Mathematics 1070 SOLUTIONS
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1. A seven card rummy hand is dealt. Let A be the event that the hand contains either 3 of a kind (3 cards with the same face value) or 4 of a kind. Let B be the event that the hand contains 3 of one kind and 4 of another kind. Find $P(B|A)$.

   Solution: One can get 3 of a kind or 4 of a kind in the following ways. For each way, we include the number of ways of obtaining such a hand:
   4 of a kind and 3 of a kind: $13 \cdot 12\binom{4}{3}$
   4 of a kind, a pair and a singleton: $13 \cdot 12\binom{4}{2}44$
   4 of a kind and three singletons: $13\binom{12}{3}4^3$
   3 of each of two kinds along with a singleton: $\binom{13}{2}\binom{4}{3}^2 44$
   3 of a kind and two pairs: $13\binom{4}{3}\binom{12}{2}\binom{4}{2}^2$
   3 of a kind, a pair and two singletons: $13\binom{4}{3}12\binom{4}{2}\binom{11}{2}4^2$
   3 of a kind and four singletons: $13\binom{4}{3}\binom{12}{4}4^4$.

   Thus, $P(B|A) =$
   $$\frac{13 \cdot 12\binom{4}{3}}{13 \cdot 12\binom{4}{3} + 13 \cdot 12\binom{4}{2}44 + 13\binom{12}{3}4^3 + \binom{13}{2}\binom{4}{3}^2 44 + 13\binom{4}{3}\binom{12}{2}\binom{4}{2}^2 + 13\binom{4}{3}12\binom{4}{2}\binom{11}{2}4^2 + 13\binom{4}{3}\binom{12}{4}4^4}.$$

   Alternatively, one can calculate $\#A$ by finding the number of ways a hand can contain no more than two of a kind (7 singletons, 1 pair and 5 singletons, 2 pairs and 3 singletons, 3 pairs and a singleton) and subtracting that from the number of 7 card hands, getting $\#A = \binom{52}{7} - (\binom{13}{7}4^7 + \binom{13}{1}\binom{4}{2}\binom{12}{5}4^5 + \binom{13}{2}\binom{4}{2}^2\binom{12}{3}4^3 + \binom{13}{3}\binom{4}{2}^3\binom{12}{1}4)$ and using that for the denominator.

2. A loan is to be paid off in monthly installments of \$275 each over a period of five years. Assuming an annual interest rate of 7 percent, what is the amount of the loan? (Assume interest on the prior balance is calculated at the same time a payment is made. Your conclusion, which should be a statement in plain English, may involve rounding off the amount of the loan to the nearest cent, but you should show any calculations with as much accuracy as your calculator displays.)

   Solution: We use the formula $P = a_{\overline{n}|i}R$, with $a_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i(1+i)^n}$, where P represents the amount of the loan, $R = 275$ is the monthly payment, $i = \frac{.07}{12}$ is the interest rate applied at the time of each payment and $n = 5 \cdot 12 = 60$ is the number of payments.

   We thus get $P = \dfrac{(1+.07/12)^{60} - 1}{\frac{.07}{12}(1+.07/12)^{60}} \cdot 275 \approx 13,888.04828062$ and conclude the loan was for \$13,888.05.

3. Sketch the feasible set for the following system of inequalities.

$$\begin{aligned} x - y &\leq 3 \\ x &\leq 2 \\ x \geq 0,\ y &\geq 0 \end{aligned}$$

Solution: The feasible set is the vertical strip in the first quadrant between the $y-$axis and the line $x = 2$, including its boundary.

4. Perform the indicated elementary row operation on the matrix $\begin{pmatrix} 5 & 3 & -2 & 7 \\ -2 & 5 & 1 & 8 \\ 3 & -2 & 6 & 11 \end{pmatrix}$:

Subtract 4 times the third row from the second row.

