Real Numbers and their Properties

Types of Numbers

- $\mathbb{Z}^+$ Natural numbers - counting numbers - $1, 2, 3, \ldots$ The textbook uses the notation N.

- $\mathbb{Z}$ Integers - $0, \pm 1, \pm 2, \pm 3, \ldots$ The textbook uses the notation J.

- Q Rationals - quotients (ratios) of integers.

- $\mathbb{R}$ Reals - may be visualized as corresponding to all points on a number line.

The reals which are not rational are called irrational.

$\mathbb{Z}^+ \subset \mathbb{Z} \subset Q \subset \mathbb{R}$.

$\mathbb{R} \subset \mathbb{C}$, the field of *complex numbers*, but in this course we will only consider *real numbers*.

Properties of Real Numbers

There are four binary operations which take a pair of real numbers and result in another real number:

Addition $(+)$, Subtraction $(-)$, Multiplication $(\times$ or $\cdot)$, Division $(\div$ or $/)$.

These operations satisfy a number of rules. In the following, we assume $a, b, c \in \mathbb{R}$. (In other words, a, b and c are all real numbers.)

- Closure: $a + b \in \mathbb{R}$, $a \cdot b \in \mathbb{R}$.

This means we can add and multiply real numbers. We can also subtract real numbers and we can divide as long as the denominator is not 0.

- Commutative Law: $a+b=b+a$, $a \cdot b=b \cdot a$.

This means when we add or multiply real numbers, the order doesn't matter.

- Associative Law: $(a+b)+c=a+(b+c)$, $(a \cdot b) \cdot c=a \cdot (b \cdot c)$.

We can thus write $a+b+c$ or $a \cdot b \cdot c$ without having to worry that different people will get different results.

- Distributive Law: $a \cdot (b+c)=a \cdot b+a \cdot c$, $(a+b) \cdot c=a \cdot c+b \cdot c$.

The distributive law is the one law which involves both addition and multiplication. It is used in two basic ways: to multiply two factors where one factor has more than one term and to factor out a common factor when we add or subtract a number of terms, all of which contain a common factor.

• 0 is the *additive identity*, 1 is the *multiplicative identity*.

$a + 0 = 0 + a = a$, $a \cdot 1 = 1 \cdot a = a$

• Additive Inverse: Every $a \in \mathbb{R}$ has an *additive inverse*, denoted by $-a$, such that $a+(-a) = 0$, the additive identity.

• Multiplicative Inverse: Every $a \in \mathbb{R}$ except for 0 has a *multiplicative inverse*, denoted by a^{-1} or $\frac{1}{a}$, such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$, the multiplicative inverse.

• Cancellation Law for Addition: If $a+c = b+c$, then $a = b$. This follows from the existence of an additive inverse (and the other laws), since if $a+c=b+c$, then $a+c+(-c)=b+c+(-c)$, so $a+0=b+0$ and hence $a=b$.

• Cancellation Law for Multiplication: If $a \cdot c = b \cdot c$ and $c \neq 0$, then $a=b$. This follows from the existence of an multiplicative inverse for c (and the other laws), since if $a \cdot c = b \cdot c$, then $a \cdot c \cdot c^{-1} = b \cdot c \cdot c^{-1}$, so $a \cdot 1 = b \cdot 1$ and hence $a=b$.

From these rules, we can see why multiplication by 0 gives 0: $a \cdot 0 + 0 = a \cdot 0 = a \cdot (0+0) = a \cdot 0 + a \cdot 0$. Thus $a \cdot 0 + 0 = a \cdot 0 + a \cdot 0$ and from the cancellation law it follows that $0 = a \cdot 0$.

We can now see why multiplication by -1 yields the additive inverse of a number: $a + (-1) \cdot a = 1 \cdot 1 + (-1) \cdot a = (1 + (-1)) \cdot a = 0 \cdot a = 0$.

We can also see why the product of a positive number and a negative number must be negative, and the product of two negative numbers is positive. More generally, we can see that $(-a) \cdot b = -a \cdot b$ as follows: $a \cdot b + (-a) \cdot b = (a + (-a)) \cdot b = 0 \cdot b = 0$, so $(-a) \cdot b$ must be the additive inverse of $a \cdot b$, in other words, $-a \cdot b$.

Subtraction and Division

All the above rules concern addition and multiplication. Those are the basic operations; subtraction and division are really special cases of addition and multiplication.

Definition 1 (Subtraction). $a - b = a + (-b)$.

