Roulette

The game of ***roulette*** serves as a nice introduction to some of the key ideas relating to probability. We will look at some of the possible outcomes when a roulette wheel is spun and relate those outcomes to probability in general.

A roulette wheel contains 38 slots, numbered 0, 00, and $1, 2, 3, \ldots 36$. When the wheel is spun, a ball eventually falls into one of the slots. Assuming the wheel is balanced and the slots are the same size, as is supposed to be the case, there are 38 possible outcomes, each of probability $\frac{1}{38}$.

This is an example of an ***Equiprobable Probability Space***.

Terminology

We consider experiments with a finite set of possible outcomes.

We call the set of possible outcomes the ***sample space*** and the individual outcomes are called ***sample points***.

A subset of the sample space is referred to as an ***event***. We'll spend time on events later on.

Straight Bets

There is a variety of ways one can bet at roulette. We will see the casino doesn't care how one bets. The simplest bet is a straight bet, where one bets on a specific number and wins, with a payoff at 35-1, if that number comes up. This means that if a player bets \$1 and wins, the player will get \$36 back, the dollar he bet along with \$35 more.

In our analyses, let's assume every bet is for \$1.

Straight Bets – The Big Picture

Suppose someone repeatedly makes a straight bet for \$1. Although the results will vary, on average the player will win about $\frac{1}{38}$ of the time and lose the rest of the time.

If he (or she) plays 38 times, he could expect to win about one time, coming out \$35 ahead that time, but losing about 37 times, coming out \$1 behind those times, so after playing 38 times he can expect to be about \$2 behind. That works out to losing about $\$\frac{2}{38} = \$\frac{1}{19}$ per bet.

Random Variables

Random variables are numerical values associated with outcomes of experiments.

The letter X is the most common symbol used to represent a random variable.

If we consider a straight bet of \$1 to be an experiment and X to be the net winnings for the player, then X can take on either the value of 35 (if the player wins) or -1 (if the player loses).

If we denote the probability X takes on a certain value k by $P(X = k)$, then $P(X = 35) = \frac{1}{38}$ and $P(X = -1) = \frac{37}{38}$.

Mathematical Expectation

Mathematical expectation is essentially what we can expect the average value of a random variable to be close to. We denote it by $E(X)$.

In this case, we expect $E(X) = -\frac{1}{19}$, since we expect to lose an average of $\$\frac{1}{19}$ per bet.

Calculating Mathematical Expectation

If we look at our earlier calculation, we can infer a reasonable definition of *mathematical expectation*.

Imagining playing 38 times, we took the 1 expected win, with X being 35 that time, the 37 expected losses, with X being -1 those times, and took $35 - 37 \cdot 1 = -2$. We then divided by 38 to get $\frac{-2}{38} = -\frac{1}{19}$.

If we unravel the calculation, we can write $-\frac{1}{19} = \frac{-2}{38} = \frac{35 - 37 \cdot 1}{38} = \frac{35 \cdot 1 - 1 \cdot 37}{38} = \frac{35 \cdot 1 + (-1) \cdot 37}{38} = 35 \cdot \frac{1}{38} + (-1) \cdot \frac{37}{38} = 35P(X = 35) + (-1)P(X = -1)$.

Abstracting to Get a Formula

The calculation $E(X) = 35P(X = 35) + (-1)P(X = -1)$ suggests we calculate mathematical expectation by taking each possible value of the random variable, multiply it by the probability the random variable takes on that value, and add the products together.

Symbolically, we may write this as

$E(X) = \sum kP(X = k)$,
where the sum is taken over all the possible values for X.

Probability Spaces

Recall:

We consider experiments with a finite set of possible outcomes.

We call the set of possible outcomes the *sample space* and the individual outcomes are called *sample points*.

We're interested in the likelihood of each possibility and assign, hopefully in a meaningful way, a probability to each.

Definition 1 (Probability). *A probability is a number between 0 and 1.*

Definition 2 (Probability Space). *A probability space is a sample space $\mathcal{S}$ for which a probability $p(x)$ has been assigned to each sample point x such that the sum $\sum_{x\in\mathcal{S}} p(x)$ of the probabilities is 1.*

Examples

Consider the sample space {H,T}, with probabilities $p(H) = p(T) = \dfrac{1}{2}$. This may be considered a model for flipping a coin a single time.

Consider the sample space $\{H_0, H_1, H_2\}$, with probabilities $p(H_0) = \dfrac{1}{4}$, $p(H_1) = \dfrac{1}{2}$, $p(H_2) = \dfrac{1}{4}$. This may be considered a model for flipping a coin twice.

The first example is known as an *Equiprobable Space*; the second is not an equiprobable space.

Equiprobable Spaces

Definition 3 (Equiprobable Space). *A probability space $\mathcal{P}$ is called an* equiprobable space *if each sample point has the same probability.*

If an equiprobabe space $\mathcal{P}$ has n points, *we'll write* $|\mathcal{P}| = n$, then $p(x) = \dfrac{1}{n} \forall x \in \mathcal{P}$.

In an equiprobable space, if we have an event $E \subset \mathcal{P}$, then $p(E) = \dfrac{|E|}{n}$.

It's obviously important to be able to count the number of sample points comprising events. The science of counting the size of a set is called *combinatorics*.