Linear Functions and Models
Functions and Mathematical Modeling

Definition 1 (Function). *A function $f : A \to B$ is a rule or correspondence which assigns a specific element of B to each element of A.*

The set A is called the domain of the function.

The element assigned to an element $a \in A$ is called the value of the function f at a and is denoted by $f(a)$.

The set B is called the co-domain of the function. Sometimes it is referred to as the range, but technically the range is the set of values taken on by the function.

Example: $f(x) = x^2$ or $y = x^2$

These are alternate notations for giving the correspondence defining a function. In each, x is called the *independent variable*. In the second case, y is called the *dependent variable*.

Everyday functions:

Population is a function of time.

Postage is a function of weight.

The balance in a bank account is a function of time.

The area of a circle is a function of its radius.

Functions are used to model problems.

Example: Dog pen problem (page 9): A dog pen is to be made from 60 feet of fencing, enclosing three sides of a rectangular region. (The fourth side is an existing wall.) The area of the pen can be represented as a function of the length of the side opposite the wall.

Let x be the length (in feet) of the side opposite the wall. The other two sides, of equal length, have to total $60 - x$ feet, so each must have length $\frac{60-x}{2}$. If we represent the area by A, we have $A = x \cdot \left(\frac{60-x}{2}\right)$.

Problem: Suppose we want the area to be 400 square feet?

Solution: Since the area is equal to $= x \cdot \left(\frac{60-x}{2}\right)$, it follows that we need to have $x \cdot \left(\frac{60-x}{2}\right) = 400$. We can find the dimensions of the pen by solving this equation.

Linear Functions

A linear function is a function of the form $f(x) = mx + b$.

m is called the slope.

b is called the vertical intercept. *If the dependent variable is y, it is called the $y-$intercept.*

Graphs

The graph of a function $f : A \to B$ is $\{(x, y) : y = f(x)\}$.

Equations of Lines

Slope-Intercept Form

$y = mx + b$

Point-Slope Form

$y - y_0 = m(x - x_0)$, where the slope is m and (x_0, y_0) is a point on the graph.

Linear Growth

We have linear growth when there is a constant rate of change. Linear growth in a population may be modeled with the formula $P = P_0 + mt$.

Some situations are best modeled by piecewise linear functions, whose graphs are the unions of line segments.

Fitting Linear Models to Data

Definition 2 (Average Rate of Change)**.** *The average rate of change of a population P is $\frac{\Delta P}{\Delta t}$, where ΔP is the change in the population and Δt is the length of time.*

An obvious model is to create a linear function which agrees with the first size of the population and has a slope equal to the average rate of change.

Problem: It may not be ideal.

It's reasonable to try to minimize the total error.

Problem: Positive and negative errors cancel each other out.

Solution: Look at the squares of the errors.

Definition 3 (Sum of Squares of the Errors)**.** $SSE =$ *the sum of the squares of the errors at all the data points.*

Our models will minimize SSE. The computations will be performed by calculators or computers.

Definition 4 (Average Squared Error)**.** *The average squared error is the average of the squares of the errors, obtained by dividing* SSE *by the number of data points.*

Definition 5 (Average Error)**.** *The average error is defined as the square root of the average squared error.*

Note: The Average Error, as defined, is not literally the *average error*.

Regression on a Calculator

Store the t and P values in a table as lists $L1$ and $L2$, choosing **STAT** and then choosing **EDIT**.

Calculate the coefficients in the regression $P = P_0 + mt$, displayed on the calculator as $y = ax + b$, by choosing **STAT**, then choosing **CALC**, then choosing **LinReg**$(ax + b)$ L_1**,** L_2**,** Y_1.

To plot the regression line along with the data points, choose the **STAT PLOT** menu directly from the calculator keypad and set **Plot 1** to **On**.

The errors or differences between the actual data values and the values predicted by the linear regression, called *residuals*, are stored in the **LIST** menu under the name **RESID**.

Percentages

Percentages are generally used to describe a *rate* rather than an absolute amount. One finds an absolute amount by multiplying a rate by another quantity.

Examples:

Speed is a rate. Distance is average speed $\times$ time.

Acceleration is a rate (the rate at which speed changes). Speed is average acceleration$\times$time.

