Mathematics 105 — SOLUTIONS

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Due Wednesday, November 28, 2006 — This problem set is worth 100 points.

1. Use Gauss-Jordan Elimination to find the inverse of the matrix $\begin{pmatrix} 1 & 5 & 2 \\ 3 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}$.

   Solution: We begin by forming a matrix with the entries from the original matrix on the left and the entries from the identity matrix on the right and then perform Gauss-Jordan Elimination. The entries we wind up with on the right will be the inverse matrix.

$$\begin{pmatrix} 1 & 5 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 0 & 1 & 0 \\ -2 & 2 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \substack{R_2 \leftarrow R_2 - 3R1 \\ R_3 \leftarrow R_3 + 2R_1}: \begin{pmatrix} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & -14 & -6 & -3 & 1 & 0 \\ 0 & 12 & 5 & 2 & 0 & 1 \end{pmatrix}$$

$$R_2 \leftarrow R_2/(-14): \begin{pmatrix} 1 & 5 & 2 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{7} & \frac{3}{14} & -\frac{1}{14} & 0 \\ 0 & 12 & 5 & 2 & 0 & 1 \end{pmatrix} \quad \substack{R_1 \leftarrow R_1 - 5R_2 \\ R_3 \leftarrow R_3 - 12R_2}: \begin{pmatrix} 1 & 0 & -\frac{1}{7} & -\frac{1}{14} & \frac{5}{14} & 0 \\ 0 & 1 & \frac{3}{7} & \frac{3}{14} & -\frac{1}{14} & 0 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} & \frac{6}{7} & 1 \end{pmatrix}$$

$$R_3 \leftarrow -7R_3: \begin{pmatrix} 1 & 0 & -\frac{1}{7} & -\frac{1}{14} & \frac{5}{14} & 0 \\ 0 & 1 & \frac{3}{7} & \frac{3}{14} & -\frac{1}{14} & 0 \\ 0 & 0 & 1 & 4 & -6 & -7 \end{pmatrix} \quad \substack{R_1 \leftarrow R_1 + \frac{1}{7}R_3 \\ R_2 \leftarrow R_2 - \frac{3}{7}R_3}: \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & 0 & -\frac{3}{2} & \frac{5}{2} & 3 \\ 0 & 0 & 1 & 4 & -6 & -7 \end{pmatrix}$$

   We conclude $\begin{pmatrix} 1 & 5 & 2 \\ 3 & 1 & 0 \\ -2 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ -\frac{3}{2} & \frac{5}{2} & 3 \\ 4 & -6 & -7 \end{pmatrix}$

2. Consider a maximum problem with the following initial simplex tableau.

$$\begin{pmatrix} 2 & 3 & 1 & 1 & 0 & 0 & 5 \\ 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ -5 & -4 & -3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Assuming the variables are x, y, z and the objective function is M, find the values of the variables at which the objective function is maximized and find that maximum value.

Solution: Since -5 is the negative entry in the bottom row with the largest magnitude, we pivot about the first column. Since $\frac{5/2}{1}$ is the smallest quotient among entries in the last column divided by corresponding entries in the pivot column, we pivot about the first row.

$$R_1 \leftarrow R_1/2\colon \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\ 4 & 1 & 2 & 0 & 1 & 0 & 11 \\ 3 & 4 & 2 & 0 & 0 & 1 & 8 \\ -5 & -4 & -3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{matrix} R_2 \leftarrow R_2 - 4R_1 \\ R_3 \leftarrow R_3 - 3R_1\colon \\ R_4 \leftarrow R_4 + 5R_1 \end{matrix} \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\ 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & 0 & 1 & \frac{1}{2} \\ 0 & \frac{7}{2} & -\frac{1}{2} & \frac{5}{2} & 0 & 0 & \frac{25}{2} \end{pmatrix}$$

Next, we pivot about the third column, third row:

$$R_3 \leftarrow 2R_3\colon \begin{pmatrix} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\ 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & -1 & 1 & -3 & 0 & 2 & 1 \\ 0 & \frac{7}{2} & -\frac{1}{2} & \frac{5}{2} & 0 & 0 & \frac{25}{2} \end{pmatrix}$$

$$\begin{matrix} R_1 \leftarrow R_1 - \frac{1}{2}R_3 \\ R_4 \leftarrow R_4 + \frac{1}{2}R_3 \end{matrix}\colon \begin{pmatrix} 1 & 2 & 0 & 2 & 0 & -1 & 2 \\ 0 & -5 & 0 & -2 & 1 & 0 & 1 \\ 0 & -1 & 1 & -3 & 0 & 2 & 1 \\ 0 & 3 & 0 & 1 & 0 & 1 & 13 \end{pmatrix}$$

We are finished pivoting and conclude the maximum value $M = 13$ occurs when $x = 2$, $y = 0$, $z = 1$.

