt{2}\cos(t/2)$.

Hence, $s = 5\sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos t}\, dt + 5\sqrt{2} \int_{2\pi}^{4\pi} \sqrt{1 - \cos t}\, dt =$
$5\sqrt{2} \cdot \left[-2\sqrt{2}\cos(t/2)\right]_0^{2\pi} + 5\sqrt{2} \cdot \left[2\sqrt{2}\cos(t/2)\right]_{2\pi}^{4\pi} =$
$-20[\cos\pi - \cos 0] + 20[\cos 2\pi - \cos\pi] = -20(-1 - 1) + 20(1 - (-1)) = 80.$

7. Sketch the polar curve $r = 2 + \sin(4\theta)$, $0 \leq \theta \leq 2\pi$ and find the area of the region it encloses.

**Solution:**



The area is $\frac{1}{2}\int_0^{2\pi}(2+\sin 4\theta)^2\,d\theta = \frac{1}{2}\int_0^{2\pi} 4 + 4\sin 4\theta + \sin^2 4\theta\,d\theta =$

$\frac{1}{2}\int_0^{2\pi} 4 + 4\sin 4\theta + \dfrac{1-\cos 8\theta}{2}\,d\theta = \int_0^{2\pi}\dfrac{9}{4} + 2\sin 4\theta - \dfrac{\cos 8\theta}{2}\,d\theta = \dfrac{9\theta}{4} - \dfrac{\cos 4\theta}{2} - \dfrac{\sin 8\theta}{16}\Big|_0^{2\pi} =$

$[\dfrac{9\cdot 2\pi}{4} - \dfrac{\cos 8\pi}{2} - \dfrac{\sin 16\pi}{16}] - [\dfrac{9\cdot 0}{4} - \dfrac{\cos 0}{2} - \dfrac{\sin 0}{16}] = \dfrac{9\pi}{2} - \dfrac{1}{2} - 0 - 0 + \dfrac{1}{2} + 0 = \dfrac{9\pi}{2}.$