Mathematics 109 <u>SOLUTIONS</u>
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Monday, October 24, 2005 This problem set is worth 50 points.

1. Multiply and simplify: $(5x^2 - 4x - 2)(3x - \sqrt{x} + 5)$.

   Solution: $(5x^2 - 4x - 2)(3x - \sqrt{x} + 5)$
   $= 5x^2 \cdot 3x + 5x^2 \cdot (-\sqrt{x}) + 5x^2 \cdot 5 - 4x \cdot 3x - 4x \cdot \sqrt{x} - 4x \cdot 5 - 2 \cdot 3x - 2 \cdot \sqrt{x} - 2 \cdot 5$
   $= 15x^3 - 5x^2\sqrt{x} + 25x^2 - 12x^2 + 4x\sqrt{x} - 20x - 6x + 2\sqrt{x} - 10$
   $= 15x^3 - 5x^2\sqrt{x} + 13x^2 + 4x\sqrt{x} - 26x + 2\sqrt{x} - 10.$

2. Divide: $(x^3 - 7x^2 - 20x + 96) \div (x - 8)$.

   Solution: $(x^3 - 7x^2 - 20x + 96) \div (x - 8) = x^2 + x - 12$

3. Find an equation for the line through the point $(2, 5)$ which is perpendicular to the line $3x + 12y = 17$

   Solution: We may simplify the equation of the perpendicular line as follows:

   $3x + 12y = 17$

   $12y = -3x + 17$

   $y = -\frac{1}{4}x + \frac{17}{13}$

   Thus, the slope of the perpendicular line is $\frac{1}{4}$. Since the product of the slopes of perpendicular lines is -1, the slope of the line we want is 4. Since it goes through $(2, 5)$, we may write its equation in the form $y - 5 = 4(x - 2)$. *If we wish, we may simplify as follows:* $y - 5 = 4x - 8$, $y = 4x - 3$.

4. Solve: $|x| > 15$.

   Solution: The solution set is $\{x | x < -15 \text{ or } x > 15\}$.

5. Solve: $|x - 9| = 3$.

   Solution: The solution set is $\{6, 12\}$.

6. Solve: $|x - 9| \geq 3$.

   Solution: The solution set is $\{x | x \leq 6 \text{ or } x \geq 12\}$.

7. Solve $x^2 + 4x + 1 = 0$ using the Quadratic Formula.

   Solution: $x = \dfrac{-4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 1}}{2 \cdot 1} = \dfrac{-4 \pm \sqrt{12}}{2} = \dfrac{-4 \pm 2\sqrt{3}}{2} = \dfrac{2(-2 \pm \sqrt{3})}{2} = -2 \pm \sqrt{3}$.

   The solution set is thus $\{-2 - \sqrt{3}, -2 + \sqrt{3}\}$.

8. Solve $x^2 + 4x + 1 = 0$ by Completing the Square.

   Solution: Since $(x+2)^2 = x^2 + 4x + 4$, it follows that $x^2 + 4x + 1 = (x+2)^2 - 3$, so the equation may be written in the form $(x+2)^2 - 3 = 0$. This is equivalent to $(x+2)^2 = 3$. Clearly, this is satisfied if and only if $x+2 = \pm\sqrt{3}$, that is, if $x = -2 \pm \sqrt{3}$. We conclude the solution set is $\{-2-\sqrt{3}, -2+\sqrt{3}\}$.

9. Factor $x^3 - 7x^2 - 20x + 96$ completely.

   Solution: Checking through the divisors of 96, we find 3 is a zero of the polynomial, so that $x - 3$ is a factor. Dividing $(x^3 - 7x^2 - 20x + 96) \div (x - 3) = x^2 - 4x - 32$, we thus factor $x^3 - 7x^2 - 20x + 96 = (x-3)(x^2 - 4x - 32)$.

   We may factor $x^2 - 4x - 32 = (x+4)(x-8)$ easily by trial and error to conclude $x^3 - 7x^2 - 20x + 96 = (x-3)(x+4)(x-8)$.

10. Solve: $x^3 + 96 = 7x^2 + 20x$.

    Solution: We may rewrite the equation in the form $x^3 - 7x^2 - 20x + 96 = 0$. The factorization obtained in the previous question makes it clear the solution set is $\{3, -4, 8\}$.

11. Solve: $x^3 + 96 > 7x^2 + 20x$.

    Solution: Similarly, we may rewrite the inequality in the form $x^3 - 7x^2 - 20x + 96 > 0$ and factor to write it in the form

    $(x-3)(x+4)(x-8) > 0$.

    Since the zeros of the polynomial on the left divide the real line into the intervals $(-\infty, -4)$, $(-4, 3)$, $(3, 8)$ and $(8, \infty)$, we look at the factors on each of those intervals.

    When $x < -4$, each of the factors is negative, so the product is negative and x is not a solution.

    When $-4 < x < 3$, the factor $x + 4$ is positive while $x - 3$ and $x - 8$ are negative, so the product is positive and x is a solution.

    When $3 < x < 8$, the factors $x + 4$ and $x - 3$ are positive while $x - 8$ is negative, so the product is negative and x is not a solution.

    When $x > 8$, each of the factors is positive, so the product is positive and x is a solution.

    We conclude the solution set is $\{x | -4 < x < 3 \text{ or } x > 8\}$. We may also write the solution in the form $(-4, 3) \cup (8, \infty)$.

