Equations in One Variable

Definition 1 (Equation)**.** *An equation is a statement that two algebraic expressions are equal.*

Definition 2 (Solution)**.** *A solution or root is a value which yields a true statement when it replaces the variable.*

Definition 3 (Solution Set)**.** *The solutions set to an equation is the set of solutions to the equation.*

Definition 4 (Equivalent Equations)**.** *Two equations are said to be equivalent if they have the same solution set.*

The process of solving an equation generally consists of finding a sequence of equations which are all equivalent until we have an equation for which the solution set is readily apparent.

Generally – there are some minor exceptions – we may treat the two sides of an equation the same way to get an equivalent equation.

Given an equation $A = B$, where A and B are algebraic expressions, and a real number $C \neq 0$, the following equations will all be equivalent to $A = B$:

$A + C = B + C$

$A - C = B - C$

$AC = BC$

$\frac{A}{C} = \frac{B}{C}$

In other words, we can add the same number to both sides of an equation, or subtract the same number from both sides of an equation, or multiply both sides of an equation by the same number, or divide both sides of an equation by the same number, and we will get an equivalent equation.

Rather than adding, subtracting, multiplying or dividing a real number, we can generally use an expression as long as we recognize the new equation we get may have some solutions the original did not have. This can occur if there are values of the independent variable for which the expression is equal to 0. These extra solutions are sometimes called *extraneous.*

Similarly, we can raise both sides of an equation to a power, with the understanding we may be introducing extraneous solutions.

The general principle is that we need to do the same thing to both sides of an equation.

We will later observe the same principle holds for inequalities, which may be treated very similarly to the way we treat equations.

Types of Equations

Identity – An equation which is satisfied by every value for the variable for which either side may be evaluated is called an identity.

Conditional Equation – An equation which has at least one solution but is not an identity.

Inconsistent Equation – An equation which has no solutions.

Solving Linear Equations

The simplest type of equation is a linear equation, an equation in which the variable only occurs to the first power.

The most general form of a linear equation is

$$ax + b = cx + d.$$

Linear equations may be solved by finding equivalent equations where the variable only occurs on the left and the constants only on the right, at which point we can divide both sides by the coefficient of the variable.

Example: $11x + 3 = 8x + 24$

Solution: Get all the terms involving x on the left by subtracting $8x$ from both sides to get $3x + 3 = 24$. Then get all the terms involving constants on the right by subtracting 3 from both sides to get $3x = 21$. Finally, divide both sides by 3, the coefficient of the variable x, to get $x = 7$.

This latter equation, equivalent to the first, clearly has just one solution, 7. We should write our conclusion in the form *the solution set is* $\{7\}$, but most people will simply write $x = 7$ and the conclusion will be understood correctly.

Linear Inequalities

Inequalities, like equations, are mathematical statements which may be true for some values of a variable and false for other values. Inequalities may be solved in a manner similar to the manner in which equations may be solved.

As with equations, we may add the same thing to both sides, subtract the same thing to both sides, multiply both sides by the same thing or divide both sides by the same thing to get an equivalent inequality. As with equations, ideally we get a string of equivalent inequalities until one of them is easy to solve.

Caution: The one thing we must be aware of is that if we multiply or divide both sides by a negative number, the sense of the inequality reverses.

Example: $3x + 1 < 19$.

The steps are really the same as for the equation $3x + 1 = 19$:

a. Subtract 1 from both sides to get $3x < 18$.

b. Divide both sides by 3 to get $x < 6$.

So the solution set is $\{x : x < 6\} = (-\infty, 6)$.

Example: $5x + 8 > 10x - 20$.

Solution:

$-5x + 8 > -20$

$-5x > -28$

$x < \frac{-28}{-5}$

$x < \frac{28}{5}$

Note: When dividing by -5, we had to reverse the sense of the inequality. We finally obtain the solution set $\{x : x < \frac{28}{5}\} = (-\infty, -\frac{28}{5})$.

Equations Involving Rational Expressions

These may be solved in a manner similar to the way complex rational expressions are simplified. One can multiply both sides by any denominator that appears. One can keep doing this until one is left with an equation which has no denominator.

