Mathematics 109 SOLUTIONS
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November 7, 2005 This problem set is worth 50 points.

1. Find the set of values for which $x^3 - 7x^2 - 6x + 72$ is non-negative.

   Solution: Looking at the divisors of 72, we find -3 is a zero of the polynomial, so $x + 3$ is a factor. Dividing $(x^3 - 7x^2 - 6x + 72) \div (x + 3) = x^2 - 10x + 24$, we have $x^3 - 7x^2 - 6x + 72 = (x + 3)(x^2 - 10x + 24)$. We easily factor $x^2 - 10x + 24 = (x - 4)(x - 6)$ to get $x^3 - 7x^2 - 6x + 72 = (x + 3)(x - 4)(x - 6)$.

   Thus, the zeros of the polynomial are -3, 4, 6, so we look at the intervals $(-\infty, -3)$, $(-3, 4)$, $(4, 6)$ and $(6, \infty)$.

   When $x > 6$, all the factors are positive, so the product is positive.

   When $4 < x < 6$, $x + 3$ and $x - 4$ are positive while $x + 6$ is negative, so the product is negative.

   When $-3 < x < 4$, $x + 3$ is positive while $x - 4$ and $x + 6$ are negative, so the product is positive.

   When $x < -3$, all the factors are negative, so the product is negative.

   When x is either -3, 4 or 6, the product is 0, which is non-negative.

   Thus, the set of values for which the polynomial is non-negative is $\{x | -3 \leq x \leq 4 \text{ or } x \geq 6\}$. This may also be written as $[-3, 4] \cup [6, \infty)$.

2. Complete each of the following statements:

   (a) The logarithm of a product is equal to

   Solution: The logarithm of a product is equal to sum of the logarithms.

   (b) The logarithm of a quotient is equal to

   Solution: The logarithm of a quotient is equal to the difference of the logarithms.

   (c) The logarithm of a number raised to a power is equal to **Solution:** The logarithm of a number raised to a power is equal to the power times the logarithm of the number.

3. Assuming $b > 0$, $b \neq 1$, find $\log_b b^{7.3}$.

   Solution: $\log_b b^{7.3} = 7.3$.

4. Assuming $b > 0$, $b \neq 1$, find $b^{\log_b 8.4}$.

   Solution: $b^{\log_b 8.4} = 8.4$.

5. Solve: $2^{3x-1} = 15$.

   Solution: $\ln(2^{3x-1}) = \ln 15$

   $(3x - 1)\ln 2 = \ln 15$

   $3x \ln 2 - \ln 2 = \ln 15$

   $3x \ln 2 = \ln 15 + \ln 2$

   $3x \ln 2 = \ln 30$

   $x = \dfrac{\ln 30}{3 \ln 2} \approx 1.63563019854$.

6. Assume you have deposited a significant sum of money in a bank account paying iterest at an annual rate of 3.4%, compounded continuously. How long will it take for your balance to double? *Extra Credit: What interest rate would be needed for the balance to double in ten years?*

   Solution: If P_0 is the initial balance and P is the balance after t years, we have $P = P_0 e^{0.034t}$. We need to find when $P = 2P_0$, so we may set $P_0 e^{0.034t} = 2P_0$. Dividing both sides by 2, we get $e^{0.034t} = 2$. We may solve this as follows:

   $\ln(e^{0.034t}) = \ln 2$

   $0.034t = \ln 2$

   $t = \dfrac{\ln 2}{0.034}$.

   It will thus take $\dfrac{\ln 2}{0.034}$ years, or ≈ 20.3866817812 years for the balance to double.

   Extra Credit: Let r be the interest rate needed for the balance to double in ten years. We would have $P = P_0 e^{rt}$, with the same meanings for the variables given above. In this case, we would need $P = 2P_0$ when $t = 10$, so we would have to have $P_0 e^{r \cdot 10} = 2P_0$. Solving, we get

   $e^{r \cdot 10} = 2$

   $\ln(e^{10r}) = \ln 2$

   $10r = \ln 2$

   $r = \dfrac{\ln 2}{10}$

   So the annual interest rate would have to be $\dfrac{\ln 2}{10}$, or $\approx 6.93147181\%$.

