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Inclusion-Exclusion Principle

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This is really a special case of a more general *Inclusion-Exclusion Principle* which may be used to find the cardinality of the union of more than two sets.

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Fundamental Principle of Counting

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If we have to make a sequence of choices for which the first choice can be made in n_1 ways, the second choice can be made in n_2 ways, the third choice can be made in n_3 ways, and so on, then the entire sequence of choices can be made in $n_1 \cdot n_2 \cdot n_3 \cdot \ldots$ ways.

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Example: There are 36 ways of rolling a pair of dice, since there are 6 ways the first die can come out and 6 ways the second can come out, so there are $6 \cdot 6 = 36$ ways the two dice can come out.

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A permutation is a list or arrangement of elements chosen from some set.

Permutations may be either with replacement or without replacement. In a permutation with replacement, there may be repetitions of elements within an arrangement. In a permutation without replacement, no such repetitions may occur.

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For example, if we shuffle a deck of cards and, one at a time, choose five cards and write down the cards we have chosen, in order, we have a permutation without replacement of length five chosen from a set of size 52.

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Many sample spaces which generate equiprobable spaces contain either combinations or permutations of elements of other sets.

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There is no special notation for the number of permutations with replacement.

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From the Fundamental Principle of Counting, if we choose k elements from a set of size n, with replacement, each of the elements can be chosen in n ways, so the sequence of elements can be chosen in $n \cdot n \cdot n \cdot \cdots \cdot n = n^k$ ways.

We thus easily see the number of permutations, with replacement, of length k chosen from a set of size n is n^k.

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Similarly, the third element can be chosen in $n - 2$ ways, the fourth in $n - 3$ ways, and so on until we get to the last, or k^{th} element, which can be chosen in $n - [k - 1]$ ways.

We thus get $P(n, k) = n(n - 1)(n - 2) \ldots (n - [k - 1])$.

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For example, $1! = 1$, $2! = 2 \cdot 1$, $3! = 3 \cdot 2 \cdot 1$, $\ldots$
$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1.$

Permutations

As a result of the cancellation law, if n and k are integers with $0 \leq k < n$, As a result of the cancellation law, if n and k are integers with $0 \leq k < n$,

$$\frac{n!}{(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \dots (n-[k-1])}{} \frac{(n-k)(n-k-1) \dots 3 \cdot 2 \cdot 1}{(n-k)(n-k-1) \dots 3 \cdot 2 \cdot 1} = n \cdot (n-1) \cdot (n-2) \dots (n-[k-1]) = P(n,k).$$

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This gives the alternate formula $P(n,k) = \dfrac{n!}{(n-k)!}$ if n is a positive integer and $k < n$.

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We therefore make the special definition $0! = 1$, so that the formula $P(n, k) = \frac{n!}{(n-k)!}$ holds whenever n is a positive integer and $0 \le k \le n$.

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We thus get the formula $C(n,k) = \frac{n!}{k!(n-k)!}$, and this holds even when $n = 0$, $k = 0$ or $k = n$.