Mathematics 109 — SOLUTIONS

Professor Alan H. Stein

Wednesday, December 7, 2005 — This problem set is worth 50 points.

1. Solve the inequality $x^4 - 27x^2 + 59x \leq 3x^3 + 30$.

   **Solution:** Subtract $3x^3 + 30$ from each side to get the equivalent inequality $x^4 - 3x^3 - 27x^2 + 59x - 30 \leq 0$. We then look for zeros of the polynomial $x^4 - 3x^3 - 27x^2 + 59x - 30$ from among the divisors of 30. We find 1 is a zero of the polynomial, so $x - 1$ must be a factor. Dividing $(x^4 - 3x^3 - 27x^2 + 59x - 30) \div (x - 1) = x^3 - 2 * x^2 - 29 * x + 30$, so $x^4 - 3x^3 - 27x^2 + 59x - 30 = (x - 1)(x^3 - 2 * x^2 - 29 * x + 30)$.

   We now look for zeros of $x^3 - 2 * x^2 - 29 * x + 30$ and find 1 is a zero of it, so $x - 1$ must be a factor. Dividing $(x^3 - 2 * x^2 - 29 * x + 30) \div (x - 1) = x^2 - x - 30$, we have $x^3 - 2 * x^2 - 29 * x + 30 = (x - 1)(x^2 - x - 30)$ and thus $x^4 - 3x^3 - 27x^2 + 59x - 30 = (x - 1)^2(x^2 - x - 30)$.

   We can factor $x^2 - x - 30 = (x + 5)(x - 6)$ by trial and error to obtain $x^4 - 3x^3 - 27x^2 + 59x - 30 = (x - 1)^2(x + 5)(x - 6)$. We may now rewrite the inequality as $(x - 1)^2(x + 5)(x - 6) \leq 0$.

   When $x > 6$, each of the factors is positive, so the polynomial is positive and $x$ is not a solution.

   When $1 < x < 6$, $x - 6$ is negative but the other factors are positive, so the polynomial is negative and $x$ is a solution.

   When $-5 < x < 1$, $x + 5$ and $(x - 1)^2$ are both positive while $x - 6$ is negative, so the polynomial is negative and $x$ is a solution.

   When $x < -5$, $x + 5$ and $x - 6$ are both negative while $(x - 1)^2$ is positive, so the polynomial is positive and $x$ is not a solution.

   Since the polynomial is 0 at each of $-5$, 1 and 6, they are all solutions and the solution set is $\{x | -5 \leq x \leq 6\}$. This may also be written using interval notation as $[-5, 6]$.

2. John is driving east towards Waterbury at a speed of 55 miles per hour, while his wife Mary is driving south towards Waterbury at a speed of 65 miles per hour. Assuming the earth is flat and the highways they are driving on are both perfectly straight, their speeds do not vary, John is 200 miles west of Waterbury at noon while his wife is 100 miles north of Waterbury at 1 p.m., express the distance between them as a function of time. *Introduce appropriate variable with appropriate units. Make sure you clearly explain what each variable represents.*

   **Solution:** Let $t$ represent time, measured by the number of hours past noon and let $s$ represent the distance between them.

   Since John is 200 miles west of Waterbury at noon and is driving at a speed of 55 miles per hour, he will be a distance $200 - 55t$ from Waterbury at time $t$.

   Since Mary is 100 miles north of Waterbury at 1 p.m. and is driving at a speed of 65 miles per hour, she will be a distance $100 - 65(t-1) = 165 - 65t$ from Waterbury at time $t$.

   Since Waterbury and the positions of John and Mary form the vertices of a right triangle, with their distances from Waterbury being the legs and their distance apart forming the hypotenuse, we may apply the Pythagorean Theorem to get $s^2 = (200 - 55t)^2 + (165 - 65t)^2$ and conclude $s = \sqrt{(200 - 55t)^2 + (165 - 65t)^2}$.