Definition 2 (Division). $a \div b = a \cdot b^{-1}$.

Alternate Notations: $a \div b = a/b = \frac{a}{b}$.

This explains why division by 0 is undefined: 0 does not have a multiplicative inverse.

We also get a *Cancellation Law* for division: If $b \neq 0$ and $c \neq 0$, then $\frac{ac}{bc} = \frac{a}{b}$.

It's important to use the Cancellation Law correctly; one may only cancel a factor which is common to both the numerator and the denominator. Often, students incorrectly try to cancel something that is a factor of a term of the numerator or denominator, but not a factor of the numerator or denominator itself.

Terms and Factors

There is a technical difference between *terms* and *factors*, and the word *term* is often misused when one is actually referring to a *factor*.

Terms are added together.

Factors are multiplied together.

$x^3 + 5x^2 - 3x + 2$ has four terms: x^3, $5x^2$, $3x$ and 2.

Technically, one might want to think of $-3x$ *rather than* $3x$ *as the term, thinking of* $x^3 + 5x^2 - 3x + 2$ *as* $x^3 + 5x^2 + (-3x) + 2$*, but the common practice is to call* $3x$ *a term.*

$x^3 + 5x^2 - 3x + 2$ consists of just one factor.

$(x^2 + 5x - 3)(2x + 1)$ has just one term, but two factors. The first factor, $x^2 + 5x - 3$ has three terms and the second factor, $2x + 1$, has two terms, but the entire expression, looked at as a whole, has just one term.

The Substitution Principle

A basic principle in algebra is sometimes called *substitution*. The basic idea is that, in any algebraic expression, anthing can be replaced by anything else that is equal to it.

This is used extensively in solving equations, but is also used a lot in just simplifying algebraic (and trigonometric) expressions.

Absolute Value

Definition 3 (Absolute Value)**.**

$$|a| = \begin{cases} a & \textit{if } a \geq 0, \\ -a & \textit{if } a < 0. \end{cases}$$

Properties of Absolute Value

$$|a| \geq 0$$

$$|-a| = |a|$$

$$|a \cdot b| = |a| \cdot |b|$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \text{ if } b \neq 0.$$

Exponents

Positive integer exponents: If $n \in \mathbb{Z}^+$, $a^n = a \cdot a \cdot a \ldots a$, where the product consists of n identical factors, all equal to a.

Negative exponents: $a^{-n} = \frac{1}{a^n}$ if $a \neq 0$.

Zero exponent: $a^0 = 1$ if $a \neq 0$.

Rational Exponents: If $m, n \in \mathbb{Z}$, $n > 0$, $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$.

$\sqrt[n]{a}$ stands for the n^{th} root of a, the number which, when raised to the n^{th} power, yields a. In other words, $(\sqrt[n]{a})^n = a$.

Rules for Exponents

$$a^m a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Rules for Radicals

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

Important: We cannot simplify sums of radicals.

Order of Operations

Exponentiation

Multiplication and Division

Addition and Subtraction

If we want to change the order, we use parentheses.

Scientific Notation

It's sometimes convenient to write a very large or a very small number as a number between 1 and 10 times a power of 10. This is called *scientific notation*.

Examples:

$52379.281 = 5.2379281 \cdot 10^4$

$0.00003578 = 3.578 \cdot 10^{-5}$

$-857.9 = -8.579 \cdot 10^2$

Many calculators display $E \pm xx$ rather than $10^{\pm xx}$. For example, instead of displaying $3.578 \cdot 10^{-5}$, many calculators would show $3.578E{-5}$. *We should still write the number down using scientific notation, not the way the calculator displays it.*

Polynomials

Definition 4 (Polynomial)**.** *A polynomial is a mathematical expression of the form* $a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$*, where* $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ *and* x *is a variable.*

$a_0, a_1, a_2, \ldots a_n$ are constants and called coefficients.

a_0 is called the constant term.

a_n is called the leading coefficient.

n is the degree of the polynomial.

The variable doesn't have to be x.

A polynomial of degree 1 is called linear.

A polynomial of degree 2 is called quadratic.

A polynomial of degree 3 is called cubic.

Polynomials can be added and subtracted in the obvious way.

Multiplication of Polynomials

Polynomials may be multiplied through the repeated use of the Distributive Law, leading to what's sometimes called the Generalized Distributive Law:

To multiply two polynomials together, one pairs each term of the first factor with each term of the second, multiplying each pair together, and then adds all those individual products together.

