Problems With Notation

Mathematical notation is very precise. This contrasts with both oral communication and some written English.

Correct mathematical notation:

$\lim_{x\to 4} \dfrac{x^2-2x-8}{x-4} = \lim_{x\to 4} \dfrac{(x-4)(x+2)}{x-4} = \lim_{x\to 4}(x+2) = 6.$

Explanation in colloquial English:

The limit as x approaches 4 of x^2-2x-8 over $x-4$ is equal to the limit of $(x-4)(x+2)$ over $(x-4)$, which equals the limit of $(x+2)$, so the limit is 6.

Translation from colloquial English to improper mathematical notation:

$\lim_{x\to 4} \dfrac{x^2-2x-8}{x-4} = \lim \dfrac{(x-4)(x+2)}{x-4} = \lim(x+2).$
$\lim = 6.$

Calculating Derivatives

There are two types of formulas for calculating derivatives, which we may classify as (a) formulas for calculating the derivatives of elementary functions and (b) structural type formulas.

Formulas for Derivatives of Elementary Functions

(1) $\dfrac{d}{dt}(t^n) = nt^{n-1}$

(2) $\dfrac{d}{dt}(e^t) = e^t$

(3) $\dfrac{d}{dt}(\ln t) = \dfrac{1}{t}$

(4) $\dfrac{d}{dt}(\sin t) = \cos t$

(5) $\dfrac{d}{dt}(\cos t) = -\sin t$

(6) $\dfrac{d}{dt}(\tan t) = \sec^2 t$

Structural Type Formulas

them for granted and use them as nonchalantly as we use the power rule:

- $\frac{dc}{dt} = 0$
- $\frac{d}{dt}(ct) = c$
- $\frac{d}{dt}(at+b) = a$

The Second Group

The last three rules are somewhat more difficult. They are called the product rule, the quotient rule and the chain rule. Of these, the product and quotient rules can be used routinely, since it is easy to recognize when you have a product or quotient, but it is more difficult and takes more practice to use the chain rule correctly.

The Product and Quotient Rules in Words

The product rule may be thought of as *the derivative of a product equals the first factor times the derivative of the second plus the second factor times the derivative of the first.*

The quotient rule may be thought of as *the derivative of a quotient equals the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

The Product and Quotient Rules – Symbolically

Symbolically, we express these rules as follows:

Formula 1 (Product Rule)**.**

$$\frac{d}{dt}(uv) = u\frac{dv}{dt} + v\frac{du}{dt}$$

Formula 2 (Quotient Rule)**.**

$$\frac{d}{dt}(u/v) = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$

The Chain Rule

The chain rule is a little trickier to use. Fortunately, its formula is easier to remember than some of the others.

Formula 3 (Chain Rule)**.** $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

The chain rule is used for calculating the derivatives of composite functions. The easiest way to recognize that you are dealing with a composite function is by the process of elimination:

If none of the other rules apply, then you have a composite function.

Overall Strategy

(1) Differentiate term by term. Deal with each term separately and, for each term, recognize any constant factor.
(2) For each term, recognize whether it is one of the elementary functions (power, trigonometric, exponential or natural logarithm of the independent variable). If it is, you can easily apply the appropriate formula and you will be done. If it's not, go on to (3).
(3) Decide whether the term is a product or a quotient. If it is, use the appropriate formula. *Note that the appropriate formula will have you calculating two other derivatives and you will have to go back to (1) to deal with those.* If it isn't, go to (4).
(4) If you've gotten this far, you have to use the Chain Rule. Try to use it correctly. *The note attached to (3) applies here as well.*

Properties of Exponents

- $b^x b^y = b^{x+y}$
- $\dfrac{b^x}{b^y} = b^{x-y}$
- $(b^x)^y = b^{xy}$

- $(ab)^x = a^x b^x$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
- $b^0 = 1$
- $b^{-x} = \frac{1}{b^x}$
- $b^{m/n} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$

Definition of a Logarithm

$$y = \log_b x \text{ if and only if } x = b^y \tag{1}$$

Essentially, the logarithm to the base b of a number x is the power which b must be raised to in order to obtain x.

This immediately leads to the two very useful formulas

$$b^{\log_b x} = x \text{ and } \log_b b^x = x. \tag{2}$$

Properties of Logarithms

Each of the properties of exponential functions has an analog for logarithmic functions.

- $\log_b(xy) = \log_b x + \log_b y$
- $\log_b(x/y) = \log_b x - \log_b y$
- $\log_b(x^r) = r \log_b x$
- $\log_b 1 = 0.$

In other words, The logarithm of a product or quotient is the sum or difference of logarithms and the logarithm of a number to a power is the power times the logarithm of that number.

The Natural Logarithm Function and The Exponential Function

One specific logarithm function is singled out and one particular exponential function is singled out.

Definition 1. $e = \lim_{x \to 0}(1 + x)^{1/x}$

Definition 2 (The Natural Logarithm Function)**.** $\ln x = \log_e x$

Definition 3 (The Exponential Function)**.** $\exp x = e^x$

The following special cases of properties of logarithms and exponential functions are worth remembering separately for the natural log function and the exponential function.

