Rates

Rates are used to find absolute amounts.

Examples:

- Sales tax rate – used to find sales tax
- Interest Rate – used to find interest
- Speed – used to find distance

Percent

The language of percents is often used to describe rates, such as tax rates or interest rates.

When one says the sales tax is 6%, one really means the *sales tax rate* is 6%. The sales tax rate is used to compute the actual sales tax.

When one says a bank pays 5% interest, one really means the bank's *interest rate* is 5%. The interest rate is used to calculate the actual interest.

Percent

Percent literally means *per 100*, or *divided by 100*.

Saying Connecticut's sales tax rate is 0.06 means exactly the same thing as saying Connecticut's sales tax rate is 6%.

How many times have you heard someone say "Connecticut's sales tax rate is 0.06?"

When you hear the word *percent*, it's a tipoff that a rate is being given rather than an absolute amount.

Compound Interest

Suppose an initial amount P_0 is placed in some financial institution, such as a bank, and left on deposit, collecting interest at an annual rate r, compounded quarterly.

After three months, the account will be credited with a quarter of a year's worth of interest, or $\frac{1}{4} \cdot r \cdot P_0 = \frac{r}{4} \cdot P_0$, leaving a balance of $P_0 + \frac{r}{4} \cdot P$

After nine months, the balance will be $\left(1+\frac{r}{4}\right)^3 P_0$ and after one year the balance will be $\left(1+\frac{r}{4}\right)^4 P_0$.

After two years, the balance will be $\left(1+\frac{r}{4}\right)^8 P_0$.

After t years, the balance will be $\left(1+\frac{r}{4}\right)^{4t} P_0$.

If, instead of compounding quarterly, or four times per year, interest was compounded n times per year, the formula would have n in place of 4, yielding the *Compound Interest Formula*:

$P = \left(1+\frac{r}{n}\right)^{nt} P_0,$
where P represents the balance after t years.

Continuous Interest

Suppose we rewrite the Compound Interest Formula as follows:

$$P = \left(1+\frac{r}{n}\right)^{nt} P_0 = \left(1+\frac{1}{n/r}\right)^{(n/r)rt} P_0 = \left[\left(1+\frac{1}{n/r}\right)^{n/r}\right]^{rt} P_0.$$

It turns out that, regardless of the value of r, when n is very big, $\left(1+\frac{1}{n/r}\right)^{n/r}$ is very close to the special mathematical constant $e \approx 2.17828182846$, so P gets very close to $e^{rt} \cdot P_0$.

This gives the *Continuous Interest Formula*, $P = P_0 e^{rt}$.

Future Value

The value of an account at some future date, taking compound interest into account, is called *Future Value*.

If we denote future value by F, the compound interest formula implies $F = P_0\left(1+\frac{r}{n}\right)^{nt}$, where the variables represent what they have in the past.

Present Value

Suppose we know we will have access to some amount F at some time in the future. The *Present Value* of that money is the amount that would need to be deposited now to grow to that amount in the future.

If we denote the present value by P, the compound interest formula implies $F = P\left(1+\frac{r}{n}\right)^{nt}$, so $P = F\left(1+\frac{r}{n}\right)^{-nt}$.

Annuities

In an annuity, periodic payments, called *rent* and usually denoted by R are made.

Annuities are used in two basic ways:

- Periodic payments are made and collect interest for a number of years, essentially as a means of saving, and at the end of that period the account has grown to an amount F, the future value.
- A fixed amount P, called the present value of the annuity, is paid, generally to an insurance company, which then makes periodic payments to the owner of the account.

Future Value of an Annuity

Consider the future value of an annuity where n periodic rent payments R are made and each time a payment is made interest at a rate i is made to the previous balance. Typically, the payments are made monthly, so if there is an annual rate r, the interest rate applied each month will be $i = \frac{r}{12}$. Let's assume the payments are made each month. The calculations will be exactly the same if the period between payments is different.

The first payment will collect interest $n-1$ times, so it will grow to a future value of $R(1+i)^{n-1}$.

The second payment will collect interest $n-2$ times, so it will grow to a future value of $R(1+i)^{n-2}$.

Similarly, the third payment will grow to a future value of $R(1+i)^{n-3}$.
Continuing, when we get to the last payment, it doesn't collect any interest at all, so it *grows* to a *future value* of just R.