Example:

$(x^2 - 5x + 3)(4x - 7)$

The terms of the first factor are x^2, $-5x$ and 3, while the terms of the second are $4x$ and -7. *One may wish to visualize the product as* $(x^2 + (-5x) + 3)(4x + (-7))$.

The pairs of terms may be listed, in an organized way, in either of the following two ways:

$(x^2, 4x)$, $(x^2, -7)$, $(-5x, 4x)$, $(-5x, -7)$, $(3, 4x)$, $(3, -7)$

or

$(x^2, 4x)$, $(-5x, 4x)$, $(3, 4x)$, $(x^2, -7)$, $(-5x, -7)$, $(3, -7)$.

Using the first listing, one gets the following products:

$4x^3$, $-7x^2$, $-20x^2$, $35x$, $12x$, -21.

Adding the products together, one gets:

$4x^3 + (-7x^2) + (-20x^2) + 35x + 12x + (-21)$.

Combining like terms, one obtains the product:

$4x^3 - 27x^2 + 47x - 21.$

Caution:

Many students have learned the evil acronym FOIL. FOIL is simply the special case of the Generalized Distribution Law for the easiest case of all, a binomial multiplied by a binomial. It is no easier to use than the Generalized Distributive Law and its use detracts from the understanding of the much more important Generalized Distributive Law. It is advised that students completely forget about FOIL and avoid it at all costs.

Division of Polynomials

Division of polynomials may be done essentially the same way long division of ordinary decimals is performed. The most common type of division is dividing a linear polynomial (something in the form $ax+b$) into a higher degree polynomial. Each trial quotient is obtained by dividing the ax term into the term in the dividend of highest degree.

When dividing a polynomial by a linear polynomial $x-c$, if we get a remainder then that remainder is actually the value of the polynomial when $x=c$. We can see this if we write

$\frac{p(x)}{x-c} = q(x) + \frac{r}{x-c}$

Multiplying both sides by $x-c$ yields

$p(x) = q(x)(x-c) + r$

Plugging in $x = c$ yields

$$p(c) = q(c)(c - c) + r = q(c) \cdot 0 + r = r.$$

Example: If we try dividing $x^3 + 3x - 7$ by $x + 5$ (which may be thought of as $x - (-5)$), we get $x^2 - 5x + 28$ with a remainder of -147, meaning that $\frac{x^3+3x-7}{x+5} = x^2 - 5x + 28 + \frac{-147}{x+5}$, and $x^3 + 3x - 7 = -147$ when $x = -5$.
Corollary. *Given a polynomial $p(x)$, $x - c$ is a factor if and only if $p(c) = 0$.*

This comes in very handy when solving equations. It is the basis of *solving by factoring*.

Example:

Solve $x^2 + 3x = 10$.

$x^2 + 3x - 10 = 0$

$(x + 5)(x - 2) = 0$

Since $x + 5$ and $x - 2$ are factors, -5 and 2 are solutions of the equation $x^2 + 3x - 10 = 0$ and thus of the equivalent original equation.

Rationalizing

Suppose one has a fractional expression like $\frac{5+\sqrt{3}}{2}$. One can rationalize the numerator as follows:

$$\frac{5+\sqrt{3}}{2} = \frac{5+\sqrt{3}}{2} \cdot \frac{5-\sqrt{3}}{5-\sqrt{3}} = \frac{25-9}{2(5-\sqrt{3})} = \frac{8}{5-\sqrt{3}}.$$

You may remember learning to rationalize a denominator using this method; in practice, it turns out that one rarely if ever needs to rationalize a denominator, but one often needs to rationalize a numerator.

Historical Note: Before the widespread use of calculators, rationalizing a denominator was a useful technique to make some calculations easier. For example, if one needed a decimal approximation to $\frac{1}{\sqrt{2}}$, one used to look up a decimal approximation to $\sqrt{2}$ in a table, getting 1.4142 (if one was interested in four decimal places).

Without rationalizing the denominator, one would then have to calculate $\frac{1}{1.4142}$ by hand, which would be rather tedious. However, if one rationalizes the denominator, one finds $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, so one could equivalently calculate $\frac{1.4142}{2}$ by hand, a much easier calculation.

With the current availability of calculators, such contortions are now anachronisms.

On the other hand, the *skills* learned in rationalizing denominators turn out to be important in rationalizing numerators. One example of this is the following.