- $y = \ln x$ if and only if $x = e^y$
- $\ln(e^x) = x$
- $e^{\ln x} = x$

Suppose $y = b^x$. By the properties of logarithms, we can write $\ln y = \ln(b^x) = x \ln b$. It follows that $e^{\ln y} = e^{x \ln b}$. But, since $e^{\ln y} = y = b^x$, it follows that

- $b^x = e^{x \ln b}$

This important identity is very useful.

Similarly, suppose $y = \log_b x$. Then, by the definition of a logarithm, it follows that $b^y = x$. But then $\ln(b^y) = \ln x$. Since $\ln(b^y) = y \ln b$, it follows that $y \ln b = \ln x$ and $y = \dfrac{\ln x}{\ln b}$, yielding the following equally important identity.

- $\log_b x = \dfrac{\ln x}{\ln b}$

Derivatives

$\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

$\dfrac{d}{dx}(e^x) = e^x$

If there are logs or exponentials with other bases, one may still use these formulas after rewriting the functions in terms of natural logs or the exponential function.

Example: Calculate $\dfrac{d}{dx}(5^x)$

Using the formula $a^x = e^{x \ln a}$, write 5^x as $e^{x \ln 5}$. We can then apply the Chain Rule, writing:

$y = 5^x = e^{x \ln 5}$
$y = e^u$
$u = x \ln 5$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= e^u \cdot \ln 5 = 5^x \ln 5 \end{aligned}$$

Example: Calculate $\dfrac{d}{dx}(\log_7 x)$

Using the formula $\log_b x = \dfrac{\ln x}{\ln b}$, we write $\log_7 x = \dfrac{\ln x}{\ln 7}$, so we can proceed as follows:

$y = \log_7 x = \dfrac{\ln x}{\ln 7} = \dfrac{1}{\ln 7} \cdot \ln x$

$y' = \dfrac{1}{\ln 7} \cdot \dfrac{d}{dx}(\ln x) = \dfrac{1}{\ln 7} \cdot \dfrac{1}{x} = \dfrac{1}{x \ln 7}$

Definition of a Logarithm

$$y = \log_b x \text{ if and only if } x = b^y \tag{3}$$

Essentially, the logarithm to the base b of a number x is the power which b must be raised to in order to obtain x.

This immediately leads to the two very useful formulas

$$b^{\log_b x} = x \text{ and } \log_b b^x = x. \tag{4}$$

Properties of Logarithms

Each of the properties of exponential functions has an analog for logarithmic functions.

- $\log_b(xy) = \log_b x + \log_b y$
- $\log_b(x/y) = \log_b x - \log_b y$
- $\log_b(x^r) = r \log_b x$
- $\log_b 1 = 0.$

In other words, The logarithm of a product or quotient is the sum or difference of logarithms and the logarithm of a number to a power is the power times the logarithm of that number.

The Natural Logarithm Function and The Exponential Function

One specific logarithm function is singled out and one particular exponential function is singled out.

Definition 4. $e = \lim_{x\to 0}(1+x)^{1/x}$

Definition 5 (The Natural Logarithm Function)**.** $\ln x = \log_e x$

Definition 6 (The Exponential Function)**.** $\exp x = e^x$

Special Cases

The following special cases of properties of logarithms and exponential functions are worth remembering separately for the natural log function and the exponential function.

- $y = \ln x$ if and only if $x = e^y$
- $\ln(e^x) = x$
- $e^{\ln x} = x$

Other Bases

Suppose $y = b^x$. By the properties of logarithms, we can write $\ln y = \ln(b^x) = x \ln b$. It follows that $e^{\ln y} = e^{x \ln b}$. But, since $e^{\ln y} = y = b^x$, it follows that

$$b^x = e^{x \ln b}$$

This important identity is very useful.

Similarly, suppose $y = \log_b x$. Then, by the definition of a logarithm, it follows that $b^y = x$. But then $\ln(b^y) = \ln x$. Since $\ln(b^y) = y \ln b$, it follows that $y \ln b = \ln x$ and $y = \dfrac{\ln x}{\ln b}$, yielding the following equally important identity.

$$\log_b x = \frac{\ln x}{\ln b}$$

Derivatives

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\frac{d}{dx}(e^x) = e^x$

If there are logs or exponentials with other bases, one may still use these formulas after rewriting the functions in terms of natural logs or the exponential function.

Example: Calculate $\frac{d}{dx}(5^x)$

Using the formula $a^x = e^{x \ln a}$, write 5^x as $e^{x \ln 5}$. We can then apply the Chain Rule, writing:

$y = 5^x = e^{x \ln 5}$
$y = e^u$
$u = x \ln 5$
$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot \ln 5 = 5^x \ln 5$

Example: Calculate $\frac{d}{dx}(\log_7 x)$

Using the formula $\log_b x = \frac{\ln x}{\ln b}$, we write $\log_7 x = \frac{\ln x}{\ln 7}$, so we can proceed as follows:

$y = \log_7 x = \frac{\ln x}{\ln 7} = \frac{1}{\ln 7} \cdot \ln x$
$y' = \frac{1}{\ln 7} \cdot \frac{d}{dx}(\ln x) = \frac{1}{\ln 7} \cdot \frac{1}{x} = \frac{1}{x \ln 7}$