Polynomial Equations and Inequalities

We will consider polynomial equations first and assume each equation has been put in the form $p(x) = 0$, where $p(x)$ is a polynomial of degree n. Most polynomials we consider will have integer coefficients and most equations we consider will have integer solutions. Most of the properties discussed hold for arbitrary polynomials, but it will generally not be feasible to solve arbitrary polynomial equations by hand.

Key Properties:

- $x - r$ is a factor of $p(x)$ if and only if r is a solution of the equation $p(x) = 0$. *Equivalently, $x - r$ is a factor of $p(x)$ if and only if r is a zero of $p(x)$.*

- If $p(x)$ is a polynomial with integer coefficients and r is an integer solution of $p(x) = 0$, then r must be a divisor of the constant term of $p(x)$.

These properties lead to a strategy for solving polynomial equations. It will always work provided the polynomial has integer coefficients and the solutions are integers.

Strategy

We assume the equation is in the form $p(x) = 0$ and $p(x)$ has integer coefficients.

1. Check all the divisors of the constant term of $p(x)$ until you find a solution r.

2. Factor $p(x) = (x - r)p^*(x)$.

3. Continue the same way, now with $p^*(x)$, until $p(x)$ has been factored completely.

4. Pick out all the solutions of the equation $p(x) = 0$. *Once $p(x)$ has been factored completely, you should be able to find the solutions at sight.*

Once you have a factor $x - r$ of $p(x)$, one can factor in any way one prefers. As a last resort, one can always use long division to find $\frac{p(x)}{x-r}$ and factor $p(x) = (x - r) \cdot \frac{p(x)}{x-r}$.

Special Cases

Linear and quadratic equations are special cases. The strategy given will work in those cases, but it's probably overkill – especially for linear equations.

Remember: If you ever see anything special you can make use of, do so. Very often, a problem can be solved routinely but can be solved more easily if one observes a special circumstance.

Polynomial Inequalities

Everything learned about solving polynomial equations applies to polynomial inequalities; polynomial inequalities just require a few more steps.

A polynomial inequality may be written in the form $p(x)$ R 0, where R is one of $<$, $\leq$, $>$, $\geq$.

Procedure

1. Use the strategy for solving an equation to factor $p(x)$ completely, writing $p(x) = (x-r_1)(x-r_2)\cdots(x-r_n)$. *For convenience, we will assume $r_1 < r_2 < \cdots < r_n$.*

2. Recognize the zeros of $p(x)$, r_1, r_2, . . . ,r_n, divide the real line into $n+1$ intervals.

3. Consider each interval separately. On each interval:

(a) Analyze the sign of each factor $x - r_k$.

(b) Use these signs to determine the sign of $p(x)$ on that interval.

(c) Determine whether the points in that interval are in the solution set of the inequality.

4. Determine which, if any, of the zeros r_1, r_2, . . . ,r_n are in the solution set.

5. Put everything together and write down the solution set.