Mathematics 1070 SOLUTIONS
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1. Sketch the feasible set for the following system of inequalities:

$$\begin{aligned} x + y &\le 5 \\ 3x + y &\ge 9 \\ x &\ge 0 \\ y &\ge 0 \end{aligned}$$

Solution: The feasible set is a triangle with vertices $(3, 0)$, $(5, 0)$ and $(2, 3)$.

2. Solve the following system of linear equations using Gaussian Elimination on the augmented matrix for the system.

$$\begin{aligned} x - 2y + z &= 6 \\ 2x + y - 3z &= -3 \\ x - 3y + 3z &= 10 \end{aligned}$$

Solution: The augmented matrix is $\begin{pmatrix} 1 & -2 & 1 & 6 \\ 2 & 1 & -3 & 3 \\ 1 & -3 & -3 & 10 \end{pmatrix}$.

Pivoting along the rst row, rst column:

$$\begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix}: \begin{pmatrix} 1 & -2 & 1 & 6 \\ 0 & 5 & -5 & -15 \\ 0 & -1 & 2 & 4 \end{pmatrix}$$

Pivoting on the second row, second column:

$$R_2/5: \begin{pmatrix} 1 & -2 & 1 & 6 \\ 0 & 1 & -1 & -3 \\ 0 & -1 & 2 & 4 \end{pmatrix} \qquad \begin{matrix} R_1 + 2R_2 \\ R_3 + R_2 \end{matrix}: \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Pivoting on the third row, third column:

$$\begin{matrix} R_1 + R_3 \\ R_2 + R_3 \end{matrix}: \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

We conclude the solution is $x = 1$, $y = -2$, $z = 1$.

(3-4): Let $A = \begin{pmatrix} 3 & 5 & 2 \\ 5 & 18 & 9 \\ 5 & 3 & 1 \end{pmatrix}$.

3. Perform the Elementary Row Operation $R_2 \to R_2 - 3R_1$ on A.

   Solution: $\begin{pmatrix} 3 & 5 & 2 \\ -4 & 3 & 3 \\ 5 & 3 & 1 \end{pmatrix}$

4. Pivot about the 3rd row, 1st column of A.

   Solution: $R_3/5$: $\begin{pmatrix} 3 & 5 & 2 \\ 5 & 18 & 9 \\ 1 & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$ $\quad$ $\begin{matrix} R_1 - 3R_3 \\ R_2 - 5R_3 \end{matrix}$: $\begin{pmatrix} 0 & \frac{16}{5} & \frac{7}{5} \\ 0 & 15 & 8 \\ 1 & \frac{3}{5} & \frac{1}{5} \end{pmatrix}$

5. Consider the following problem.

   > The Silver State Mining Company operates two mines for the purpose of extracting silver and iron ore. One costs $\$15,000$ per day to operate and yields 100 ounces of silver and 20 tons of iron ore each day. The other costs $\$20,000$ per day to operate and yields 120 ounces of silver and 15 tons of iron ore each day. The company needs to deliver at least 800 ounces of silver and 100 tons of iron ore. How many days should each mine be operated in order to enable the company to meet its commitments at the least cost?

   Determine appropriate variables and set up appropriate constraints, non-negativity conditions and an objective function.

   Solution: Let x represent the number of days the rst mine is operated and let y represent the number of days the second mine is operated. Let C represent the cost of operating the mines.

$$100x + 120y \geq 800$$
$$20x + 15y \geq 100$$
$$x \geq 0, y \geq 0$$
$$C = 15000x + 20000y$$

6. Consider a linear programming problem with the following constraints, non-negativity conditions and objective function. Assume the objective function is to be maximized. Set up the initial simplex tableau.

$$\begin{aligned} 4x + 3y + 2z &\le 80 \\ 4x + 8y + 9z &\le 65 \\ 4x + 4y + 5z &\le 73 \\ x \ge 0, y &\ge 0 \\ p = 2x + 3y + 2z \end{aligned}$$

Solution: $\begin{pmatrix} 4 & 3 & 2 & 1 & 0 & 0 & 0 & 80 \\ 4 & 8 & 9 & 0 & 1 & 0 & 0 & 65 \\ 4 & 4 & 5 & 0 & 0 & 1 & 0 & 73 \\ -2 & -3 & -2 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

7. Solve the following linear programming problem geometrically, where C is to be minimized.

$$\begin{aligned} x + y &\ge 6 \\ 2x + y &\ge 10 \\ x \ge 0, y &\ge 0 \\ C = 3x + 5y \end{aligned}$$

Solution: The feasible set is an unbounded region in the rst quadrant with vertices $(6, 0)$, $(4, 2)$ and $(0, 10)$. The values of the objective function at those three vertices are 18, 22 and 50, respectively. Thus the minimal value of the objective function is 18 and occurs when $x = 6$ and $y = 0$.