3. Let $f(x) = x^3 - 5x + 1$. Simplify $\dfrac{f(x) - f(8)}{x - 8}$ as much as possible.

   **Solution:** $\dfrac{f(x) - f(8)}{x - 8} = \dfrac{(x^3 - 5x + 1) - 473}{x - 8} = \dfrac{x^3 - 5x - 472}{x - 8}$.

   Conveniently, 8 is a zero of $x^3 - 5x - 472$, so $x - 8$ is a factor. We may divide to get $\dfrac{x^3 - 5x - 472}{x - 8} = x^2 + 8x + 59$. We conclude $\dfrac{f(x) - f(8)}{x - 8} = x^2 + 8x + 59$.

4. Solve $|x + 5| \leq 12$.

   **Solution:** $|x + 5| = |x - (-5)|$ represents the distance between $x$ and $-5$ on a number line. This will be less than or equal to 12 if $x$ is within 12 of $-5$, i.e. if $x$ is between $-17$ and 7. Thus, the solution set is $\{x \mid -17 \leq x \leq 7\}$. This may also be written, using interval notation, as $[-17, 7]$.

5. A meteorite contains 0.038 grams of radioactive kryptonite, a substance which Superman is highly allergic to. Kryptonite has a half-life of two years and it won't be safe for Superman to touch the meteorite until it contains no more than 0.0013 grams of kryptonite. If today is Superman's eighteenth birthday, how old will be be when the meteorite is safe to touch? *Determine his age to the nearest month.*

   **Solution:** Let $x$ represent the amount of kryptonite and $t$ represent time, measure from now in years. We know $x$ is an exponential function of $t$, so we may write $x = a\exp(bt)$. *It is convenient to use the notation* exp *for the exponential function. For example, writing* $\exp(x)$ *rather than* $e^x$, *or* $\exp(bt)$ *rather than* $e^{bt}$.

   Since $x = 0.038$ when $t = 0$, we may plug that into the formula to get $0.038 = a\exp(b{\cdot}0)$, so $a\exp(0) = 0.038$, $a = 0.038$ and thus $x = 0.038\exp(bt)$.

   Since the half-life is 2 years, we know $x = \frac{1}{2}{\cdot}0.038$ when $t = 2$, so $\frac{1}{2}{\cdot}0.038 = 0.038\exp(b{\cdot}2)$, $\exp(2b) = \frac{1}{2}$.

   Taking logs of both sides, we have $\ln(\exp(2b)) = \ln(\frac{1}{2})$, $2b = -\ln 2$, $b = -\frac{\ln 2}{2}$. It follows that $x = 0.038\exp\left(-\frac{t\ln 2}{2}\right)$.

   We want to know when there will be 0.0013 grams of kryptonite, so we solve $0.038\exp\left(-\frac{t\ln 2}{2}\right) = 0.0013$ for $t$.

   Again, we may take logs of both sides to convert this exponential equation into an equivalent linear equation.

   $$\ln(0.038\exp\left(-\tfrac{t\ln 2}{2}\right)) = \ln(0.0013)$$
   $$\ln 0.038 - \tfrac{t\ln 2}{2} = \ln 0.0013$$
   $$\tfrac{t\ln 2}{2} = \ln 0.038 - \ln 0.0013$$
   $$t = \frac{2(\ln 0.038 - \ln 0.0013)}{\ln 2} \approx 9.73883178038.$$

   Since Superman is now 18 years old, he will be 27 years old when the kryptonite is safe for him to touch.

   Since $0.73883178038 \cdot 12 \approx 8.86598136456$, he will be 27 years and 8 months old.

   If each month had 30 days, since $0.86598136456 \cdot 30 \approx 25.9794409368$, he would be 27 years, 8 months and 25 days old. Since some months have 28, 29 or 31 days, he could actually be anywhere between 27 years, 8 months, 24 days old and 27 years, 8 months, 26 days old.

6. A right triangle has an acute angle measuring 39° and the leg adjacent to that angle has length 49. Solve the triangle.

   **Solution:** The other angle is clearly 51° since the measurements of the angles of a triangle must add up to 180°.

   Call the side opposite the 39° angle $x$ and the hypotenuse $h$.

   $\cos 39° = \dfrac{49}{h}$, so $h \cos 39° = 49$, $h = \dfrac{49}{\cos 39°} \approx \dfrac{49}{0.777145961} \approx 63.0512187288$. So the hypotenuse has length $\approx 63.0512187288$.