A sales tax rate is a rate. Sales tax is tax rate$\times$ price.

Interest rate is a rate. Interest is interest rate$\times$ balance.

Percent means per cent or per 100. It is actually away of representing a fraction.

6% means 6 per 100 or 6/100 or 0.06.

In Connecticut, the sales tax rate is 6%. It would be equally correct to say the sales tax rate is 6/100, or the sales tax rate is 3/50, or the sales tax rate is 0.06.

Party Fun: Get up on a chair during a party and tell everyone the sales tax rate in Connecticut is 3/50.

Caution: When mentioning rates, the word rate is often omitted. Thus, we often say that in Connecticut the sales tax is 6% rather than saying the sales tax *rate* is 6%. *It must be understood, when percent is mentioned, that the figure represents a rate even if the term rate is left unsaid. Thus, while the statements "the sales tax is 6%" and "the sales tax is 3/50" literally mean the same thing, they are understood to have different meanings.*

Let P be the price of an item, r the sales tax rate, C the net cost including sales tax.

The sales tax is rP, so the total cost is $P + rP$ (the price plus the sales tax). Since $P + rP = P(1 + r)$, we get the formula $C = P(1 + r)$.

Interest

Interest is similar to sales tax. If interest on a balance P_0 was computed once a year at an annual rate r, the interest after a year would be rP_0, so the balance P after a year would be $P_0 + rP_0 = P_0(1+r)$ (the original balance plus the interest), so we'd get $P = P_0(1+r)$. This should look very similar to the formula for net cost including sales tax.

After two years, interest would be calculated again. The amount of interest would be $r \times P_0(1+r)$, so the new balance would be $P_0(1+r) + r \times P_0(1+r) = P_0(1+r)(1+r)$, giving us the balance $P = P_0(1+r)^2$.

We can continue indefinitely and find that after t years, the balance will be $P = P_0(1+r)^t$.

We have built a *mathematical model*.

Compound Interest

Suppose that instead of interest being computed once per year, it was computed twice per year. Each time, the amount of interest computed would be half the amount it would be had it been computed for a full year. Thus, after a half year, the interest would be $\frac{1}{2} \times r \times P_0$, so the balance would be $P = P_0 + \frac{1}{2} \times rP_0 = P_0(1 + \frac{r}{2})$.

After another half year, we'd get interest in the amount $\frac{1}{2} \times r \times P_0(1 + \frac{r}{2})$ so the balance would be $P_0(1 + \frac{r}{2}) + \frac{1}{2} \times r \times P_0(1 + \frac{r}{2}) = P_0(1 + \frac{r}{2})^2$.

After two years, the balance would be $P_0(1 + \frac{r}{2})^4$.

After t years, the balance would be $P = P_0(1 + \frac{r}{2})^{2t}$.

If, instead of interest being compounded twice a year, it was compounded n times per year, the balance would be $P = P_0(1 + \frac{r}{n})^{nt}$.

This is called the *Compound Interest Formula*.

Continuous Interest

Using the properties of exponents, one may rewrite the Compound Interest Formula $P = P_0(1+\frac{r}{n})^{nt}$ in the form

$$P = P_0\left[(1+\tfrac{r}{n})^{n/r}\right]^{rt}$$

or

$$P = P_0\left[(1+\tfrac{1}{n/r})^{n/r}\right]^{rt}.$$

When interest is compounded very frequently, the quotient n/r will be very large. It turns out that the larger n/r gets, the closer $(1+\frac{1}{n/r})^{n/r}$ gets to the number e, and thus the closer P gets to P_0e^{rt}. This leads to the *Continuous Interest Formula*:

$$P = P_0e^{rt}.$$

The number e is a special mathematical constant, which is also the base of the natural logarithm function. A decimal approximation to e is $e \approx 2.7182818$. For many purposes, it suffices to know $2 < e < 3$, that is, e is between 2 and 3.

Paying Off a Loan

Notation

P	Initial loan balance
P_n	Balance after n months
m	Monthly payment
r	Monthly interest rate

Note: $P = P_0$

Since the balance P_n after n months will be the balance P_{n-1} the month before, less the monthly payment m, plus the interest $P_{n-1}r$ for one month on the balance the month before, it follows that

$$P_n = P_{n-1} - m + P_{n-1}r = P_{n-1}(1 + r) - m.$$

So . . .

