

Exponential Functions

Functions of the form $f(x) = a^x$, $a > 0$, $a \neq 1$, are called *exponential functions*.

More generally, functions of the form $f(x) = a \cdot b^x + c$ will also be referred to as exponential functions.

Key Properties

- $a^x > 0$ (exponential functions are always positive)
- a^x is defined for all real x
- $a^0 = 1$

The Exponential Function

One exponential function, $\exp(x) = e^x$, is called *the exponential function*. $e \approx 2.718281828$ is a special mathematical constant.

Applications

Exponential functions come up naturally in applications involving interest, population growth and radioactive decay.

Interest

Let

P_0 represent the initial balance placed in an account

r represent the annual interest rate

t represent the amount of time, measured in years, the money is left in the account

n represent the number of times interest is compounded annually

P represent the balance in the account

Compound Interest Formula:

$$P = P_0 \left(1 + \frac{r}{n}\right)^{nt}.$$

Continuous Interest Formula:

$$P = P_0 e^{rt}$$

This formula applies when interest is compounded continuously. This is the limiting situation when interest is compounded more and more frequently, as $n \rightarrow \infty$.

Population Growth

The formula for population growth is the same as the formula for compound interest, with the variables simply represent the population and the annual growth rates.

Radioactive Decay

Let:

x_0 represent the initial amount of a radioactive substance

t represent the amount of time the substance is left to decay

h represent the half-life of the substance, the amount of time it takes for half the substance to disintegrate

x represent the amount of the substance.

$$x = x_0 \left(\frac{1}{2}\right)^{t/h}.$$

We can use these formulas to find a balance, the size of a population or the amount of a radioactive substance which is left at a given time.

Logarithmic Functions

Another important question is to determine when a balance or a population or the amount of a radioactive substance will be a certain size. In order to determine these, logarithms are helpful.

Definition 1 (Logarithm). $\log_b x$ is the power b must be raised to in order to obtain x .

In other words, $y = \log_b x$ if and only iff $b^y = x$. This leads to a key property of logarithms:

$$b^{\log_b x} = x.$$

A related property, also immediate from the definition, is

$$\log_b b^x = x.$$

With the notation used, b is called the base of the logarithm and $\log_b x$ is called the *logarithm to the base b of x* .

Looked at as a function, the logarithm to the base b can be thought of as the inverse of the exponential function to the base b .

The logarithm to the base e , the inverse of the exponential function $\exp(x) = e^x$, is called the *natural logarithm function* and is often denoted by \ln .

The properties of logarithms can be inferred almost directly from the properties of exponentials.

Properties of Logarithms

The following is a list of key properties, with the special case for natural logarithms listed next to the general property for all logarithms. Some of the rules are justified, either in general or in the case of natural logs.

- $b^{\log_b x} = x, e^{\ln x} = x$
- $\log_b b^x = x, \ln(e^x) = x$
- The domain of any log function is the set of positive reals. In other words, $\log_b x$ is defined for $x > 0$.
- $\log_b 1 = 0, \ln 1 = 0$

Justification: $e^0 = 1$.

- $\log_b(xy) = \log_b x + \log_b y$, $\ln(xy) = \ln x + \ln y$

(The log of a product is the sum of the logs.)

Justification: Let $X = \ln x$, $Y = \ln y$. It follows that $e^X = x$, $e^Y = y$, so $xy = e^X e^Y = e^{X+Y}$. From the definition of logs, it follows that $X + Y = \ln(xy)$, so $\ln(xy) = \ln x + \ln y$.

- $\log_b(x/y) = \log_b x - \log_b y$, $\ln(x/y) = \ln x - \ln y$

(The log of a quotient is the quotient of the logs.)

- $\log_b(x^y) = y \log_b x$, $\ln(x^y) = y \ln x$

(The log of something to a power is the power times the log.)

Justification: Let $X = \ln x$. It follows that $e^X = x$, so $x^y = (e^X)^y = e^{(Xy)}$, so by the definition of a logarithm $Xy = \ln(x^y)$ and thus $y \ln x = \ln(x^y)$.

Logarithms used to be very useful for calculations. The fact that a log of a product equals the sum of logs was used, in conjunction with extensive tables of logarithms, to turn multiplication problems into addition problems and the fact that a log of a quotient equals the difference of logs was used to turn division problems into subtraction problems.

Today, the properties of logarithms are used to solve exponential equations, that is, equations where the unknown occurs in an exponent. These equations come up in problems involving interest, population growth and radioactive decay.

Generally, natural logs are used to solve exponential equations, although in theory any base may be used.

The basic idea is to take an exponential equation and equate the logarithms of the two sides, using the properties of logarithms to obtain an ordinary equation.

In the applications involving interest, population growth and radioactive decay, the ordinary equation obtained is generally linear and thus easily solved.

Example: Solve $5^x = 3$.

Solution: $\ln(5^x) = \ln 3$, so $x \ln 5 = \ln 3$ and thus $x = \frac{\ln 3}{\ln 5}$.

Example: \$700 is placed in an account paying interest at an annual rate of 3%. How long will it take for the balance to double.

Solution: Using the Continuous Interest Formula, we know $P = 700e^{0.03t}$, where P is the balance and t is the amount of time the money is left in the account. We need to know the value of t for which $P = 2 \cdot 700 = 1400$, so we solve the equation $700e^{0.03t} = 1400$.

It's convenient, although unnecessary, to first divide both sides by 700 to get $e^{0.03t} = 2$. We may then equate the logarithms of the two sides to get $\ln(e^{0.03t}) = \ln 2$, so $0.03t = \ln 2$ and $t = \frac{\ln 2}{0.03} \approx 23.0149060187$.

So it takes approximately 23.0149060187 years for the balance to double.

Exercise: Convert 23.0149060187 years to years, days, hours and minutes, rounding off to the nearest minute.