The Natural Logarithm Function and The Exponential Function

One specific logarithm function is singled out and one particular exponential function is singled out.

Definition 1. $e = \lim_{x \to 0}(1+x)^{1/x}$

Definition 2 (The Natural Logarithm Function)**.** $\ln x = \log_e x$

Definition 3 (The Exponential Function)**.** $\exp x = e^x$

The following special cases of properties of logarithms and exponential functions are worth remembering separately for the natural log function and the exponential function.

- $y = \ln x$ if and only if $x = e^y$
- $\ln(e^x) = x$
- $e^{\ln x} = x$

Suppose $y = b^x$. By the properties of logarithms, we can write $\ln y = \ln(b^x) = x \ln b$. It follows that $e^{\ln y} = e^{x \ln b}$. But, since $e^{\ln y} = y = b^x$, it follows that

- $b^x = e^{x \ln b}$

This important identity is very useful.

Similarly, suppose $y = \log_b x$. Then, by the definition of a logarithm, it follows that $b^y = x$. But then $\ln(b^y) = \ln x$. Since $\ln(b^y) = y \ln b$, it follows that $y \ln b = \ln x$ and $y = \dfrac{\ln x}{\ln b}$, yielding the following equally important identity.

- $\log_b x = \dfrac{\ln x}{\ln b}$

Derivatives

$\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

$\dfrac{d}{dx}(e^x) = e^x$

If there are logs or exponentials with other bases, one may still use these formulas after rewriting the functions in terms of natural logs or the exponential function.

Example: Calculate $\dfrac{d}{dx}(5^x)$

Solution: Using the formula $a^x = e^{x \ln a}$, write 5^x as $e^{x \ln 5}$. We can then apply the Chain Rule, writing:

$y = 5^x = e^{x \ln 5}$

$y = e^u$

$u = x \ln 5$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$
$$= e^u \cdot \ln 5 = 5^x \ln 5$$

Example: Calculate $\dfrac{d}{dx}(\log_7 x)$

Solution: Using the formula $\log_b x = \dfrac{\ln x}{\ln b}$, we write $\log_7 x = \dfrac{\ln x}{\ln 7}$, so we can proceed as follows:

$y = \log_7 x = \dfrac{\ln x}{\ln 7} = \dfrac{1}{\ln 7} \cdot \ln x$

$y' = \dfrac{1}{\ln 7} \cdot \dfrac{d}{dx}(\ln x) = \dfrac{1}{\ln 7} \cdot \dfrac{1}{x} = \dfrac{1}{x \ln 7}$