3. Consider the following linear programming problem:

   $3x + y \geq 6$
   $x + y \geq 4$
   $x + 3y \geq 6$
   $x \geq 0,\ y \geq 0$
   $m = 200x + 160y$

   The objective function m is to be minimized.

   (a) Solve the problem geometrically.

   (b) Set up the initial simplex tableau.

   (c) Solve the problem using the Simplex Method.

   Obviously, the minimum values you get for m using the two different methods should be in agreement.

   Solution:

   To solve geometrically, we sketch the feasible set. The vertices are:
   $(0, 6)$: the y-intercept of the line $3x + y = 6$
   $(1, 3)$: the intersection of the lines $3x + y = 6$ and $x + y = 4$
   $(3, 1)$: the intersection of the lines $x + y = 4$ and $x + 3y = 6$
   $(6, 0)$: the x=intercept of the line $x + 3y = 6$

   The lines $x + 3y = 6$ and $3x + y = 6$ intersect at $(2/3, 2/3)$, which does not satisfy the constraint $x + y \geq 4$ and thus is not in the feasible set and is not a vertex.

   At $(0, 6)$, $m = 960$
   At $(1, 3)$, $m = 680$
   At $(3, 1)$, $m = 760$
   At $(6, 0)$, $m = 1200$

   The minimum value $m = 680$ obviously occurs when $x = 1$, $y = 3$.

   To use the Simplex Method, we multiply both sides of each inequality by -1 to get them in the proper form before introducing slack variables:

   $-3x - y \leq -6$
   $-x - y \leq -4$
   $-x - 3y \leq -6$

   We also let $M = -m = -200x - 160y$, so we may write $200x + 160y + M = 0$.

   We thus set up the following initial simplex tableau:

   $$\begin{pmatrix} -3 & -1 & 1 & 0 & 0 & -6 \\ -1 & -1 & 0 & 1 & 0 & -4 \\ -1 & -3 & 0 & 0 & 1 & -6 \\ 200 & 160 & 0 & 0 & 0 & 0 \end{pmatrix}$$

   We first have to get rid of the negative entries in the last column. Since, aside from the bottom row, each entry in the last column is negative, we can look for a negative entry in any of the first three rows. Since there are negative entries in each of the first two columns in each of those rows, we can choose either of the first two columns as the pivot column. We'll (arbitrarily) choose to pivot using the first column. Looking at the appropriate quotients, the smallest positive quotient is $\frac{-6}{-3}$, so we pivot about the first row.

$$R_1 \leftarrow R_1/(-3): \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 2 \\ -1 & -1 & 0 & 1 & 0 & -4 \\ -1 & -3 & 0 & 0 & 1 & -6 \\ 200 & 160 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{r} R_2 \leftarrow R_2 + R_1 \\ R_3 \leftarrow R_3 + R_1 \\ R_4 \leftarrow R_4 - 200 * R_1 \end{array} : R_1 \leftarrow R_1/(-3): \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 2 \\ 0 & -\frac{2}{3} & -\frac{1}{3} & 1 & 0 & -2 \\ 0 & -\frac{8}{3} & -\frac{1}{3} & 0 & 1 & -4 \\ 0 & \frac{280}{3} & \frac{200}{3} & 0 & 0 & -400 \end{pmatrix}$$

We still have negative entries in the last column, so we need at least one more preliminary pivot. We can use either the second or third columns. We'll try the third column, recognizing that will ensure eliminating fractions in the pivot row.

For the third column, the smallest positive quotient is $\frac{-2}{-1/3}$, so we pivot about the second row.

$$R_2 \leftarrow -3R_2: \begin{pmatrix} 1 & \frac{1}{3} & -\frac{1}{3} & 0 & 0 & 2 \\ 0 & 2 & 1 & -3 & 0 & 6 \\ 0 & -\frac{8}{3} & -\frac{1}{3} & 0 & 1 & -4 \\ 0 & \frac{280}{3} & \frac{200}{3} & 0 & 0 & -400 \end{pmatrix}$$

$$\begin{array}{r} R_1 \leftarrow R_1 + R_2/3 \\ R_3 \leftarrow R_3 + R_2/3 \\ R_4 \leftarrow R_4 - \frac{200}{3} R_2 \end{array} : \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 4 \\ 0 & 2 & 1 & -3 & 0 & 6 \\ 0 & -2 & 0 & -1 & 1 & -2 \\ 0 & -40 & 0 & 200 & 0 & -800 \end{pmatrix}$$

We still have a negative entry in the last column, so we still need another preliminary pivot. If we choose the fourth column, we will obviously pivot on the third row and not introduce and fractions, so we'll do that.