   $\tan 39° = \dfrac{x}{49}$, so $49 \tan 39° = x$, $x \approx 49 \cdot 0.809784033 \approx 39.6794176266$. So the side opposite the 39° angle has length $\approx 39.6794176266$.

7. A right triangle has an acute angle measuring 39° and the hypotenuse has length 49. Solve the triangle.

   **Solution:** The other angle is clearly 51° since the measurements of the angles of a triangle must add up to 180°.

   Call the leg adjacent to the 39° angle $x$ and the opposite side $y$.

   $\cos 39° = \dfrac{x}{49}$, so $x = 49 \cos 39° \approx 49 \cdot 0.777145961 \approx 38.0801521114$.

   So the leg adjacent to the 39° angle has length $\approx 38.0801521114$.

   $\sin 39° = \dfrac{y}{49}$, so $y = 49 \sin 39° \approx 49 \cdot 0.62932039105 \approx 30.8366991614$.

   So the side opposite the 39° angle has length $\approx 30.8366991614$.

Answer one of questions (8-9) without using either the Law of Sines or the Law of Cosines. Answer the other without constructing any altitudes.

8. A triangle has angles of 53° and 58° with the length of the included side being 25. Solve the triangle.

   **Solution:** Since the measurements of the angles of a triangle must total 180°, the other angle must be 69°.

   Let $a$ be the length of the side opposite the 53° angle and $b$ the lenght of the side opposite the 58° angle.

   **Solution Using the Law of Sines:**

   Using the Law of Sines, $\dfrac{\sin 53°}{a} = \dfrac{\sin 69°}{25}$, so $a = \dfrac{25 \sin 53°}{\sin 69°} \approx \dfrac{25 \cdot 0.79863551}{0.933580426} \approx 21.3863607082$.

   Also using the Law of Sines, $\dfrac{\sin 58°}{b} = \dfrac{\sin 69°}{25}$, so $b = \dfrac{25 \sin 58°}{\sin 69°} \approx \dfrac{25 \cdot 0.848048096}{0.933580426} \approx 22.709561814$.

   Thus, the side opposite the 53° angle has length $\approx 21.3863607082$ and the side opposite the 58° angle has length $\approx 22.709561814$.

   **Solution Without Using the Law of Sines or the Law of Cosines:**

   Draw the altitude from the 58° angle to the side opposite it. Let $h$ be the length of the altitude. Let $c$ be the length of the segment from the 53° angle to the altitude and let $d = b - c$ be the length of the other segment.

   We now have two more right triangles.

   The leg of length 25 is the hypotenuse of one of the other triangles, so we start with that triangle and observe $\cos 53° = \dfrac{c}{25}$, so $c = 25 \cos 53° \approx 0.601815023 \approx 15.0453755788$.

   Similarly, $\sin 53° = \dfrac{h}{25}$, so $h = 25 \sin 53° \approx 25 \cdot 0.79863551 \approx 19.9658877512$.

   $h$ is a leg of the other new triangle and $a$ is the hypotenuse. $\sin 69° = \dfrac{h}{a}$, $a = \dfrac{h}{\sin 69°} \approx \dfrac{19.9658877512}{0.933580426} \approx 21.3863607082$.

   $\cos 69° = \dfrac{d}{h}$, so $d = h \cos 69° \approx 19.9658877512 \cdot 0.35836795 \approx 7.15513425424$.

   We thus have $b = c + d \approx 15.0453755788 + 7.15513425424 \approx 22.200509833$

   Thus, the side opposite the 53° angle has length $\approx 21.3863607082$ and the side opposite the 58° angle has length $\approx 22.709561814$.

9. A triangle has sides of length 53 and 58 and the angle between them measures 25°. Solve the triangle.

   **Solution:** Let $a$ be the length of the other side, let $B$ be the angle opposite the side of length 53 and let $C$ be the angle opposite the side of length 58.

   **Solution Using the Law of Sines and the Law of Cosines:**

   Using the Law of Cosines, $a^2 = 53^2 + 58^2 - 2 \cdot 53 \cdot 58 \cos 25° \approx 2809 + 3364 - 2 \cdot 53 \cdot 58 \cdot 0.906307787 \approx 601.0197253$, so $a \approx 24.5157036468$.

