Verifying Trigonometric Identities

A trigonometric identity is simply an identity involving trigonometric functions. We learn to verify and spend time verifying trigonometric identities because the practice we get verifying trigonometric identities gives us practice simplifying trigonometric expressions.

Verifying trigonometric identities depends on the standard basic trigonometric identities. Everyone needs to be intimately familiar with each of these.

The Basic Trigonometric Identity

$$\cos^2\theta + \sin^2\theta = 1$$

This is *the* basic trigonometric identity, upon which most of the others are based. It is essentially the *Pythagorean Theorem* in disguise and is the basis of a total of roughly nine different identities, each obtained by taking this basic identity and subtracting either $\cos^2\theta$ or $\sin^2\theta$ from both sides, or taking one of those identities and subtracting the square of one of the six trigonometic functions from both sides.

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If we subtract 1 from both sides, we get the identity

$\tan^2\theta = \sec^2\theta - 1.$

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Subtracting $\cot^2\theta$ from both sides gives the variation

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Subtracting $\cot^2\theta$ from both sides gives the variation
$\csc^2\theta - \cot^2\theta = 1.$
Subtracting 1 from both sides gives the variation
$\cot^2\theta = \csc^2\theta - 1.$

The Really Basic Trigonometric Identities

These identities come immediately from the definition of the trigonometic functions.

- $\tan\theta = \frac{\sin\theta}{\cos\theta}$
- $\sec\theta = \frac{1}{\cos\theta}$
- $\csc\theta = \frac{1}{\sin\theta}$
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$
- $\cot\theta = \frac{1}{\tan\theta}$

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$$\text{Left Hand Side} = \text{Something} = \text{Something Else} = \ldots$$

$$= \cdots = \text{Right Hand Side}$$

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$(1+\sin\theta)(1-\sin\theta) = 1-\sin^2\theta = \cos^2\theta$
$(1+\cos\theta)(1-\cos\theta) = 1-\cos^2\theta$
These are analogous to $(1+\sqrt{x})(1-\sqrt{x}) = 1-x$.

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Notice how each calculation is clearly true and doesn't really need any explanation. There is no question about the correctness of this verification.

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This could have been done many other ways. That is generally the case.