$$R_3 \leftarrow -R_3: \begin{pmatrix} 1 & 1 & 0 & -1 & 0 & 4 \\ 0 & 2 & 1 & -3 & 0 & 6 \\ 0 & 2 & 0 & 1 & -1 & 2 \\ 0 & -40 & 0 & 200 & 0 & -800 \end{pmatrix}$$

$$\begin{array}{r} R_1 \leftarrow R_1 + R_3 \\ R_2 \leftarrow R_2 + 3R_3 \\ R_4 \leftarrow R_4 - 200R_3 \end{array} : \begin{pmatrix} 1 & 3 & 0 & 0 & -1 & 6 \\ 0 & 8 & 1 & 0 & -3 & 12 \\ 0 & 2 & 0 & 1 & -1 & 2 \\ 0 & -440 & 0 & 0 & 200 & -1200 \end{pmatrix}$$

Note this corresponds to the feasible solution $x = 6$, $y = 0$, $m = 1200$. There is a negative entry in the bottom row, so we need to pivot on the second column. The smallest quotient involves the third row, so we pivot on the third row, second column.

$$R_3 \leftarrow R_3/2: \begin{pmatrix} 1 & 3 & 0 & 0 & -1 & 6 \\ 0 & 8 & 1 & 0 & -3 & 12 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & -440 & 0 & 0 & 200 & -1200 \end{pmatrix}$$

$$\begin{array}{r} R_1 \leftarrow R_1 - 3R_3 \\ R_2 \leftarrow R_2 - 8R_3 \\ R_4 \leftarrow R_4 + 440R_3 \end{array} : \begin{pmatrix} 1 & 0 & 0 & -\frac{3}{2} & \frac{1}{2} & 3 \\ 0 & 0 & 1 & -4 & 1 & 4 \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & 220 & -20 & -760 \end{pmatrix}$$

This corresponds to the feasible solution $x = 3$, $y = 1$, $m = 760$, which is still not optimal as we see since there is a negative entry in the bottom row, fifth column, so we pivot about the fifth column. Looking at the positive quotients, we see we need to pivot about the second row, fifth column. Conveniently, there's already a 1 in that place.

$$\begin{array}{l} R_1 \leftarrow R_1 - R_2/2 \\ R_3 \leftarrow R_3 + R_2/2: \\ R_4 \leftarrow R_4 + 20R_2 \end{array} \begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 & -4 & 1 & 4 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{2} & 0 & 3 \\ 0 & 0 & 20 & 140 & 0 & -680 \end{pmatrix}$$

With no negative entries in the bottom row except for the last column, we are finished and obtain the optimal solution $x = 1$, $y = 3$, $m = 680$. *Note this is the same solution we obtained, albeit much more quickly, geometrically.*

For each of the following exercises, assume an annual interest rate of 5 percent, compounded monthly. For each question, clearly show any formulas you use and clearly define any variables.

4. What is the effective annual yield of any funds invested?

   Solution: We may use the formula $\text{EAY} = (1+r/n)^n - 1$, where r is tha annual rate, n is the number of times interest is compounded annually and EAY is the effective annual yield. Since $r = 0.05$ and $n = 12$ (since interest is compounded monthly), we have:

   $\text{EAY} = (1{+}0.05/12)^{12}{-}1 \approx 0.05116189792$, so the effective annual yield is approximately 5.116189792%.

5. What is the future value, in twenty years, of \$500?

   Solution: We may use the formula $F = P(1 + r/n)^{nt}$, where r and n are as before, t is the amount of time, in years (in this case, 20), P is the initial balance (in the case 500) and F is the future value. Thus, we have

   $F = 500(1 + 0.05/12)^{12\cdot 20} \approx 1,356.32014382$, so the future value will be \$1,356.32.

6. You are entitled to receive a lump sum of \$40,000 in ten years. What is its present value?

   Solution: This is similar to the previous question, except the future value $F = 40,000$, the time $t = 10$ and we need to find the present value P.

   Since $F = P(1 + r/n)^{nt}$, we get $40000 = P(1 + 0.05/12)^{12\cdot 10}$, $P = \dfrac{40000}{(1 + 0.05/12)^{12\cdot 10}} \approx 24,286.4416022$, so the present value is \$24,286.44.

7. If you save \$200 each month for twenty years, what will the balance be at the end of twenty years?

   Solution: We may use the annuity formula $F = s_{\overline{n}|i}R$, where F is the future value, R is the monthly payment or rent (in this case 200), n is the number of payments (in this case 240, since there are 12 payments per year for 20 years) and i is the interest rate applied each time interest is compounded, in this case 0.05/12, since there is an annual rate of 5 percent but interest is compounded twelve times per year. Also, $s_{\overline{n}|i} = \dfrac{(1+i)^n - 1}{i}$.

   Thus, $F = \dfrac{(1 + 0.05/12)^{240} - 1}{0.05/12} \cdot 200 \approx 82,206.7338068$, so the balance after twenty years will be \$82,206.73.

8. If you save \$200 each month for twenty years and then leave the balance alone for ten more years, what will the balance be at the end of thirty years?

   Solution: We may take the balance obtained after twenty years in the previous question and use the *Compound Interest Formula* to determine what it will grow to in another ten years. The *Compound Interest Formula* yields $82,206.73(1 + 0.05/12)$