Mathematics 109 Highlights and Study Guide

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Listed here are some of the highlights of the semester, with some of the central ideas you should keep in mind. This list is incomplete and there are certain to be some items on the final examination which are not mentioned here.

Algebraic and Trigonometric Manipulations

You need to be familiar with the basic properties of real numbers, including things like the associative, commutative and distributive laws, and the proper use of parentheses, and you need to be able to use them to manipulate and simplify algebraic and trigonometric expressions.

One technicality that often causes problems is the convention regarding the order of operations. Basically, unless indicated otherwise by the clever use of parentheses, exponentiation is always performed before multiplication or division, which is always performed before addition or subtraction. Thus, for example, $2 + 3 \cdot 5$ has a completely different meaning and value than $(2 + 3) \cdot 5$, the former equalling 17 and the latter equalling 25.

Of particular importance is the cancellation rule.

Theorem 1 (Cancellation Rule). $c \neq 0$, then $\dfrac{ac}{bc} = \dfrac{a}{b}$.

This rule is often used incorrectly. It says that non-zero, common *factors* of a numerator and denominator can be cancelled.

You need to be able to routinely do things like:

- Multiply together algebraic expressions.

   This includes, but is not restricted to, polynomials. Many students try to use the infamous FOIL method for all multiplications; however, that method only works for products of binomials and most students would be better off completely forgetting about it.

- Factor polynomials.

   This is essential for solving both equations and inequalities. Its importance for solving equations is readily apparent from the *Factor Theorem.*

   Theorem 2 (Factor Theorem). $x - c$ is a factor of a polynomial $p(x)$ if and only if $p(c) = 0$.

   It's importance for solving inequalities is readily apparent from the fact that the sign of a product is determined by the signs of its factors.

- Simplify complex algebraic expressions.

   This includes sums of rational expressions (quotients of polynomials) as well as quotients where the numerator or denominator also involves a quotient.

- Solve algebraic equations.

   You need to be able to solve equations involving expressions that can be factored. There are certain techniques that can be applied to specific types of equations, such

as quadratic equations, but your first attempt should almost always involve obtaining an equivalent equation for which one side is 0 and trying to factor the other side.
- Solve inequalities.

    You need to be able to solve any inequality that, if the inequality symbols was changed to an equal sign, would look like one of the types of equations described above.

    Remember that one method is to find where the two sides of the inequality are actually equal and use those points to divide the real line into intervals, each of which is either contained in or disjoint from the solution set.
- Use the rules for exponents and radicals.
- Verify trigonometric identities.

    You need to be able to verify trigonometric identities by taking one side of the identity and, using known identities along with the basic rules of algebra, transforming it into the other side. You should be adept with the trick of taking an expression such as $1 \pm \sin x$ and multiplying by $\dfrac{1 \mp \sin x}{1 \mp \sin x}$ in order to take advantage of the basic trigonometric identity $\cos^2 x + \sin^2 x = 1$.

Functions

You need to understand the definition of a function and properly use functional notation.

Given a formula for a function f (note: I said f, not $f(x)$, in order to emphasize the fact that the name of the independent variable is almost irrelevant), you need to be able to routinely evaluate things like $f(x)$, $f(x^2)$, $f(x+3)$, $f(t)$ and $\dfrac{f(x+h) - f(x)}{h}$.

You need to understand the concept of the graph of a function.

Definition 1 (Graph). *The graph of a function f consists of the set of points (x, y) such that $y = f(x)$.*

You need to be able to sketch graphs of simple functions, including lines, parabolas and the basic trigonometric functions. You need to be able to recognize shifts (both horizontal and vertical) and use them to sketch graphs accurately while plotting as few specific points as possible.

Recall how the same technique of *completing the square* which leads to the quadratic formula can often be indispensable when trying to sketch the graph of an equation.

Trigonometry

You need to be familiar with the two primary methods of measuring the size of an angle, radian measure and the virtually obsolete but still common method of degree measure, and be able to switch back and forth between the two.

You need to be familiar with the definitions of the trigonometric functions, both the classical definitions involving right triangles and the more general definitions involving the wrapping function and arbitrary angles. You should be able to use these definitions to sketch the graphs of the basic trigonometric functions and also to evaluate them *exactly* at certain points, such as multiples of $\pi/6$ and $\pi/4$.

You need to be able to use your knowledge of the basic trigonometric functions to obtain the dozens of rather simple but important trigonometric identities involving pairs of angles that are related in a fairly straightforward way, such as being complementary or supplementary, or differing by multiples of $\pi/2$ or π or 2π.

You need to be familiar with the inverses of the basic trigonometric functions.

You need to be able to *solve* triangles if you are given enough information to tell whether they are congruent.

You need to be familiar with the basic trigonometric identities and able to use them not only to simplify trigonometric expressions but also to find exact values of all the trignometric functions at a given point given sufficient information.

You need to be familiar with the addition and subtraction formulas, particularly the double angle formulas involving the cosine function.

Problem Solving

You need to be able to set up simple word problems. Remember to concentrate on understanding the statement of the problem, looking for variables and unknowns, and looking for connections between them.

Remember that the solution to a word problem usually comes out if you understand the information given, so don't concentrate too hard on exactly what is asked of you. However, remember that what is asked of you usually gives a clue about a variable or unknown. For example, if you are asked to find the volume of a sphere, then the volume of the sphere is a variable or unknown that needs to be represented by a symbol.

Calculators

You should be able to use a calculator appropriately and effectively, minimizing round off errors by avoiding writing down intermediate results and then reentering those results with reduced precision.

One key is to visualize (or even write down) how the expression to be calculated could be written in a linear form, writing quotients using a slash (/) rather than writing the numerator over the denominator.

Always try to estimate the result of a calculation in advance, since most calculator errors are gross errors that will be caught if you have a rough idea of about how big the result should be.

Random Thoughts

(1) Try to be organized and use proper notation. Mathematical notation has evolved over the years and the proper notation can be a terrific aid in helping guide you to appropriate paths. If you pay attention to notation, and stay organized, you're also likely to notice when you've made a careless mistake.
(2) Pay attention to whether your results make sense.
(3) Check your answers yourself. For example, if you're getting a numerical answer, try to estimate how large it should be as early in the process as you can, and then see if you come up with something close.

(4) Remember that on a final exam you won't have the opportunity to explain what you were thinking if what you write isn't clear.

(5) Get enough sleep before your final exam.