Mathematics 105 SOLUTIONS
Professor Alan H. Stein
October 31, 2007

1. For most maximum problems, the contraints are in the form "$l(x) \leq k$," where $l(x)$ is a linear polynomial and k is a positive constant. Explain how one modifies setting up the initial simplex tableau to deal with constraints of the form "$l(x) \geq k$."

   Solution: One multiplies both sides of the inequality by -1 to get $-l(x) \leq -k$, which is in the $\leq$ form. One then introduces a slack variable just as if the inequality started out in that form.

2. Explain how, in the "Simplex Method," one chooses where to pivot when the simplex tablaux has no negative entries in the right hand column but does have negative entries in the bottom row, other than in the lower right hand corner.

   Solution: From among the negative entries in the bottom row, one chooses the smallest entry, that is, the negative entry whose absolute value is largest. The column containing that entry is the pivot column.

   For each row where it can be calculated and the quotient is positive, one then checks the quotient of the entry in the last column divided by the entry in the pivot column. The row for which that quotient is the smallest is the pivot row.

3. Suppose a pound of meat contains one unit of carbohydrates, three units of vitamins, twelve units of proteins and costs \$6.50, while a pound of delicious cabbage contains three units of carbohydrates, four units of vitamins and just one unit of protein while costing a mere \$1.50. You are allergic to all other foods but must consume at least eight units of carbohydrates, nineteen units of vitamins and seven units of protein each day. You are also strapped for cash and need to spend as little as possible on food. *Clearly and unambiguously* define appropriate variables and set up the necessary contraints, non-negativity conditions and objective function.

   Solution: Let: x be the number of pounds of meat consumed daily and let y be the number of pounds of cabbage. Let C be the total cost.

   The number of units of carbohydrates will equal the number of units of carbohydrates per pound of meat times the number of pounds of meat plus the number of units of carbohydrates of per pound of cabbage times the number of pounds of carbohydrates. Thus the number of units of carbohydrates is $x + 3y$. Since we need at least 8 units of carbohydrates, we have the constraint $x + 3y \geq 8$.

   Similarly, since we need at least 19 units of vitamins, we get a constraint $3x + 4y \geq 19$ and since we need at least 7 units of protein we have a constraint $12x + y \geq 7$.

   Since we cannot eat a negative amount of either meat or cabbage, no matter how much we might like to, we have the non-negativity conditions $x \geq 0$, $y \geq 0$.

   The cost of x pounds of meat is obviously $6.5x$ while the cost of y pounds of cabbage is $1.5y$, so we have $C = 6.5x + 1.5y$.

   We thus get the following set of constraints, along with the non-negativity conditions and formula for the objective function:

   $$x + 3y \geq 8$$
   $$3x + 4y \geq 19$$
   $$12x + y \geq 7$$
   $$x \geq 0,\ y \geq 0$$
   $$C = 6.5x + 1.5y$$

4. Consider the linear programming problem with the following set of contraints and objective function. Find both the minimum and maximum values of the objective function subject to the constraints and non-negativity.

   $$5x + 2y \geq 40$$
   $$x + 2y \geq 20$$
   $$x \geq 0$$
   $$y \geq 3$$
   $$p = 2x + 10y$$

   Solution: There clearly is no maximum, since any point (x, y) where $x \geq 0$ and $y \geq 20$ is clearly in the feasible set and $2x + 10y$ can be made arbitrarily large by making y arbitrarily large.

   The minimum must occur at one of the vertices of the feasible set. The vertices are the following:

   $(0, 20)$, where the line $5x + 2y = 40$ intersects the y-axis.

   The point where $5x + 2y = 40$ and $x + 2y = 20$ intersect. Using elimination, we immediately get $4x = 20$, $x = 5$, so $2y = 15$ and $y = \frac{15}{2}$. So we have the vertex $(5, 15/2)$.

   The point where $x + 2y = 20$ meets $y = 3$. We have $x + 6 = 20$, $x = 14$, so we have the vertex $(14, 3)$.

   At $(0, 20)$, $p = 2 \cdot 0 + 10 \cdot 20 = 200$.

   At $(5, 15/2)$, $p = 2 \cdot 5 + 10 \cdot 15/2 = 85$.

   At $(14, 3)$, $p = 2 \cdot 14 + 10 \cdot 3 = 58$.

   The minimum value is obviously 58 and it occurs when $x = 14$ and $y = 3$.

5. Consider the linear programming problem with the following set of contraints, non-negativity conditions and objective function, which is to be maximized. Set up the initial simplex tableau.

   $$5x + 3y + 8z \leq 10$$
   $$9x - 2y + 4z \leq 5$$
   $$x \geq 0,\ y \geq 0$$
   $$p = 4x + 3y + 12z$$

   Solution: The initial simplex tableau is:

   $$\begin{pmatrix} 5 & 3 & 8 & 1 & 0 & 10 \\ 9 & -2 & 4 & 0 & 1 & 5 \\ -4 & -3 & -12 & 0 & 0 & 0 \end{pmatrix}$$

6. Consider the linear programming problem with the following set of contraints, non-negativity conditions and objective function, which is to be minimized. Set up the initial simplex tableau.

