Mathematics 105 <u>SOLUTIONS</u>

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December 12, 2007 Final Examination

1. Use Gaussian Elimination to solve the following system of equations.

$$
\begin{aligned}
5x + 3y &= 5 \\
3x + 4y &= 14
\end{aligned}
$$

Solution: $\begin{pmatrix} 5 & 3 & 5 \\ 3 & 4 & 14 \end{pmatrix}$ $\quad R_1 \leftarrow R_1/5$: $\begin{pmatrix} 1 & 3/5 & 1 \\ 3 & 4 & 14 \end{pmatrix}$

$R_2 \leftarrow R_2 - 3R_1$: $\begin{pmatrix} 1 & 3/5 & 1 \\ 0 & 11/5 & 11 \end{pmatrix}$ $\quad R_2 \leftarrow (5/11)R_2$: $\begin{pmatrix} 1 & 3/5 & 1 \\ 0 & 1 & 5 \end{pmatrix}$

$R_1 \leftarrow R_1 - (3/5)R_2$: $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 5 \end{pmatrix}$ $\quad$ We conclude $x = -2$, $y = 5$.

2. Consider the following linear programming problem. Set up appropriate variables and set up the linear programming problem in standard form with constraints, non-negativity conditions and an objective function. State whether the objective function is to be minimized or maximized. *Do not solve the linear programming problem.*

> The Yukon Mining Company operates two mines for the purpose of extracting gold, silver and copper. One mine, in the Yukon, costs $17,000 per day to operate and yields 50 ounces of gold, 3,500 ounces of silver and 7,000 ounces of copper each day. The other mine, in the Napa Valley, costs $15,000 per day to operate and yields 70 ounces of gold, 2,000 ounces of silver and 6,500 ounces of copper each day. The company has contracts requiring it to deliver at least 700 ounces of gold, 18,000 ounces of silver and 30,000 ounces of copper. How many days should it operate each mine in order to meet its contracts while minimizing its costs?

Solution: Let x be the number of days the Yukon mine is operated and y the number of days the Napa Valley mine is operated. Let c be the cost of operating the mines.

We get:

$50x + 70y \geq 700$

$3,500x + 2,000y \geq 18,000$

$7,000x + 6,500y \geq 30,000$

$x \geq 0$, $y \geq 0$

$c = 17,000x + 15,000y$

3. Consider the following linear programming problem:

   Constraints:

$$\begin{aligned} 3x + 5y &\leq 78 \\ 4x + y &\leq 56 \end{aligned}$$

   Non-negativity conditions: $x \geq 0$, $y \geq 0$

   The objective function $M = 5x + 4y$ is to be maximized.

   Set up the initial simplex tableau and use the Simplex Method to maximize the objective function.

   Solution: We have the initial simplex tableau: $\begin{pmatrix} 3 & 5 & 1 & 0 & 78 \\ 4 & 1 & 0 & 1 & 56 \\ -5 & -4 & 0 & 0 & 0 \end{pmatrix}$.

   Using the Simplex Method: $R_2 \leftarrow R_2/4$: $\begin{pmatrix} 3 & 5 & 1 & 0 & 78 \\ 1 & 1/4 & 0 & 1/4 & 14 \\ -5 & -4 & 0 & 0 & 0 \end{pmatrix}$

   $R_1 \leftarrow R_1 - 3R_2$, $R_3 \leftarrow R_3 + 5R_2$: $\begin{pmatrix} 0 & 17/4 & 1 & -3/4 & 36 \\ 1 & 1/4 & 0 & 1/4 & 14 \\ 0 & -11/4 & 0 & 5/4 & 70 \end{pmatrix}$

   $R_1 \leftarrow (4/17)R_1$: $\begin{pmatrix} 0 & 1 & 4/17 & -3/17 & 144/17 \\ 1 & 1/4 & 0 & 1/4 & 14 \\ 0 & -11/4 & 0 & 5/4 & 70 \end{pmatrix}$

   $R_2 \leftarrow R_2 - (1/4)R_1$, $R_3 \leftarrow R_3 + (11/4)R_1$: $\begin{pmatrix} 0 & 1 & 4/17 & -3/17 & 144/17 \\ 1 & 0 & -1/17 & 5/17 & 202/17 \\ 0 & 0 & 11/17 & 13/17 & 1586/17 \end{pmatrix}$

   We conclude the maximum occurs when $x = 202/17$ and $y = 144/17$ and the maximum is $1586/17$.

4. What will the balance be after 35 years if \$650 is placed in an account paying interest at an annual rate of 4.8%, compounded quarterly?

   Solution: $650(1 + 0.048/4)^{4 \cdot 35} = 650 \cdot 1.012^{140}$

5. What will the balance be after 35 years if \$650 is deposited monthly in an account paying interest at an annual rate of 4.8%, compounded monthly?

   Solution: $650 s_{\overline{420}|0.048/12} = 650 s_{\overline{420}|0.004} = 650 \cdot \dfrac{1.004^{420} - 1}{0.004}$

6. What is the monthly payment on an $8,000 loan being repaid over five years if the annual interest rate is 7%?

   Solution: $\dfrac{8,000}{a_{\overline{60}|0.07/12}} = \dfrac{8,000}{\frac{1-(1+0.07/12)^{-60}}{0.07/12}} = \dfrac{8,000 \cdot 0.07}{12(1-(1+0.07/12)^{-60})}$

7. Calculate $P(7,4)$.

   Solution: $P(7,4) = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$

8. Calculate $C(7,4)$.

   Solution: $C(7,4) = \dfrac{7!}{4!3!} = \dfrac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 35.$

9. How many license plates are possible if each license plate consists of three letters, followed by two digits and ending with either a letter or a digit?

   Solution: $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 36 = 10^2 \cdot 26^3 \cdot 36$

10. Quality control chooses four phones to test from a production run of 200 phones. How many ways can this be done?

    Solution: $C(200,4) = \dfrac{200!}{4!196!}$

11. In how many ways can eight commercials be scheduled into the eight slots available during a half hour television show?

    Solution: $P(8,8) = 8! = 40,320$

12. A bridge hand consists of thirteen cards. How many different bridge hands are there?

    Solution: $C(52,13) = \dfrac{52!}{13!39!}$

13. How many different bridge hands contain exactly six hearts and exactly five diamonds?

    Solution: $C(13,6)C(13,5)C(26,2)$

14. What is the probability of being dealt a bridge hand containing exactly six hearts and exactly five diamonds?

    Solution: $\dfrac{C(13,6)C(13,5)C(26,2)}{C(52,13)}$

15. Find the mathematical expectation $E(X)$ for a random variable X with the following probability distribution:

k	Pr(X=k)
2	0.5
4	0.3
5	0.2

    Solution: $E(X) = 2 \cdot 0.5 + 4 \cdot 0.3 + 5 \cdot 0.2 = 1 + 1.2 + 1 = 3.2$