Solution: $\begin{pmatrix} 5 & 3 & -2 & 7 \\ -14 & 13 & -23 & -36 \\ 3 & -2 & 6 & 11 \end{pmatrix}$

5. Pivot about the second row, second column of the matrix $\begin{pmatrix} 4 & 2 & 2 \\ -1 & -3 & 4 \\ 3 & -1 & 6 \end{pmatrix}$. *Your arithmetic should be exact.*

Solution: $R_2/(-3): \begin{pmatrix} 4 & 2 & 2 \\ 1/3 & 1 & -4/3 \\ 3 & -1 & 6 \end{pmatrix}$

$R_1 - 2R_2: \begin{pmatrix} 10/3 & 0 & 14/3 \\ 1/3 & 1 & -4/3 \\ 3 & -1 & 6 \end{pmatrix}$

$R_3 + R_2: \begin{pmatrix} 10/3 & 0 & 14/3 \\ 1/3 & 1 & -4/3 \\ 10/3 & 0 & 14/3 \end{pmatrix}$

6. Solve the following systems of equations by setting up an augmented matrix, using Gaussian Elimination on that matrix and interpreting the result.

$$\begin{aligned} x + 2y + z &= -2 \\ -2x - 3y - z &= 1 \\ 2x + 4y + 2z &= -4 \end{aligned}$$

Solution:
$$\begin{pmatrix} 1 & 2 & 1 & -2 \\ -2 & -3 & -1 & 1 \\ 2 & 4 & 2 & -4 \end{pmatrix} \quad \begin{matrix} R_2 + 2R_1 \\ R_3 - 2R_1 \end{matrix} : \begin{pmatrix} 1 & 2 & 1 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_1 - 2R_2 : \begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
This is equivalent to the system:
$x - z = 4$
$y + z = -3.$
The solution set is $\{(x, y, z) : x = z + 4, y = -z - 3, z \in \mathbb{R}\}$.

(7-9): Consider the following linear programming problem.

Maximize $P = 2x + 5y$ subject to

$$\begin{aligned} 2x + y &\leq 16 \\ 2x + 3y &\leq 24 \\ y &\leq 6 \\ x \geq 0,\ y &\geq 0 \end{aligned}$$

7. Solve the linear programming problem geometrically.

   Solution: The feasible set is a pentagon with vertices $(0, 0), (8, 0), (6, 4), (3, 6), (0, 6)$. Evaluating the objective function at each of those points we get:
   At $(0, 0)$, $P = 0$
   At $(8, 0)$, $P = 16$
   At $(6, 4)$, $P = 32$
   At $(3, 6)$, $P = 36$
   At $(0, 6)$, $P = 30$.

   We thus find the maximum value of P is 36, occurring when $x = 3$ and $y = 6$.

8. Set up the initial simplex tableau for the linear programming problem.

   Solution: $\begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 16 \\ 2 & 3 & 0 & 1 & 0 & 24 \\ 0 & 1 & 0 & 0 & 1 & 6 \\ -2 & -5 & 0 & 0 & 0 & 0 \end{pmatrix}$

9. Solve the linear programming problem using the simplex method.

Solution: We start by pivoting about the 2nd column, 3rd row.

$$\begin{array}{l} R_1 - R_3 \\ R_2 - 3R_3: \\ R_4 + 5R_3 \end{array} \begin{pmatrix} 2 & 0 & 1 & 0 & -1 & 10 \\ 2 & 0 & 0 & 1 & -3 & 6 \\ 0 & 1 & 0 & 0 & 1 & 6 \\ -2 & 0 & 0 & 0 & 5 & 30 \end{pmatrix}$$

Next we pivot about the 1st column, 2nd row.

$$R_2/2: \begin{pmatrix} 2 & 0 & 1 & 0 & -1 & 10 \\ 1 & 0 & 0 & 1/2 & -3/2 & 3 \\ 0 & 1 & 0 & 0 & 1 & 6 \\ -2 & 0 & 0 & 0 & 5 & 30 \end{pmatrix}$$

$$\begin{array}{l} R_1 - 2R_2 \\ R_4 + 2R_2 \end{array}: \begin{pmatrix} 0 & 0 & 1 & -1 & 2 & 4 \\ 1 & 0 & 0 & 1/2 & -3/2 & 3 \\ 0 & 1 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 & 2 & 36 \end{pmatrix}.$$

Since there are no negative numbers in the bottom row, we are finished and conclude the optimal solution is $x = 3$, $y = 6$, $P = 36$.