$P_1 = P_0(1 + r) - m$ or

$P_1 = P(1+r) - m$

$P_2 = P_1(1+r) - m = (P(1+r) - m)(1+r) - m = P(1+r)^2 - m(1+r) - m$

$P_3 = P_2(1+r) - m = (P(1+r)^2 - m(1+r) - m)(1+r) - m = P(1+r)^3 - m(1+r)^2 - m(1+r) - m$

Similarly, we get

$P_4 = P(1+r)^4 - m(1+r)^3 - m(1+r)^2 - m(1+r) - m.$

We can see the pattern that in general we will get

$P_n = P(1+r)^n - m(1+r)^{n-1} - m(1+r)^{n-2} - m(1+r)^{n-3} - \cdots - m(1+r) - m$ or

$P_n = P(1+r)^n - m(1 + (1+r) + (1+r)^2 + (1+r)^3 + \cdots + (1+r)^{n-1}).$

The expression multiplied by m is in the form $1 + x + x^2 + x^3 + \cdots + x^{n-1}$. This leads to the question of whether we can simplify $1 + x + x^2 + x^3 + \cdots + x^{n-1}$.

Exploration

Factor $1 - x^2$

Result: $1 - x^2 = (1 - x)(1 + x)$

Factor $1 - x^3$

Result: $1 - x^3 = (1 - x)(1 + x + x^2)$

Factor $1 - x^4$

Result: $1 - x^4 = (1 - x)(1 + x + x^2 + x^3)$

Notice a pattern?

It appears that

$$1-x^n = (1-x)(1+x+x^2+x^3+\cdots+x^{n-2}+x^{n-1})$$

This can be verified by multiplying the right hand side out:

$(1-x)(1+x+x^2+x^3+\cdots+x^{n-2}+x^{n-1}) = 1-x+x-x^2+x^2-x^3+x^3-\cdots-x^{n-2}+x^{n-2}-x^n = 1-x^n$.

This confirms that factorization of $1-x^n$ and, dividing both sides by $1-x$ yields

$$\frac{1-x^n}{1-x} = 1+x+x^2+x^3+\cdots+x^{n-2}+x^{n-1}.$$

This is the formula we need.

Replacing x by $1+r$, we get $1+(1+r)+(1+r)^2+\cdots+(1+r)^{n-1} = \dfrac{1-(1+r)^n}{1-(1+r)} = \dfrac{1-(1+r)^n}{-r} = \dfrac{(1+r)^n-1}{r}$ and we then can rewrite

$P_n = P(1+r)^n - m(1 + (1+r) + (1+r)^2 + (1+r)^3 + \cdots + (1+r)^{n-1})$ as

$$P_n = P(1+r)^n - m \cdot \frac{(1+r)^n - 1}{r}.$$

Suppose we want to calculate a monthly payment if a loan will be paid off in n months? In that case, P_n will equal 0, so we'll have

$$P(1+r)^n - m \cdot \frac{(1+r)^n - 1}{r} = 0.$$

We can fairly easily solve this for the monthly payment m.

$$P(1+r)^n = m \cdot \frac{(1+r)^n - 1}{r}$$

Switching sides:

$$m \cdot \frac{(1+r)^n - 1}{r} = P(1+r)^n$$

$$m = \frac{P(1+r)^n}{[(1+r)^n - 1]/r}$$

$$m = \frac{Pr(1+r)^n}{(1+r)^n - 1}.$$

If we divide both numerator and denominator by $(1+r)^n$, we get

$$m = \frac{Pr}{1-(1+r)^{-n}}.$$

This should make sense, since the numerator is Pr, which is one month's interest on the original balance, and the denominator is clearly less than one. Thus the monthly payment will be more than one month's interest on the original balance. This is clearly necessary, since otherwise the balance would keep growing and the loan would never get paid off.

The larger n is, the closer $(1+r)^{-n}$ will be to 0, so the closer the denominator will be to 1 and the smaller the monthly payment will be. In other words, stretching out the repayment period results in a lower monthly payment.

On the other hand, the larger the monthly interest rate r, the larger the numerator Pr is and the larger the monthly payment.