Example: $\frac{2x+3}{x+2} = \frac{13}{7}$

One may multiply both sides by $x + 2$ to get $2x + 3 = \frac{13x+26}{7}$, and then multiply both sides by 7 to get $14x+21 = 13x+26$. This is now an ordinary linear equation, which may be solved as follows: $x + 21 = 26$, $x = 5$.

Inequalities Involving Rational Expressions

As with linear inequalities and linear equations, inequalities involving rational expressions may be solved in a manner analogous to the way equations involving rational expressions are solved.

Example: $\frac{2x+3}{x+2} < \frac{13}{7}$

If we had the equation $\frac{2x+3}{x+2} = \frac{13}{7}$, we might start by multiplying both sides by $7(x+2)$ to eliminate the denominators. For the inequality, we may do the same, but we have to pay attention to whether $x+2$ is positive or negative. This forces us to divide the process into two cases: $x > -2$ and $x < -2$.

Case 1: $x > -2$. Here, when multiplying by $7(x+2)$ we get:

$14x + 21 < 13x + 26$

$x + 21 < 26$

$x < 5.$

We thus observe that when $x > -2$, x will be a solution if $x < 5$. In other words, we have obtained the information that every number x such that $-2 < x < 5$ is a solution.

Case 2: $x < -2$. Here, since $x + 2$ is negative, when multiplying by $7(x+2)$ we have to reverse the sense of the inequality to get:

$14x + 21 > 13x + 26$

$x + 21 > 26$

$x > 5.$

We thus observe that when $x < -2$, x will be a solution if $x > 5$. Obviously, there is no such value of x.

We conclude the solution set is $\{x : -2 < x < 5\}$.

Alternative Method of Solving Rational Inequalities

We may use the following fact about inequalities to solve rational inequalities. It will also yield a very nice method for solving inequalities involving polynomials.
Theorem 1. *The solution set to an inequality consists of a union of intervals, with each end point of each interval being a point at which either the two sides of the inequality are equal or at least one of the sides does not exist.*

Example: $\frac{2x+3}{x+2} < \frac{13}{7}$

We know the two sides are equal only when $x = 5$, while the left side is not defined when $x = -2$. Thus, the only possible endpoints of intervals in the solution set are -2 and 5 and the only intervals we need to consider are $(-\infty,-2)$, $(-2,5)$ and $(5,\infty)$. Each of those intervals must be either totally within the solution set or totally disjoint from the solution set.

We may determine which possibility is the actual case by looking at a single point in each interval.

Looking at the intervals one-by-one:

$(-\infty,-2)$: We can take any point in that interval. For example, take $x=-3$. The inequality would become $\frac{2(-3)+3}{-3+2} < \frac{13}{7}$, $3 < \frac{13}{7}$. Since this is clearly false, -3 is not in the solution set and thus $(-\infty,-2)$ is disjoint from the solution set.

$(-2,5)$: Again, we can take any point in that interval. For example, take $x=0$. The inequality would become $\frac{2\cdot 0+3}{0+2} < \frac{13}{7}$, $\frac{3}{2} < \frac{13}{7}$.

Since this is true, 0 is in the solution set and thus the entire interval $(-2, 5)$ is in the solution set.

$(5, \infty)$: Again, we can take any point in that interval. For example, take $x = 12$. The inequality becomes $\frac{2 \cdot 12+3}{12+2} < \frac{13}{7}$, $\frac{27}{14} < \frac{13}{7}$. Since $\frac{13}{7} = \frac{26}{14}$, this is clearly false, so 12 is not in the solution set and thus the entire interval $(5, \infty)$ is disjoint from the solution set.

We conclude the solution set is $(-2, 5)$.

Equations Involving Absolute Value

One can almost always solve basic equations involving absolute value by using the definition of absolute value, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$.

This sometimes gets rather involved and one can often solve equations more simply by looking into the meaning of it.

$|x - 10| = 3$

One can use the definition of absolute value to recognize that $|x-10| = x-10$ when $x-10 \geq 0$, which occurs when $x \geq 10$. Thus, for $x \geq 10$, the equation may be written as

$x - 10 = 3.$

This may be solved easily, obtaining $x = 13$, so clearly 13 is a solution.

