Mathematics 1132 <u>SOLUTIONS</u>
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Due Thursday, July 30, 2009

1. How long will it take for the balance in a bank account to double if the bank pays interest at an annual rate of 2.7% compounded continuously? *Make sure you clearly define any variables you introduce. Give an exact conclusion followed by a decimal approximation.*

   Solution: Let P be the balance at time t and let P_0 be the initial balance. We know $P = ae^{0.027t}$ for some $a \in \mathbb{R}$.

   Since $P = P_0$ when $t = 0$, $ae^{0.027\cdot 0} = P_0$, so $a = P_0$ and $P = P_0 e^{0.027t}$.

   When the balance has doubled, we will have $P = 2P_0$, so $P_0 e^{0.027t} = 2P_0$, $e^{0.027t} = 2$, $0.027t = \ln 2$, $t = \frac{\ln 2}{0.027}$.

   It will thus take $\frac{\ln 2}{0.027}$ years, or approximately 25.6721177985 years, for the balance to double.

2. Find $\tan(\arccos(-0.73))$.

   Solution: Let $\theta = \arccos(-0.73)$. Then $0 \le \theta \le \pi$ and $\cos\theta = -0.73$. Since $\cos^2\theta + \sin^2\theta = 1$, $(-0.73)^2 + \sin^2\theta = 1$, $\sin^2\theta = 1 - (-0.73)^2 = 0.4671$, so $\sin\theta = \pm\sqrt{0.4671}$. Since $0 \le \theta \le \pi$, $\sin\theta \ge 0$, so $\sin\theta = \sqrt{0.4671}$.

   Hence $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{0.4671}}{-0.73} = -\frac{\sqrt{0.4671}}{0.73}$.

3. Find $\arcsin(\sin(0.97\pi))$.

   Solution: $\arcsin(\sin(0.97\pi)) = \pi - 0.97\pi = 0.03\pi$.

4. Let $f(x) = \ln(\arcsin(3x))$. Find the domain and derivative of $f(x)$.

   Solution: Since ln is de ned only for positive reals, $\arcsin(3x)$ must be positive. Since arcsin is positive on $(0, 1]$, we need $0 < 3x \le 1$, so $0 < x \le 1/3$. So the domain is $(0, 1/3]$.

   $f'(x) = \frac{1}{\arcsin 3x} \cdot \frac{1}{\sqrt{1-(3x)^2}} \cdot 3 = \frac{3}{(\arcsin 3x)\sqrt{1-9x^2}}$.

5. Calculate $\int (3x^2 + 2x + 1)\cos(x)\,dx$.

   Solution: Using Integration By Parts, let $f(x) = 3x^2 + 2x + 1$, $g'(x) = \cos x$. Then $f'(x) = 6x + 2$, $g(x) = \sin x$, so
   $\int (3x^2 + 2x + 1)\cos(x)\,dx = (3x^2 + 2x + 1)\sin x - \int (6x + 2)\sin x\,dx$.

   Using Integration By Parts again, this time let $f(x) = 6x + 2$, $g'(x) = \sin x$. Then $f'(x) = 6$, $g(x) = -\cos x$, so
   $\int (3x^2 + 2x + 1)\cos(x)\,dx = (3x^2 + 2x + 1)\sin x - [(6x + 2)(-\cos x) - \int 6(-\cos x)\,dx]$
   $= (3x^2 + 2x + 1)\sin x + (6x + 2)\cos x - 6\int \cos x\,dx$
   $= (3x^2 + 2x + 1)\sin x + (6x + 2)\cos x - 6\sin x = (3x^2 + 2x - 5)\sin x + (6x + 2)\cos x + k$.

6. Calculate $\lim_{x\to 0^+}(\ln x)(\tan x)$. *Extra Credit: Find the limit two completely different ways.*

Solution: $\lim_{x\to 0^+}(\ln x)(\tan x) = \lim_{x\to 0^+} \frac{\ln x}{\cot x} = \lim_{x\to 0^+} \frac{1/x}{-\csc^2 x} = -\lim_{x\to 0^+} \frac{\sin^2 x}{x} = -\lim_{x\to 0^+} \frac{2\sin x \cos x}{1} = 0$.

Alternatively, $\lim_{x\to 0^+}(\ln x)(\tan x) = \lim_{x\to 0^+}(\ln x)\cdot \frac{\sin x}{\cos x} = \lim_{x\to 0^+}(\ln x)\sin x = \lim_{x\to 0^+}(x\ln x)\cdot \frac{\sin x}{x} = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = -\lim_{x\to 0^+} x = 0$.

One may also note
$\lim_{x\to 0^+} \frac{\sin^2 x}{x} = \lim_{x\to 0^+} \sin x \cdot \frac{\sin x}{x} = \lim_{x\to 0^+} \sin x \cdot \lim_{x\to 0^+} \frac{\sin x}{x} = 0\cdot 1 = 0$.

7. Calculate $\lim_{x\to 0}(1-3\sin x)^{\frac{2}{x}}$.

Solution: $\lim_{x\to 0}(1-3\sin x)^{\frac{2}{x}} = \lim_{x\to 0} e^{\frac{2}{x}\ln(1-3\sin x)}$.

Since $\lim_{x\to 0} \frac{2}{x}\ln(1-3\sin x) = 2\lim_{x\to 0} \frac{\ln(1-3\sin x)}{x} = 2\lim_{x\to 0} \frac{\frac{1}{1-3\sin x}\cdot(-3\cos x)}{1} = 2\lim_{x\to 0} \frac{1}{1-3\sin x}\cdot(-3) = 2\cdot 1\cdot(-3) = -6$, it follows that
$\lim_{x\to 0}(1-3\sin x)^{\frac{2}{x}} = e^{-6} = \frac{1}{e^6}$.