All together, we can add the future values of all the payments to get the future value of the entire annuity:

$$F = R(1+i)^{n-1} + R(1+i)^{n-2} + R(1+i)^{n-3} + \cdots + R(1+i) + R.$$

Some may recognize this as a geometric series consisting of n terms, with first term $R(1+i)$ and common ratio $\frac{1}{1+i}$. In any case, we can use a trick to evaluate F. The trick is to multiply F by $1+i$ and then subtract:

$$F(1+i) = R(1+i)^n + R(1+i)^{n-1} + R(1+i)^{n-2} + \cdots + R(1+i)^2 + R(1+i).$$

$$F - F(1+i) = [R(1+i)^{n-1} + R(1+i)^{n-2} + R(1+i)^{n-3} + \cdots + R(1+i) + R] - [R(1+i)^n + R(1+i)^{n-1} + R(1+i)^{n-2} + \cdots + R(1+i)^2 + R(1+i)],$$

so $-iF = R - R(1+i)^n$, $-iF = R[1-(1+i)^n]$, $F = \frac{R(1-(1+i)^n)}{-i}$, or $F = \frac{(1+i)^n - 1}{i} \cdot R$.

We sometimes write $F = s_{\overline{n}|i}R$, where $s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$ is pronounced *s sub n angle i*.

If we want to calculate the *rent* needed so that we will wind up with a future value F, we need only solve for R to get:

$$R = \frac{1}{s_{\overline{n}|i}} \cdot F.$$

Present Value of an Annuity

Since $P = F(1+i)^{-n}$, we have $P = \frac{(1+i)^n - 1}{i} \cdot R \cdot (1+i)^{-n} = \frac{(1+i)^n - 1}{i(1+i)^n} \cdot R$.

We sometimes write $P = a_{\overline{n}|i} \cdot R$, where $a_{\overline{n}|i} = \frac{(1+i)^n - 1}{i(1+i)^n}$.

We can use this the calculate the cost of buying an annuity that will pay back a certain monthly rent. *Of course, if we're buying it from an insurance company, the company must make a profit, so the amount the insurance company charges will be greater than the present value.*

Amortization of Loans

When we take out a loan, it is mathematically the same as if the loan company was buying an annuity from us. We thus have $P = a_{\overline{n}|i} \cdot R$, where P is the amount of the loan and R is the monthly payment.

Since we are probably most interested in our monthly payment, we may want to solve for R to get

$$R = \frac{P}{a_{\overline{n}|i}}.$$

Since $a_{\overline{n}|i} = \frac{(1+i)^n - 1}{i(1+i)^n}$, we have $R = \dfrac{P}{\frac{(1+i)^n - 1}{i(1+i)^n}} = \dfrac{Pi(1+i)^n}{(1+i)^n - 1}$.

This may also be written as $R = \dfrac{Pi}{(1 - (1+i)^{-n})}$.

Note that Pi is one month's interest on the initial balance, and this gets divided by a number a little less than 1, so the monthly payment will be slightly more than the amount of interest that would be paid in a month on the original amount of the loan.

Calculating the Balance on a Loan

Let P_k be the balance after k payments have been made.

After the next payment is made, interest in the amount of iP_k will be charged, but the balance will also be decreased by R, so

$P_{k+1} = P_k + iP_k - R$, or

$P_{k+1} = (1 + i)P_k - R$.

It is convenient to do this calculation using a spreadsheet.

Calculating the Balance on a Loan

We can also develop a formula for the balance after a certain number of payments. Noting $P = P_0$,

$P_1 = (1+i)P - R.$

$P_2 = (1+i)P_1 - R = (1+i)((1+i)P - R) - R = (1+i)^2P - (1+i)R - R$

$P_3 = (1+i)P_2 - R = (1+i)((1+i)^2P - (1+i)R - R) - R = (1+i)^3P - (1+i)^2R - (1+i)R - R = (1+i)^3P - [(1+i)^2 + (1+i) + 1]R$

Continuing, we get

$$P_k = (1+i)^kP - [(1+i)^{k-1} + (1+i)^{k-2} + \cdots + 1]R.$$

The coefficient of R is precisely what we found to be the value of an annuity for which the *rent* was 1 after k payments, $s_{\overline{k}|i} = \dfrac{(1+i)^k - 1}{i}$.

We thus have $P_k = (1+i)^kP - s_{\overline{k}|i}R$.

Note $P_n = (1+i)^nP - s_{\overline{n}|i}R$. But $(1+i)^nP$ represents the future value of P and $s_{\overline{n}|i}R$ represent the future value of an annuity after n payments. If the loan is to be paid off after n payments, these should be equal, so their difference will be 0. Obviously, if the loan is to be paid off after n payments, $P_n = 0$.