On the other hand, when $x-10<0$, $|x-10| = -(x-10) = 10-x$. Thus, in the case $x<10$, the equation may be written as

$$10-x=3.$$

This may also be solved easily, obtaining $x = 7$, so clearly 7 is a solution and the original equation has solution set $\{7, 13\}$.

On the other hand, one may observe $|x-10| = 3$ if $x-10$ is either 3 or -3.

Clearly, $x-10=3$ if $x=13$, while $x-10 = -3$ if $x=7$, so we more easily get the same solution set.

Alternatively, one may observe that $|x-10| = 3$ if x lies exactly 3 units from 10 on a number line. Clearly, this is the case for two numbers, 7 and 13, and they comprise the same solution set.

Note the basic ideas behind the three methods. One was to look at the absolute value of a difference as the distance, on a number line, between the two numbers; one was to recognize that the absolute value of a number can take on a particular value if that number equals either that value or its negative; one was to use the definition of absolute value.

Inequalities Involving Absolute Value

Each of the three ideas for solving equations involving absolute value works in a similar way for inequalities involving absolute value. Consider the following similar example.

Example: $|x - 10| < 3$.

Method 1: Using the idea that the absolute value of a difference is the distance between the points on the number line.

With this interpretation, $|x-10| < 3$ translates to *the distance between x and 10 is less than 3*.

This may also be expressed as x *is less than 3 units from 10*.

With the latter interpretation, it's clear that, since 7 is 3 units to the left of 10 and 13 is

3 units to the right of 10, x will be within 3 units of 10 if it's between 7 and 13. We thus conclude the solution set is $\{x : 7 < x < 13\} = (7, 13)$.

Method 2: Just thinking of $|x-10|$ in terms of absolute value, we can translate $|x-10| < 3$ to *the absolute value of* $x - 10$ *is less than 3*.

If we think about the numbers whose absolute values are less than 3, it's clear that the numbers between -3 and 3 have absolute values less than 3, so that the absolute value of $x - 10$ will be less than 3 if $x - 10$ is between -3 and 3.

In other words, x will be a solution if $-3 < x - 10 < 3$, which is shorthand for $-3 < x - 10$ and $x - 10 < 3$. Adding 10 to all parts, we get $-3 + 10 < x - 10 + 10 < 3 + 10$, or $7 < x < 13$, so we again get the solution set $\{x : 7 < x < 13\} = (7, 13)$.

Method 3: We may use the definition of absolute value to observe $|x - 10| = x - 10$ if $x \geq 10$ while $x - 10 = -(x - 10) = 10 - x$ if $x < 10$. We thus divide the calculations into two cases.

Case 1: $x \geq 10$. In this case, the inequality may be written $x - 10 < 3$, so $x < 13$. We thus see that when $x \geq 10$, x is a solution if $x < 13$. In other words, all numbers x such that $10 \leq x < 13$ are solutions.

Case 2: $x < 10$. In this case, the inequality may be written $10 - x < 3$, so $-x < -7$, $x > 7$. We thus see that when $x < 10$, x is a solution if $x > 7$. In other words, all numbers x such at $7 < x < 10$ are solutions.

Putting the two cases together, we see that x is a solution if $7 < x < 13$, obtaining the same solution set we obtained with the other two methods.

Caution Regarding Notation

In most of the examples so far, one could be somewhat sloppy about the notation used in describing the solution without confusing anyone. That is not always the case, as is shown in the following example. In other words, it's always a good idea to use correct notation.

Example: $|2x - 8| > 20$.

We may solve this as follows:

$|2(x - 4)| > 20$

$2|x - 4| > 20$

$|x - 4| > 10$.

If we read the last inequality as x *is more than 10 units from 4*, it's clear that x is a solution if either x is bigger than 14 or x is less than -6.

Using set notation, the solution set may be expressed as $\{x : x < -6 \text{ or } x > 14\}$.

Using interval notation, the solution set may be expressed as $(-\infty, -6) \cup (14, \infty)$.

Either is correct and unambiguous.

Almost anything else would likely be interpreted in a way different from what is intended. As just one example, if one writes, $x < -6$, $x > 14$, the only reasonable interpretation would be $\{x : x < -6, x > 14\}$. Another way of writing that is $\{x : x < -6 \text{ and } x > 14\}$, which is clearly $\emptyset$, the empty set, which is clearly not the solution set.

