Mathematics 105 SOLUTIONS
Professor Alan H. Stein
December 13, 2006 Final Examination

1. Consider the arithmetic expression $\dfrac{2.1^{5\cdot 1.3-8.1}+22}{2.4\cdot 3.1-2.6^2}$.

   (a) Use your calculator to calculate the expression. Have your calculator do all the calculations, without doing any mental arithmetic, without using the memory in the calculator, without pressing the enter key except at the very and, and without doing any intermediate calculations that need to be reentered.

   Solution: 32.801624791

   (b) Write down the sequence of keys you pressed in order to do the calculation.

   For example, on many calculators, one might calculate $\dfrac{2+3^{5\cdot 2}}{7}$ *by pressing "(2+3∧(5 × 2)) ÷ 7 enter".*

   Solution: $(2.1 \wedge (5 \times 1.3 - 8.1) + 22) \div (2.4 \times 3.1 - 2.6 \wedge (2))$ enter

(4-5): Consider the following system of equations:

$$\begin{aligned} x + 2y + 3z &= 19 \\ 2x + 3y + 4z &= 28 \\ 5x - y + 2z &= 21 \end{aligned}$$

4. Set up the augmented matrix for the system.

   Solution: $\begin{pmatrix} 1 & 2 & 3 & 19 \\ 2 & 3 & 4 & 28 \\ 5 & -1 & 2 & 21 \end{pmatrix}$

5. Use Gaussian elimination to solve the system.

   Solution: $\begin{pmatrix} 1 & 2 & 3 & 19 \\ 2 & 3 & 4 & 28 \\ 5 & -1 & 2 & 21 \end{pmatrix}$

   $R_2 - 2R_1 \to R_2$, $R_3 - 5R_1 \to R_3$:

   $$\begin{pmatrix} 1 & 2 & 3 & 19 \\ 0 & -1 & -2 & -10 \\ 0 & -11 & -13 & -74 \end{pmatrix}$$

   $-R_2 \to R_2$:

   $$\begin{pmatrix} 1 & 2 & 3 & 19 \\ 0 & 1 & 2 & 10 \\ 0 & -11 & -13 & -74 \end{pmatrix}$$

   $R_1 - 2R_2 \to R_1$, $R_3 + 11R_2 \to R_3$:

   $$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 9 & 36 \end{pmatrix}$$

   $R_3/9 \to R_3$:

   $$\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

   $R_1 + R_3 \to R_1$, $R_2 - 2R_3 \to R_2$:

   $$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

   So the solution is $x = 3$, $y = 2$, $z = 4$.

6. Consider the following linear programming problem:

> A bank grants mortgages and makes business loans. The bank has \$50 million available to invest in this way. Suppose state banking laws require that the bank invest at least three times as much money in mortgages as in business loans. If the bank charges 8% interest on mortgages and 10% on business loans, how much should it invest in each in order to earn as much as possible from its investments?

Introduce appropriate variables and set up the constraints, non-negativity conditions and objective function in standard form.

Solution: Let x be the amount invested in mortgages, let y the amount invested in business loans and let p be the amount of interest it earns annually. (Let the units be in millions of dollars.)

Since the bank has \$50 million to invest, $x + y \leq 50$.

Since the amount it invests in mortgages is at least three times the amount it invests in business loans, $x \geq 3y$, or $-x + 3y \leq 0$.

It will earn $0.08x + 0.10y$, so $p = 0.08x + 0.1y$.

We may thus express the linear programming problem in the following form:

$$\begin{aligned} x + y &\leq 50 \\ -x + 3y &\leq 0 \\ x \geq 0,\ y &\geq 0 \\ p &= 0.08x + 0.1y \end{aligned}$$

(7-9): Consider the following linear programming problem:

$$\begin{aligned} x + 2y &\leq 16 \\ x + y &\leq 10 \\ x \geq 0,\, y &\geq 0 \\ p = 2x &+ 3y, \end{aligned}$$

where the objective function p is to be maximized.