Suppose one wants to get the slope of the line tangent to $y = \sqrt{x}$ at the point $(1, 1)$. One can't find it directly, but one might expect the slope of a line between $(1, 1)$ and a point $(x, \sqrt{x})$ on the curve, close to the point $(1, 1)$, to be close to the slope of the tangent.

The slope of that line is $\frac{\sqrt{x}-1}{x-1}$, and when x is close to 1, one expects this to be close to the slope of the tangent. Unfortunately, it's hard to directly estimate the value of $\frac{\sqrt{x}-1}{x-1}$. However, if one rationalizes the numerator, one observes $\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{x-1}\frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{x-1}{(x-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1}$.

The last expression is obviously close to $\frac{1}{2}$ when x is close to 1, and we thus expect the slope of the tangent line to be $\frac{1}{2}$, as indeed it is.

Rational Expressions

The key rules of algebra relating to rational expressions are the following.

Cancellation Law: $\frac{ac}{bc} = \frac{a}{b}$ if $c \neq 0$.

(A common mistake is to think one is using the Cancellation Law when it does not apply.)

Addition: $\frac{a}{d} + \frac{b}{d} = \frac{a+b}{d}$.

Subtraction: $\frac{a}{d} - \frac{b}{d} = \frac{a-b}{d}$.

(If one does not have the same denominator, one finds a common denominator.)

Example: $\frac{3}{10} + \frac{2}{5} = \frac{3}{10} + \frac{4}{10} = \frac{7}{10}$.

If we cannot find a common denominator easily, we can always use the product of the denominators. If the product isn't the least common denominator, we may be able to reduce the sum we get to lower terms, cancelling a common factor of the numerator and denominator.

Example: $\frac{3}{4}+\frac{1}{6}=\frac{3}{4}\cdot\frac{6}{6}+\frac{1}{6}\cdot\frac{4}{4}=\frac{18}{24}+\frac{4}{24}=\frac{22}{24}=\frac{2\cdot 11}{2\cdot 12}=\frac{11}{12}$.

We can find the least common denominator by factoring each denominator into a product of powers of primes and then use each prime to the highest power it appears.

In the preceding example, the two denominators are 4 and 6. We may factor them as:

$4 = 2^2$

$6 = 2 \cdot 3$

The prime factors that appear are 2 and 3. Since the highest power 2 appears as is 2^2 and the highest power 3 appears as is $3 = 3^1$, the least common denominator is $2^2 \cdot 3$. We may then calculate

$\frac{3}{4} + \frac{1}{6} = \frac{3}{4} \cdot \frac{3}{3} + \frac{1}{6} \cdot \frac{2}{2} = \frac{9}{12} + \frac{1}{12} = \frac{11}{12}$, the same sum we obtained before.

The same basic method may be used even if the rational expressions involve variables.

Example: $\frac{x}{2x+1} + \frac{5}{3x} = \frac{x}{2x+1} \cdot \frac{3x}{3x} + \frac{5}{3x} \cdot \frac{2x+1}{2x+1} = \frac{3x^2}{(2x+1)(3x)} + \frac{10x+5}{3x(2x+1)} = \frac{3x^2+10x+5}{3x(2x+1)}$.

Multiplication $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Division: $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$

Division is really multiplication by the multiplicative inverse of the divisor.

Simplifying Complex Rational Expressions

Sometimes one is confronted with rational expressions where the numerator and/or the denominator are themselves rational expressions. There are two equally effective approaches to dealing with these complex fractional expressions.

Method 1: Write the numerator and denominator so that each is a single term, perhaps with a numerator and denominator, and then treat it as a quotient.

Example: $\frac{3x}{\frac{5}{x+1}-2} = \frac{3x}{\frac{5}{x+1}-\frac{2(x+1)}{x+1}} = \frac{3x}{\frac{5}{x+1}-\frac{2x+2}{x+1}} = \frac{3x}{\frac{3-2x}{x+1}} = \frac{3x}{1} \cdot \frac{x+1}{3-2x} = \frac{3x(x+1)}{3-2x}$.

Method 2: Multiply the numerator and denominator by the product of each factor in the denominator of either the numerator or the denominator.

Example: $\frac{3x}{\frac{5}{x+1}-2} = \frac{3x}{\frac{5}{x+1}-2} \cdot \frac{x+1}{x+1} = \frac{3x(x+1)}{5-2(x+1)} = \frac{3x(x+1)}{3-2x}$.