   Using the Law of Sines, $\dfrac{\sin B}{53} = \dfrac{\sin 25°}{a}$, so $\sin B = \dfrac{53 \sin 25°}{a} \approx \dfrac{53 \cdot 0.422618262}{24.5157036468} \approx 0.913649806$. Since $B$ must be acute (it is not the largest angle in the triangle), $B \approx \arcsin 0.913649806 \approx 66.0147034307°$.

   It follows that $C \approx 88.9852965693°$.

   **Solution Without Using the Law of Sines or the Law of Cosines:**

   Draw an altitude from angle $C$. Let $h$ be the length of the altitude, $d$ the length of the segment between the 25° angle and the altitude and let $e = 58 - d$ be the length of the rest of that side.

   The altitude becomes a leg of two new triangles. The side of length 53 is the hypotenuse of one of these new triangles, so we start with that triangle.

   $\sin 25° = \dfrac{h}{53}$, so $h = 53 \sin 25° \approx 53 \cdot 04.22618262 \approx 22.987678723$.

   $\cos 25° = \dfrac{d}{53}$, so $d = 53 \cos 25° \approx 53 \cdot 0.906307787 \approx 48.034312713$.

   It follows that $e = 58 - d \approx 58 - 48.034312713 \approx 9.965687287$.

   We may now use the Pythagorean Theorem to see $h^2 + e^2 = a^2$, so $a^2 \approx 22.987678723^2 + 9.965687287^2 \approx 601.019725299$, so $a \approx 24.5157036468$.

   We also know $\tan B = \dfrac{h}{e} \approx \dfrac{22.987678723}{9.965687287} \approx 2.24758887443$, so $B \approx \arctan 2.24758887443 \approx 66.0147034306$

   It follows that $C \approx 88.9852965694°$.

10. Find $\sin(-8\pi/3)$, $\cos(-8\pi/3)$ and $\tan(-8\pi/3)$ exactly without using a calculator.

    **Solution:**

    $\sin(-8\pi/3) = -\sqrt{3}/2$
    $\cos(-8\pi/3) = -1/2$
    $\tan(-8\pi/3) = \sqrt{3}$

11. Find $\sin(7\pi/4)$, $\cos(7\pi/4)$ and $\tan(7\pi/4)$ exactly without using a calculator.

    **Solution:**

    $\sin(7\pi/4) = -1/\sqrt{2}$
    $\cos(7\pi/4) = 1/\sqrt{2}$
    $\tan(7\pi/4) = -1$

12. Suppose $\cos\theta = w$. Find $\cos(-\theta)$.

    **Solution:** $\cos(-\theta) = \cos\theta = w$.

13. Suppose $\sin\theta = w$. Find $\sin(\pi - \theta)$.

    **Solution:** $\sin(\pi - \theta) = \sin\theta = w$

14. Verify the identity $\dfrac{\sin\theta}{1+\cos\theta} = \csc\theta - \cot\theta$ by simplifying one of the sides until you obtain the other side.

    **Solution:** $\dfrac{\sin\theta}{1+\cos\theta} = \dfrac{\sin\theta}{1+\cos\theta}\cdot\dfrac{1-\cos\theta}{1-\cos\theta} = \dfrac{(\sin\theta)(1-\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} = \dfrac{(\sin\theta)(1-\cos\theta)}{1-\cos^2\theta} = \dfrac{(\sin\theta)(1-\cos\theta)}{\sin^2\theta} = \dfrac{1-\cos\theta}{\sin\theta} = \dfrac{1}{\sin\theta} - \dfrac{\cos\theta}{\sin\theta} = \csc\theta - \cot\theta$.

15. Starting with the identity $\cos^2\theta + \sin^2\theta = 1$, perform the following steps in sequence, simplifying at each step as appropriate, to obtain one of the variations of this fundamental trigonometric identity.

    (a) Divide both sides by $\cos^2\theta$.

    **Solution:**

    $\dfrac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}$
    $\dfrac{\cos^2\theta}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta} = \sec^2\theta$
    $1 + \tan^2\theta = \sec^2\theta$

    (b) Subtract $\tan^2\theta$ from both sides.

    **Solution:**

    $1 + \tan^2\theta - \tan^2\theta = \sec^2\theta - \tan^2\theta$
    $1 = \sec^2\theta - \tan^2\theta$.