7. Solve the problem geometrically.

   Solution: The line $x + 2y = 16$ intersects the axes at $x = 16$ and $y = 8$, while the line $x + y = 10$ intersects the axes at $x = 10$ and $y = 10$, so the points $(10, 0)$ and $(0, 8)$ are vertices of the feasible set along with the origin $(0, 0)$. The other vertext will be the point where the lines $x + 2y = 16$ and $x + y = 10$ intersect.

   From the second equation, we get $y = 10 - x$. Plugging that into the first equation yields $x + 2(10 - x) = 16$, $20 - x = 16$, $x = 4$. Since $y = 10 - x = 6$, the fourth vertex is $(4, 6)$.

   Evaluating the objective function at the vertices:

(x, y)	$p = 2x + 3y$
$(0, 0)$	0
$(10, 0)$	20
$(0, 8)$	24
$(4, 6)$	26

   .

   We thus find the maximum value for the objective function is 26 and it occurs when $x = 4$ and $y = 6$.

8. Set up the initial simplex tableau for the problem.

   Solution: $\begin{pmatrix} 1 & 2 & 1 & 0 & 16 \\ 1 & 1 & 0 & 1 & 10 \\ -2 & -3 & 0 & 0 & 0 \end{pmatrix}$

9. Solve the problem using the Simplex Method.

   Solution: Since $|-3| > |-2|$, we pivot on the second column. Since $\frac{16}{2} < \frac{10}{1}$, we pivot on the first row.

   $R_1/2 \to R_1$:

   $$\begin{pmatrix} 1/2 & 1 & 1/2 & 0 & 8 \\ 1 & 1 & 0 & 1 & 10 \\ -2 & -3 & 0 & 0 & 0 \end{pmatrix}$$

   $R_2 - R_1 \to R_2$, $R3 + 3R_1 \to R_3$:

   $$\begin{pmatrix} 1/2 & 1 & 1/2 & 0 & 8 \\ 1/2 & 0 & -1/2 & 1 & 2 \\ -1/2 & 0 & 3/2 & 0 & 24 \end{pmatrix}$$

   Since there's a negative number in the first column of the bottom row, we pivot on the first column. Since $\frac{2}{1/2} < \frac{8}{1/2}$, we pivot on the second row.

   $2R_2 \to R_2$:

   $$\begin{pmatrix} 1/2 & 1 & 1/2 & 0 & 8 \\ 1 & 0 & -1 & 2 & 4 \\ -1/2 & 0 & 3/2 & 0 & 24 \end{pmatrix}$$

   $R_1 - R_2/2 \to R_1$, $R_3 + R_1/2 \to R_3$:

   $$\begin{pmatrix} 0 & 1 & 1 & -1 & 6 \\ 1 & 0 & -1 & 2 & 4 \\ 0 & 0 & 1 & 1 & 26 \end{pmatrix}$$

   We conclude the maximum value of the objective function is 26 and it occurs when $x = 4$ and $y = 6$.

10. Find the mathematical expectation $E(X)$ for a random variable X with the following probability distribution:

k	P(X=k)
0	0.2
5	0.3
7	0.4
8	0.1

Solution: $E(X) = 0 \cdot 0.2 + 5 \cdot 0.3 + 7 \cdot 0.4 + 8 \cdot 0.1 = 5.1$

11. List all combinations of three elements chosen from the set {business, english, history, math, sports}.

Solution:

{business, english, history}
{business, english, math}
{business, english, sports}
{business, history, math}
{business, history, sports}
{business, math, sports}
{english, history, math}
{english, history, sports}
{english, math, sports}
{history, math, sports}

12. How many seven card rummy hands are possible? *A standard deck has fifty-two cards.*

Solution: $C(52, 7) = \frac{52!}{7!45!} = 133,784,560$, although the arithmetic doesn't have to be worked out.

13. Calculate $C(7, 4)$.

Solution: $C(7, 4) = \frac{7!}{4!3!} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35.$

14. Calculate $P(7, 4)$.

Solution: $P(7, 4) = \frac{7!}{(7-4)!} = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$