Constructing Models to Solve Problems

Strategy for Word Problems

Mathematics, including algebra, may be used to solve a variety of problems of a type sometimes called *word problems* or *verbal problems* or *applications*, which effectively call taking a problem expressed in ordinary language, modelling it with mathematics (building a mathematical model) and using the tools of mathematics to solve it.

In this course, the major tools will involve methods for solving equations or systems of equations; in other courses, the tools may differ but the approach is always the same.

Guidelines

- Read the question!

- Read the question!

- Read the question!

Has the point been made? Everything that must be done generally becomes apparent if one *reads the question*!

Questions to Ask Yourself

- What do I know?
- What don't I know?
- What can I figure out or infer?
- What do I want?
  This is actually the least important!

What I Know

Every fact translates into a mathematical statement, generally a formula, equation or statement that a particular variable takes on a certain value when another variable takes on a certain value.

The key to writing down an appropriate formula corresponding to a given fact is to write down that fact in plain language and then rewrite the fact using the descriptions of variables already defined and a verb such as *is* or *equals* which indicates that two quantities are equal.

What I Don't Know

This is often the key. Any unknown quantity can often be profitably represented by a variable.

Related to unknown quantities are variable quantities. These almost always need to be represented by variables.

What I Want

This needs to be kept in the back of your mind. One common mistake is to concentrate too hard on what you want; it's generally more advantageous to concentrate on what you know, what you don't know and what you can figure out from what you know.

Just remember to notice when you've actually figured out what you ultimately want.

Suggestions

- Draw pictures, charts, graphs or anything visual that may help you understand the problem. The key is understanding and translating facts to mathematics.

- Look for variables and unknowns. Represent them by symbols. Write down what each stands for and make sure you don't use the same symbol to represent two different quantities.

- Write down known facts in terms of the variables and unknowns you've defined. These will generally be in the form of equations and formulas.

- Solve equations where possible.

Equations and Graphs in Two Variables

The Coordinate Plane

Points in a plane may be located using a coordinate system. The most common is the Cartesian Coordinate System, also called the Rectangular Coordinate System.

Two axes, one horizontal (generally but not always called the $x-$ axis) and one vertical (generally but not always called the $y-$ axis) are drawn. The location of a point is determined by two numbers, called coordinates.

The first coordinate, called the abscissa, represents the (signed) horizontal distance from the vertical axis.

The second coordinate, called the ordinate, represents the (signed) vertical distance from the horizontal axis.

The two coordinates are placed in parentheses, separated by a comma.

Example: $(5, -3)$ would indicate a point five units to the right of the vertical axis and three units below the horizontal axis.

Note: This is very similar to the system of longitude and latitude used to locate points on the planet we reside on.

Distance Between Two Points

The distance between two points may be found using the Pythagorean Theorem. When the calculation is done in the abstract, we obtain *the Distance Formula*.

Suppose we have two points, $P_1(x_1, y_1)$, $P - 2(x_2, y_2)$. For convenience, assume P_2 is above and to the right of P_1, so $x_1 < x_2$ and $y_1 < y_2$. This makes the derivation of the Distance Formula easier; the actual formula is correct in general.

If you draw a horizontal line through P_1 and a vertical line through P_2, along with the line connecting P_1 and P_2, you get a right triangle, with the hypotenuse being the line segment from P_1 to P_2.

The length of the horizontal side will be $x_2 - x_1$, while the length of the vertical leg will be

$y_2 - y_1$, so if we call the length of the hypotenuse d the Pythagorean Theorem implies $(x_2-x_1)^2+(y_2-y_1)^2 = d^2$. In effect, this is the Distance Formula, which may also be written in the following ways:

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

or

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Midpoints

The midpoint of the line segment joining two points is the point midway between the two points on the line segment connecting them. In order to get the coordinates of the midpoint, effectively we just average the coordinates of the endpoints.

In other words, suppose we have two points, (x_1, y_1), (x_2, y_2).

The average of the first coordinates is $\frac{x_1+x_2}{2}$.

The average of the second coordinates is $\frac{y_1+y_2}{2}$.