12. Solve: $x^4 + 8x^3 + 140 = 9x^2 + 92x$.

    Solution: We may rewrite the equation in the form $x^4 + 8x^3 - 9x^2 - 92x + 140 = 0$. Looking among the divisors of 140, we find 2 is a solution to the equation, so $x - 2$ is a divisor of $x^4+8x^3-9x^2-92x+140$. Dividing, we get $(x^4+8x^3-9x^2-92x+140)\div(x-2) = x^3 - 10x^2 + 11x - 70$, so $x^4 + 8x^3 - 9x^2 - 92x + 140 = (x-2)(x^3 + 10x^2 + 11x - 70)$.

    Similarly, looking at the divisors of 70, we find 2 is a zero of $x^3 + 10x^2 + 11x - 70$, so that $x-2$ is a factor. Dividing, we get $(x^3 + 10x^2 + 11x - 70) \div (x-2) = x^2 + 12x + 35$, so we have $x^4 + 8x^3 - 9x^2 - 92x + 140 = (x-2)^2(x^2 + 12x + 35)$.

    We can easily factor $x^2 + 12x + 35 = (x+5)(x+7)$ by trial and error to get $x^4 + 8x^3 - 9x^2 - 92x + 140 = (x-2)^2(x+5)(x+7)$ and we may write the equation in the form $(x-2)^2(x+5)(x+7) = 0$.

    From this, we easily see the solution set is $\{-7, -5, 2\}$.

13. Solve: $x^4 + 8x^3 + 140 \leq 9x^2 + 92x$.

    Solution: Rewriting the inequality in the form $x^4 + 8x^3 - 9x^2 - 92x + 140 \leq 0$ and factoring, we write it in the form $(x-2)^2(x+5)(x+7) \leq 0$.

    Since the zeroes of the polynomial divide the real line into the intervals $(-\infty, -7)$, $(-7, -5)$, $(-5, 2)$ and $(2, \infty)$, we look at the factors on each of those intervals. Our work is simplified since $(x-2)^2 > 0$ on each of the intervals, so we really only have to look at the other factors.

    When $x < -7$, both $x+7$ and $x+5$ are negative, so the product is positive and x is not a solution.

    When $-7 < x < -5$, $x+7$ is positive while $x+5$ is negative, so the product is negative and x is a solution.

    For both $-5 < x < 2$ and $x > 2$, both $x+5$ and $x+7$ are positive, so the product is positive and x is not a solution.

    Clearly, -7, -5 and 2 satisfy the inequality, so the solution set is $\{x | x = 2 \text{ or } -7 \leq x \leq -5\}$. This may also be written in the form $\{2\} \cup [-7, -5]$.

14. Solve: $\dfrac{x+5}{x-2} \leq 0$.

    Solution: Since the numerator is 0 when $x = -$

15. You are able to drive to Boston without hitting any traffic and average 63 miles per hour for the entire drive. Unfortunately, you hit heavy traffic on your return trip and only aveage 47 miles per hour. Assuming the distances in both directions are exactly the same, what is your average speed for the round trip?

    Solution: Let x be the distance between your starting point and Boston, t_1 be the time it takes you to get to Boston and t_2 the time it takes you to return. Let v be your average speed for the entire round trip journey.

    Clearly, $x = 63t_1$ and $x = 47t_2$, while $v = \dfrac{2x}{t_1 + t_2}$.

    We may solve each of the first two equations for t_1 and t_2, getting $t_1 = \dfrac{x}{63}$ and $t_2 = \dfrac{x}{47}$. Plugging these values into the formula for v, we get $v = \dfrac{2x}{\dfrac{x}{63} + \dfrac{x}{47}} = \dfrac{2x}{x\left(\dfrac{1}{63} + \dfrac{1}{47}\right)} = \dfrac{2}{\dfrac{1}{63} + \dfrac{1}{47}} = \dfrac{2}{\dfrac{1}{63} + \dfrac{1}{47}} \cdot \dfrac{63 \cdot 47}{63 \cdot 47} = \dfrac{2 \cdot 63 \cdot 47}{63 + 47} = \dfrac{2 \cdot 63 \cdot 47}{110} = \dfrac{63 \cdot 47}{55} = \dfrac{2961}{55}$.

    So the average speed will be $\dfrac{2961}{55}$ miles per hour, or approximately 53.8363636364 miles per hour.

16. A $5'' \times 7''$ photograph is placed in a frame which has a total area of 43.2225 square inches. If the border surrounding the photograph is the same size on all sides, what are the dimensions of the frame?

    Solution: If we let x be the width of the border, then the dimensions of the frame will be $5 + 2x \times 7 + 2x$ and its area will be $(5 + 2x)(7 + 2x) = 4x^2 + 24x + 35$.

    We must therefore have $4x^2 + 24x + 35 = 43.2225$, or $4x^2 + 24x - 8.2225 = 0$.

    We may solve this equation using the Quadratic Formula, obtaining

    $$x = \frac{-24 \pm \sqrt{24^2 - 4 \cdot 4 \cdot (-8.2225)}}{2 \cdot 4} = \frac{-24 \pm \sqrt{576 + 131.56}}{8} = \frac{-24 \pm \sqrt{707.56}}{8}.$$

    Since x must be positive (since it represents a width), we must have $x = \dfrac{-24 + \sqrt{707.56}}{8} = \dfrac{-24 + 26.6}{8} = \dfrac{2.6}{8} = 0.325$.

    Since $2 \times 0.325 = 0.65$, the frame must be $5.65'' \times 7.65''$.