8. Use the Simplex Method to solve the linear programming problem with the following initial simplex tableau. Assume the variables are x and y, the objective function is p and the objective function is to be maximized.

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 & 0 & 40 \\ 3 & 2 & 0 & 1 & 0 & 0 & 70 \\ 4 & 3 & 0 & 0 & 1 & 0 & 90 \\ -14 & -8 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Solution: First pivot on the rst row, rst column:

$$\tfrac{1}{2}R_1: \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 20 \\ 3 & 2 & 0 & 1 & 0 & 0 & 70 \\ 4 & 3 & 0 & 0 & 1 & 0 & 90 \\ -14 & -8 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \begin{matrix} R_2 - 3R_1 \\ R_3 - 4R_1: \\ R_4 + 14R_1 \end{matrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 20 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 & 0 & 10 \\ 0 & 1 & -2 & 0 & 1 & 0 & 10 \\ 0 & -1 & 7 & 0 & 0 & 1 & 280 \end{pmatrix}$$

Now pivot on the third row, second column:

$$\begin{matrix} R_1 - \frac{1}{2}R_3 \\ R_2 - \frac{1}{2}R_3: \\ R_4 + R_3 \end{matrix} \begin{pmatrix} 1 & 0 & \frac{3}{2} & 0 & -\frac{1}{2} & 0 & 15 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 5 \\ 0 & 1 & -2 & 0 & 1 & 0 & 10 \\ 0 & 0 & 5 & 0 & 1 & 1 & 290 \end{pmatrix}$$

We conclude the maximal value for p is 290 and occurs when $x = 15$ and $y = 10$.

9. Explain the di erence between a permutation and a combination.

Solution: A combination is simply a subset while a permutation is an arrangement of elements.

10. Calculate $C(7,3)$.

Solution: $C(7,3) = \dfrac{7!}{3!4!} = \dfrac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 35.$

11. Calculate $P(7,3)$.

Solution: $P(7,3) = 7 \cdot 6 \cdot 5 = 210.$

12. How many ten (10) card hands are there which contain four of one kind, a pair and four singletons?

Solution: $13 \cdot 12 \binom{4}{2} \binom{11}{4} 4^4$

(13-16): A pair of dice are rolled. Let A be the event the sum is odd, let B be the event a 7 is rolled and let C be the event an 8 is rolled.

13. Find $Pr(A)$.

    Solution: $Pr(A) = \frac{1}{2}$.

14. Find $Pr(B)$.

    Solution: $Pr(B) = \frac{6}{36} = \frac{1}{6}$.

15. Find $Pr(B|A)$.

    Solution: $Pr(B|A) = \dfrac{Pr(B \cap A)}{Pr(A)} = \dfrac{Pr(B)}{Pr(A)} = \dfrac{\frac{1}{6}}{\frac{1}{2}} = \dfrac{1}{3}$.

16. Find $Pr(C|A)$.

    Solution: $Pr(C|A) = 0$.

17. What will the balance in an account be after 25 years if a single deposit of $15,000 is left alone, accumulating interest at an annual rate of 6%, compounded quarterly?

    Solution: $15,000\left(1 + \frac{.06}{4}\right)^{25\cdot 4} = 15,000\left(1 + \frac{.06}{4}\right)^{100}$.

18. What is the present value of 25 annual payments of $15,000, assuming an annual interest rate of 6%, compounded annually? *Assume the first payment is made in one year.*

    Solution: $P = a_{\overline{n}|i}R = \dfrac{(1 + .06)^{25} - 1}{0.06(1 + .06)^{25}} \cdot 15,000$.

19. What is the future value of 25 annual payments of $15,000, assuming an annual interest rate of 6%, compounded annually? *Assume the first payment is made in one year and you want the value as soon as the last payment is made.*

    Solution: $F = s_{\overline{n}|i}R = \dfrac{(1 + .06)^{25} - 1}{0.06} \cdot 15,000$.

20. What will the monthly payments be on a $150,000 loan to be repaid in 25 years if the annual interest rate is 6%?

$P = a_{\overline{n}|i}R$, so $R = \dfrac{P}{a_{\overline{n}|i}} = \dfrac{150,000}{\dfrac{\left(1 + .06/12\right)^{300} - 1}{(.06/12)(1 + .06/12)^{300}}}$.