So the midpoint is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Example: The midpoint between $(5, 8)$ and $(-23, 14)$ is $(-9, 11)$, since the average of 5 and -23 is $\frac{5+(-23)}{2} = \frac{-18}{2} = -9$ and $\frac{8+14}{2} = \frac{22}{2} = 11$.

Linear Equations in Two Variables

If we have an equation using two variables, say x and y, we represent the solution graphically. If we write the equation in the form $F(x,y) = G(x,y)$, where we can think of $F(x,y)$ and $G(x,y)$ as algebraic expressions which may or may not contain the variables x and y, the graph is simply the set of points $\{(x,y) : F(x,y) = G(x,y)\}$.

This may be thought of as the set of points whose coordinates satisfy the equation. This may be thought of as meaning the coordinates of (x,y) satisfy the equation if $F(x,y) = G(x,y)$, or $x = a$, $y = b$ satisfies the equation if $F(a,b) = G(a,b)$.

We particularly look at **Line**ar Equations (whose graphs are straight lines), Quadratic functions (whose graphs are parabolas), second degree equations (whose graphs are called conic sections) and circles (which happen to be conic sections).

Lines

The key property of a line is its *slope*. The slope of a line is the tangent of the angle the line makes with the horizontal. We actually define slope as follows.

Definition 5 (Slope)**.** *Consider a line through the points (x_1, y_1), (x_2, y_2). Its slope, often denoted by m, is the quotient $\frac{y_2 - y_1}{x_2 - x_1}$.*

Exercise: Show that the slope does not depend on the choice of points on the line.

Example: The slope of the line through $(2, 8)$ and $(5, 23)$ is $\frac{23-8}{5-2} = 5$.

If a line rises as we go from left to right, it will have a positive slope. The steeper the line, the bigger the slope.

If a line falls as we go from left to right, it will have a negative slope. The steeper the line, the smaller the slope. *Pay attention to what that really means; it does not mean "closer to 0."*

A line making an angle of $\pi/4$ (45° in degree measure) with the horizontal will have slope ± 1.

Some students think of slope as $\frac{\text{rise}}{\text{run}}$ or $\frac{\Delta y}{\Delta x}$.

Equations of Lines

The simplest way to obtain an equation of a line requires a point on the line and the slope of the line. It is based on the fact that if a line contains the point (x_1, y_1) and has slope m, then a point (x, y) will be on that line precisely if $m = \frac{y-y_1}{x-x_1}$, which is equivalent to $m(x-x_1) = y - y_1$. Th is $y-4=6(x-8)$. It also has equation $y-28=6(x-12)$.

Suppose we take those two different equations obtained in the last example and simplify them.

$y-4=6(x-8)$, $y-4=6x-48$, $y=6x-44$

$y-28=6(x-12)$, $y-28=6x-72$, $y=6x-44$.

In each case, we get the same equation: $y=6x-44$. This is in another standard form, called the *Slope-Intercept* form, which in this case enables us to see the slope is -6 and the $y-$intercept is -44.

The standard *Slope-Intercept Formula* is $y=mx+b$, where m is the slope and b is the $y-$intercept. This is really a special case of the *Point-Slope Formula*, where the point known is the $y-$

m and goes through the point $(0, b)$, the Point-Slope Formula gives the equation $y - b = m(x - 0)$, which easily simplifies to $y - b = mx$, $y = mx + b$.

Example: Consider the line through $(3, 7)$ with slope 10.

Using the Point-Slope Formula, we get the equation $y - 7 = 10(x - 3)$.

Simplifying, we may also get $y - 7 = 10x - 30$, $y = 10x - 23$, which is in the Slope-Intercept Form.

Circles

Definition 6 (Circle)**.** *A circle is the set of points a fixed distance, called the radius of the circle, from a fixed point, called the center of the circle.*

Suppose a circle has center (a, b) and radius r. An arbitrary point (x, y) will be on the circle if its distance from the center (a, b) is r. Using the *distance formula*, this will be the case precisely when $(x-a)^2+(y-b)^2 = r^2$. This gives an equation for the circle in standard form.

Example: A circle of radius 5 has center $(2, 7)$. Its equation may be written $(x-2)^2+(y-7)^2 = 25$.

