Mathematics 1070 <u>SOLUTIONS</u>
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Note: Most calculations are approximations, even though equal signs are used. The different variables used are used in the same way the variables with the same names were used in class except where noted.

1. Consider the formula $P = P_0\left(1 + \frac{r}{n}\right)^{nt}$. Explain the meaning of each of the variables: P, P_0, r, n and t. Do so using complete sentences and without the use of any mathematical notation other than references to the variables.

   Solution: This formula gives the balance in an account if a single deposit is made and interest is compounded periodically.
   P represents the balance in the account at time t.
   P_0 represents the initial balance in the account.
   r represents the annual interest rate.
   n represents the number of times interest is compounded annually.
   t represents the amount of time, measured in years, the money is left in the account.

2. Consider the formula $s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$. Discuss the use of the formula and explain the meaning of the variables: n and i. Do so using complete sentences and without the use of any mathematical notation other than references to the variables.

   Solution: The formula for $s_{\overline{n}|i}$ is used to obtain the future value of an increasing annuity. It is essentially a multiplier, representing the multiple of the annual \rent" which the future value will be.
   n is the number of payments which are to be made.
   i is the interest rate which is applied to the prior balance each time another \rent" payment is made.

3. Consider the formula $P = a_{\overline{n}|i}R$, where $a_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i(1+i)^n}$. Discuss the use of the formula and explain the meaning of the variables: P, n, i and R. Do so using complete sentences and without the use of any mathematical notation other than references to the variables.

   Solution: This is the formula for the present value of a decreasing annuity.
   P represents the present value.
   R represents the periodic \rent."
   n represents the number of payments that are made.
   i represents the interest rate applied to the existing balance at the time each payment is made.

4. \$573.28 is placed on deposit in an account paying interest at an annual rate of 3.72%, compounded quarterly. What will the balance be in ve years?

   Solution: We calculate $P = P_0(1 + r/n)^{nt} = 573.28(1 + .0372/4)^{4\cdot5} = 689.877936841$, so the balance will be \$689.88.

5. \$573.28 is placed on deposit in an account paying interest at an annual rate of 3.72%, compounded continuously. What will the balance be in ve years?

   Solution: We use the formula $P = P_0e^{rt} = 573.28e^{.0372\cdot5} = 690.471193431$, so the balance will be \$690.47.

6. An account pays interest at an annual rate of 4.31%, compounded monthly. What is the e ective annual yield?

   Solution: Since \$100 will grow to $100(1 + 0.431/12)^{12\cdot1} = 104.396168068$ in one year, the e ective annual yield is 4.396168068%. *Note: this can be done many other ways.*

7. An account pays interest at an annual rate of 4.31%, compounded monthly. How long will it take for the balance to at least triple?

   Solution: Using the formula $P = P_0(1 + r/n)^{nt}$, we need to solve $3P_0 = P_0(1 + .0431/12)^{12t}$ for t. We get:
   $(1 + .0431/12)^{12t} = 3$
   $12t \ln(1 + .0431/12) = \ln 3$
   $t = \dfrac{\ln 3}{12 \ln(1 + .0431/12)} \approx 25.5355924204.$
   Since $.5355924204 \cdot 12 \approx 6.4271090448$, it will take 25 years, 7 months for the balance to triple.

8. What annual interest rate is needed on an account in order for the balance to double in twelve years, assuming interest is compounded continuously?

   Solution: Using the formula $P = P_0e^{rt}$, we need to solve $2P_0 = P_0e^{r\cdot12}$ for r:
   $e^{12r} = 2$
   $12r = \ln 2$
   $r = \dfrac{\ln 2}{12} \approx 0.057762265.$
   We thus need an annual interest rate of approximately 5.7762265%.

9. \$50 is placed in an account at the end of each month in an account paying interest at an annual rate of 5%.

   (a) How much will the account contain after ten years? **Solution:** Using the formula $F = Rs_{\overline{n}|i}$, with $s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$, since there will be 120 monthly payments made in ten years, we get: $F = 50 \cdot \dfrac{(1+.05/12)^{120} - 1}{.05/12} = 7,764.1139802$. The balance will thus be \$7, 764.11.

   (b) How much interest will have been earned? **Solution:** Since $120 \cdot 50 = 6,000$ will have been made in payments, the amount of interest earned will have been \$1, 764.11.

10. An annuity pays \$50 at the end of each month for a period of ten years. Assuming an annual interest rate of 5%, what is the present value of the annuity?

    Solution: Using the formula $P = a_{\overline{n}|i}R$, where $a_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i(1+i)^n}$, we have $P = \dfrac{(1+.05/12)^{12\cdot 10} - 1}{(.05/12)\cdot(1+.05/12)^{12\cdot 10}} \cdot 50 = 4,714.06751933$, so the present value is \$4, 714.07.

11. What will the monthly payment be on a \$300, 000 mortgage to be repaid over thirty years if the annual interest rate is 6%? *Extra Credit: What will the balance on the mortgage be after fifteen years?*

    Solution: Using the same formula, we get $R = \dfrac{P}{a_{\overline{n}|i}} = \dfrac{P}{\frac{(1+i)^n-1}{i(1+i)^n}} = \dfrac{300,000}{\frac{(1+.06/12)^{360}-1}{(.06/12)(1+.06/12)^{360}}} = 1,798.65157546$. The monthly payment will thus be \$1, 798.66.

    Extra Credit: The balance will be the amount monthly payments of \$1, 798.66 *can pay off in fifteen years, so we may calculate* $P = a_{\overline{n}|i}R$, *where* $a_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i(1+i)^n}$. *to get* $P = \dfrac{(1+.06/12)^{180} - 1}{i(1+.06/12)^{180}} \cdot 1,798.66 = 213,147.531693$, *so the balance on the loan after fifteen years will be* \$213, 147.53.