7. A fossil is discovered. It is known that the sample examined would have originally contained 0.014 grams of Carbon-14, but now contains just 0.005 grams of Carbon-14. How old is the fossil? *You may assume the half-life of Carbon-14 is 5730 years.*

   Solution: Let x be the amount of Carbon-14 in the fossil t years after it started fossilizing. We know $x = x_0 e^{at}$ for some constants x_0, a.

   Since $x = 0.014$ when $t = 0$, we have $0.014 = x_0 e^{a \cdot 0} = x_0 e^0 = x_0$, so $x = 0.014 e^{at}$.

   Since the half life is 5730, we know $x = \frac{1}{2} \cdot 0.014$ when $t = 5730$, so $0.014 e^{a \cdot 5730} = \frac{1}{2} \cdot 0.014$. Solving for a, we get

   $e^{5730a} = \frac{1}{2}$

   $\ln(e^{5730a}) = \ln \frac{1}{2}$

   $5730a = -\ln 2$

   $a = -\dfrac{\ln 2}{5730}$,

   so $x = 0.014 e^{-t \ln 2/5730}$.

   When there are 0.005 grams left, we have $0.014 e^{-t \ln 2/5730} = 0.005$, so

   $e^{-t \ln 2/5730} = \dfrac{5}{14}$

   $\ln(e^{-t \ln 2/5730}) = \ln(\dfrac{5}{14})$

   $-t \ln 2/5730 = \ln(\dfrac{5}{14})$

   $t = -\dfrac{5730 \ln(\frac{5}{14})}{\ln 2}$.

   It follows that the fossil is $-\dfrac{5730 \ln(\frac{5}{14})}{\ln 2}$, or $\approx 8,511.49571967$ years old.

8. The degree measure of an angle is 47.3°. What is its radian measure?

   Solution: The radian measure is $\dfrac{47.3}{180} \cdot \pi$, or approximately 0.825540736.

9. The radian measure of an angle is 2.18π. What is its degree measure?

   Solution: The degree measure is $2.18 \cdot 180° = 392.4°$.

10. The radian measure of an angle is 2.18. What is its degree measure?

    Solution: The degree measure is $\dfrac{2.18}{\pi} \cdot 180° \approx 124.9047993°$.

11. A right triangle has hypotenuse of length 14. One of its acute angles is 37°. Find the other acute angle and the two legs. Clearly indicate the location of each leg. *One way of doing that would be, for each leg, to indicate whether it is adjacent to or opposite the* 37° *angle.*

    Solution: Let a be the length of the side adjacent to the 37° angle and let b be the length of the side opposite the 37° angle.

    We have $\sin 37^\circ = \dfrac{b}{14}$, so $b = 14\sin 37^\circ \approx 14 \cdot 0.601815023 \approx 8.42541032413$.

    Similarly, we have $\cos 37^\circ = \dfrac{a}{14}$, so $a = 14\cos 37^\circ \approx 14 \cdot 0.79863551 \approx 11.1808971407$.

    Since the sum of the two acute angles must be 90°, the other acute angle must be 53°.

12. Let $\theta = \arcsin 0.7$. Find $\cos\theta$.

    Solution: We know $\sin\theta = 0.7$. Since $\sin^2\theta + \cos^2\theta = 1$, we have $0.7^2 + \cos^2\theta = 1$, $0.49 + \cos^2\theta = 1$, $\cos^2\theta = 0.51$, $\cos\theta = \pm\sqrt{0.51}$.

    Since $\cos\theta$ must be non-negative, we have $\cos\theta = \sqrt{0.51} \approx 0.714142843$.