Example: A circle of radius 3 has center $(8, -5)$. Its equation may be written $(x-8)^2+(y+5)^2 = 9$.

Example: $(x-3)^2+(y+9)^2 = 121$ is an equation for the circle with center $(3,-9)$ and radius 11.

Example: $x^2 - 6x + y^2 = 55$

This takes a bit more work. We may *complete the squares* by noting $(x-3)^2 = x^2 - 6x + 9$, so that $x^2 - 6x = (x-3)^2 - 9$, and thus write the equation in the form $(x-3)^2 - 9 + y^2 = 55$ or $(x-3)^2 + y^2 = 64$ and thus recognize an equation of the circle with center $(3,0)$ and radius 8.

Completing the Square

The method of *completing the square* comes in handy in a number of instances. The goal in completing the square is to rewrite a quadratic as a square plus or minus a constant, effectively eliminating the linear (first degree) term.

The key observation is that when one squares a binomial of the form $(x + a)$, one gets a quadratic where the linear term has a coefficient twice the constant in the original binomial.

Another way of saying that is the constant term of the binomial is half the coefficient of the linear term in its square.

Examples:

$(x + 1)^2 = x^2 + 2x + 1$: 2 is twice 1; 1 is half of 2

$(x+5)^2 = x^2 + 10x + 25$: 10 is twice 5; 5 is half of 10

$(x-8)^2 = x^2 - 16x + 64$: -16 is twice -8; -8 is half of -16

This leads to the observation that to complete the square of a quadratic, start by calculating the square of

$x+$ half the coefficient of the linear term.

Example: Complete the square of $x^2 + 6x$.

The coefficient of the linear term is 6. Half of 6 is 3, so calculate $(x+3)^2 = x^2 + 6x + 9$. It easily follows that $x^2 + 6x = (x+3)^2 - 9$.

Example: Complete the square of $x^2 - 8x + 5$.

The coefficient of the linear term is -8. Half of -8 is -4, so calculate $(x-4)^2 = x^2 - 8x + 16$.

It easily follows that $x^2 - 8x = (x-4)^2 - 16$ and thus $x^2 - 8x + 5 = (x-4)^2 - 16 + 5 = (x-4)^2 - 11$.

In general, we may note $(x+\frac{b}{2})^2 = x^2+bx+\frac{b^2}{4}$, so $x^2 + bx = (x+\frac{b}{2})^2 - \frac{b^2}{4}$ and $x^2 + bx + c = (x+\frac{b}{2})^2 - \frac{b^2}{4} + c = (x+\frac{b}{2})^2 + c - \frac{b^2}{4}$. We may look at

$$x^2 + bx + c = (x+\tfrac{b}{2})^2 + c - \tfrac{b^2}{4}$$

as a *completing the squares* formula, although it is probably better practice to work out the calculation in each individual case.

Non-Monic Polynomials

Sometimes we need to complete the square for a quadratic that is not monic, for which the leading coefficient is not 1. In that case, we may factor out the leading coefficient, compete the square of the quadratic factor, and then go back and multiply through by the leading coefficient again. It is sometimes more convenient to deal with the constant term of the original polynomial separately.

Example: Complete the square for $3x^2+12x-5$.

Write $3x^2+12x$ as $3(x^2+4x)$. Completing the square for x^2+4x, we get $x^2+4x=(x+2)^2-4$.

Thus, $3x^2+12x=3[(x+2)^2-4]=3(x+2)^2-12$ and $3x^2+12x-5=3(x+2)^2-12-5=3(x+2)^2-17$.

Sometimes we have to deal with fractions. The

We may also get a general *Completing the Square* formula by completing the square on an arbitrary quadratic $ax^2 + bx + c$.

Writing $ax^2+bx = a(x^2+\frac{b}{a}x)$, we can complete the square on $x^2 + \frac{b}{a}x$ by looking at $x + \frac{b}{2a}$. Squaring, we get $(x + \frac{b}{2a})^2 = x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}$, so that $x^2 + \frac{b}{a}x = (x + \frac{b}{2a})^2 - \frac{b^2}{4a^2}$ and $ax^2 + bx = a[(x + \frac{b}{2a})^2 - \frac{b^2}{4a^2}] = a(x + \frac{b}{2a})^2 - \frac{ab^2}{4a^2}$.