   $5x + 3y + 8z \geq 10$

   $9x - 2y + 4z \geq 5$

   $x \geq 0,\ y \geq 0$

   $p = 4x + 3y + 12z$

   Solution: We multiply both sides of each constraint by -1 to get:

   $-5x - 3y - 8z \leq -10$

   $-9x + 2y - 4z \leq -5$

   We also let $M = -p = -4x - 3y - 12z$, so $4x + 3y + 12z + M = 0$ and we can then write down the initial simplex tableau:

   $$\begin{pmatrix} -5 & -3 & -8 & 1 & 0 & -10 \\ -9 & 2 & -4 & 0 & 1 & -5 \\ 4 & 3 & 12 & 0 & 0 & 0 \end{pmatrix}.$$

7. Explain how one begins using the Simplex Method when there are negative entries in the rightmost column, excluding the lower righthand corner.

   Solution: We pick a row with a negative entry in the rightmost column and then choose another column in that row which contains a negative entry. The column containing that negative entry becomes the pivot column. We then choose the pivot row in the usual way and pivot. We keep doing this until there are no more negative entries in the rightmost column.

8. Consider the following initial simplex tableau.

$$\begin{pmatrix} 5 & 2 & 1 & 0 & 0 & 8 \\ -2 & -4 & 0 & 1 & 0 & -10 \\ 8 & 1 & 0 & 0 & 1 & 3 \\ 2 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Do the necessary pivoting to eliminate the negative entry from the last column.

Solution: Since there's a negative entry in the last column of the second row, we look for another negative entry in that row. Either the first or second column could be used. We'll choose the second column.

Looking at the quotients of entries in the last column divided by entries in the second (pivot) column, the smallest comes from the second row, so we pivot about the second row, second column. Since the pivot entry is -4, we start by dividing the second row by -4:

$$R_2 \leftarrow R_2/(-4)\colon \begin{pmatrix} 5 & 2 & 1 & 0 & 0 & 8 \\ \frac{1}{2} & 1 & 0 & -\frac{1}{4} & 0 & \frac{5}{2} \\ 8 & 1 & 0 & 0 & 1 & 3 \\ 2 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We now get 0's elsewhere in the second column:

$$R_1 \leftarrow R_1 - 2R_2\colon \begin{pmatrix} 4 & 0 & 1 & \frac{1}{2} & 0 & 3 \\ \frac{1}{2} & 1 & 0 & -\frac{1}{4} & 0 & \frac{5}{2} \\ 8 & 1 & 0 & 0 & 1 & 3 \\ 2 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_3 \leftarrow R_3 - R_2\colon \begin{pmatrix} 4 & 0 & 1 & \frac{1}{2} & 0 & 3 \\ \frac{1}{2} & 1 & 0 & -\frac{1}{4} & 0 & \frac{5}{2} \\ \frac{15}{2} & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{2} \\ 2 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_4 \leftarrow R_4 + R_2\colon \begin{pmatrix} 4 & 0 & 1 & \frac{1}{2} & 0 & 3 \\ \frac{1}{2} & 1 & 0 & -\frac{1}{4} & 0 & \frac{5}{2} \\ \frac{15}{2} & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{2} \\ \frac{5}{2} & 0 & 0 & -\frac{1}{4} & 0 & \frac{5}{2} \end{pmatrix}$$

If one chooses to first pivot using the first column, one first pivots about the third row, first column and then must pivot a second time, about the second row, second column. One winds up with the following tableau.

$$\begin{pmatrix} 0 & 0 & 1 & \frac{11}{30} & -\frac{8}{15} & \frac{31}{10} \\ 0 & 1 & 0 & -\frac{4}{15} & -\frac{1}{15} & \frac{11}{5} \\ 1 & 0 & 0 & \frac{1}{30} & \frac{2}{15} & \frac{1}{10} \\ 0 & 0 & 0 & -\frac{1}{3} & -\frac{4}{3} & 2 \end{pmatrix}$$

9. Consider the linear programming problem with the following initial simplex tableau.

$$\begin{pmatrix} 1 & 3 & 1 & 0 & 15 \\ 4 & 1 & 0 & 1 & 16 \\ -3 & 4 & 0 & 0 & 0 \end{pmatrix}$$

Use the Simplex Method to find the maximum value for the objective function. Assuming the variables are x, y and the objective function is p, find the given the maximum value for p along with the feasible values of x and y which yield that value for p.

Solution: Since the only negative entry in the bottom row is in the first column, we pivot on the first column. Looking at the quotients of the entries in the last column divided by the corresponding entries in the first column, the smallest is $\frac{16}{4} = 4$, so we pivot about the second row. We proceed as follows:

$R_2 \leftarrow R_2/4$: $\begin{pmatrix} 1 & 3 & 1 & 0 & 15 \\ 1 & \frac{1}{4} & 0 & \frac{1}{4} & 4 \\ -3 & 4 & 0 & 0 & 0 \end{pmatrix}$

$R_1 \leftarrow R_1 - R_2$: $\begin{pmatrix} 0 & \frac{11}{4} & 1 & -\frac{1}{4} & 11 \\ 1 & \frac{1}{4} & 0 & \frac{1}{4} & 4 \\ -3 & 4 & 0 & 0 & 0 \end{pmatrix}$

$R_3 \leftarrow R_3 + 3R_2$: $\begin{pmatrix} 0 & \frac{11}{4} & 1 & -\frac{1}{4} & 11 \\ 1 & \frac{1}{4} & 0 & \frac{1}{4} & 4 \\ 0 & \frac{19}{4} & 0 & \frac{3}{4} & 12 \end{pmatrix}$

There are no negative entries in the bottom row, so we are finished pivoting. y is still a Group 1 variable, so we have $y = 0$, but x is a Group 2 variable and, from the second row, we get $x = 4$. The entry in the bottom right hand corner is 12, so we conclude the maximum value of the objective function is 12 and it occurs when $x = 4$ and $y = 0$.