It then follows that

$$ax^2 + bx + c = a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c.$$

This may be viewed as a general Completing the Squares formula.

Some resemblence to the Quadratic Formula may be observed in this formula; indeed, the Quadratic Formula may be derived by using Completing the Squares to solve an arbitrary quadratic equation.

Quadratic Equations

A quadratic equation is an equation which is equivalent to an equation in the form $ax^2 + bx + c = 0$.

Quadratic equations may be solved using factoring along with the property that the only way a product can be 0 is if one of the factors is 0.

The general strategy is to rewrite the quadratic in the standard form $ax^2 + bx + c = 0$, factor the left hand side, and observe when one of the factors on the left is 0.

Example: Solve $x^2 + 10x = 144$.

We rewrite this as the equivalent quadratic equation $x^2 + 10x - 144 = 0$. Since $x^2 + 10x - 144$ may be factored as $(x-8)(x+18)$, we may also write the equation in the form

$(x-8)(x+18)=0$. Since $x-8$ is 0 when $x=8$, while $x+18$ is 0 when $x=-18$, there are clearly two solutions, 8 and -18. We may wish to write *the solution set is* $\{8,-18\}$.

We may also solve this equation using *Completing the Squares*, in which case we don't even have to put it in standard form first.

Completing the square, we observe $(x+5)^2 = x^2+10x+25$, so $x^2+10x=(x+5)^2-25$. It follows that we may write the equation in the form $(x+5)^2-25=144$, which is equivalent to the equation $(x+5)^2=169$. Since the two square roots of 169 are ± 13, it's clear that x will satisfy the equation if and only if $x+5$ is either 13 or -13. Since $x+5$ equals 13 when $x=8$, while $x+5$ equals -13 when $x=-18$, it follows that 8 and -18 are the only solutions to the equaton.

Quadratic Formula

Suppose we have an arbitrary quadratic equation in the standard form $ax^2 + bx + c = 0$. Completing the Square, we can write it in the form

$$a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c = 0$$

and do the following manipulations to get a set of equivalent equations until we have something easier to solve.

$$a(x + \frac{b}{2a})^2 = \frac{b^2}{4a} - c$$

$$a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a}$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

Thus, x will be a solution if and only iff $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$.

This will be the case if and only if $x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

This gives us the *Quadratic Formula*, that the solutions of an equation $ax^2 + bx + c = 0$ are $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$.

Note: This is a shorthand way of writing two separate solutions, $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$.

We also note we need to be careful whenever calculating a square root, which leads to three possibilities.

If $b^2 - 4ac > 0$, the quadratic formula gives two distinct solutions.

If $b^2 - 4ac = 0$, the quadratic formula just gives one real solution, $-\frac{b}{2a}$. This is often referred to as a double root.

If $b^2 - 4ac < 0$, $\sqrt{b^2 - 4ac}$ is a pure imaginary number. In this case, there are no real solutions, although there are two distinct complex solutions.

Example: Solve $x^2 + 10x = 144$.

We may rewrite this in the form $x^2 + 10x - 144 = 0$ and can apply the Quadratic Formula with $a = 1$, $b = 10$, $c = -144$. This gives $\frac{-10 \pm \sqrt{10^2 - 4 \cdot 1 \cdot (-144)}}{2 \cdot 1} = \frac{-10 \pm \sqrt{676}}{2} = \frac{-10 \pm 26}{2} = \frac{2(-5 \pm 13)}{2} = -5 \pm 13$.

Since $-5 + 13 = 8$ and $-5 - 13 = -18$, we get the two solutions 8 and -18, the same solutions obtained previously using factoring and completing the square.

Example: Solve $x^2 - 6x + 9 = 0$.

We may apply the Quadratic Formula with $a = 1$, $b = -6$, $c = 9$ to get $\frac{-(-6) \pm \sqrt{(-6)^2 - 4 \cdot 1 \cdot 9}}{2 \cdot 1} =$

$\frac{6\pm\sqrt{0}}{2} = \frac{6}{2} = 3$, so we have the single, double solution 3.

Note this could have been solved by factoring by writing the equation in the form $(x-3)^2 = 0$.