Mathematics 105 — SOLUTIONS
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1. Consider the arithmetic expression $\dfrac{2.3^{5\cdot 1.4-8.7}+23}{2.5\cdot 3.1-2.8^2}$.

   (a) Use your calculator to calculate the expression. Have your calculator do all the calculations, without doing any mental arithmetic, without using the memory in the calculator, without pressing the enter key except at the very and, and without doing any intermediate calculations that need to be reentered.

   Solution: -258.252178151

   (b) Write down the sequence of keys you pressed in order to do the calculation.

   For example, on many calculators, one might calculate $\dfrac{2+3^{5\cdot 2}}{7}$ *by pressing "(2+3∧(5 × 2)) ÷ 7 enter".*

   Solution: $(2.3 \wedge (5\times 1.4-8.7)+23) \div (2.5\times 3.1-2.8\wedge(2))$ enter

2. \$8,000 is put into a retirement plan each year for thirty years. If interest is compounded at an annual rate of 4.3 percent, what is the balance immediately after the thirtieth deposit.

   Solution: We may use the formula $F = s_{\overline{n}|i}R$, where F is the future balance, n is the number of deposits, i is the interest rate applied with each deposit, and R is the amount of each deposit. We know $s_{\overline{n}|i} = \dfrac{(1+i)^n-1}{i}$.

   In this case, $R = 8000$, $n = 30$ and $i = 0.043$, so $F = \dfrac{(1+0.043)^{30}-1}{0.043}\cdot 8000 \approx 471,839.69986976002$, so the balance will be $\$471,839.70$.

3. Perform the elementary row operation $R_3 - 4R_1 \to R_3$ on the matrix $\begin{pmatrix} 5 & 3 & -2 \\ 4 & 6 & 5 \\ 2 & 1 & 4 \end{pmatrix}$. *In other words, subtract four times the first row from the third row.*

   Solution: $\begin{pmatrix} 5 & 3 & -2 \\ 4 & 6 & 5 \\ -18 & -11 & 12 \end{pmatrix}$

(4-5): Consider the following system of equations:

$$\begin{aligned} x + 2y + 3z &= 19 \\ 2x + 3y + 4z &= 27 \\ 5x - y + 2z &= 19 \end{aligned}$$

4. Set up the augmented matrix for the system.

   Solution: $\begin{pmatrix} 1 & 2 & 3 & 19 \\ 2 & 3 & 4 & 27 \\ 5 & -1 & 2 & 19 \end{pmatrix}$

5. Use Gaussian elimination to solve the system.

   Solution: $\begin{pmatrix} 1 & 2 & 3 & 19 \\ 0 & -1 & -2 & -11 \\ 0 & -11 & -13 & -76 \end{pmatrix}$

   $\begin{pmatrix} 1 & 2 & 3 & 19 \\ 0 & 1 & 2 & 11 \\ 0 & -11 & -13 & -76 \end{pmatrix}$

   $\begin{pmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 9 & 45 \end{pmatrix}$

   $\begin{pmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 1 & 5 \end{pmatrix}$

   $\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{pmatrix}$

   So the solution is $x = 2$, $y = 1$, $z = 5$.

6. Consider the following linear programming problem:

> A bank grants mortgages and makes business loans. The bank has \$50 million available to invest in this way. Suppose state banking laws require that the bank invest at least three times as much money in mortgages as in business loans. If the bank charges 8% interest on mortgages and 10% on business loans, how much should it invest in each in order to earn as much as possible from its investments?

Introduce appropriate variables and set up the constraints, non-negativity conditions and objective function in standard form.

Solution: Let x be the amount invested in mortgages, let y the amount invested in business loans and let p be the amount of interest it earns annually. (Let the units be in millions of dollars.)

Since the bank has \$50 million to invest, $x + y \leq 50$.

Since the amount it invests in mortgages is at least three times the amount it invests in business loans, $x \geq 3y$, or $-x + 3y \leq 0$.

It will earn $0.08x + 0.10y$, so $p = 0.08x + 0.1y$.

We may thus express the linear programming problem in the following form:

$$\begin{aligned} x + y &\leq 50 \\ -x + 3y &\leq 0 \\ x \geq 0,\ y &\geq 0 \\ p = 0.08x + 0.1&y \end{aligned}$$

(7-9): Consider the following linear programming problem:

$$
\begin{aligned}
2x + 3y &\le 24 \\
x + y &\le 10 \\
x \ge 0,\ y &\ge 0 \\
p = 2x &+ 3y,
\end{aligned}
$$

where the objective function p is to be maximized.

Solution:

7. Solve the problem geometrically.

Solution: The line $2x + 3y = 24$ intersects the axes where $x = 12$ and where $y = 8$, while the line $x + y = 10$ intersects the axes where $x = 10$ and where $y = 10$, so the points $(10, 0)$ and $(0, 8)$ are vertices of the feasible set, as is the origin $(0, 0)$. We need to find one more point, where the lines $2x + 3y = 24$ and $x + y = 10$ intersect.

From the second equation, $y = 10 - x$, which we may substitute into the first to get $2x + 3(10 - x) = 24$, $30 - x = 24$, $x = 6$. Thus $y = 10 - 6 = 4$, so the fourth vertex is $(6, 4)$.

(x, y)	$p = 2x + 3y$
$(0, 0)$	0
$(10, 0)$	20
$(0, 8)$	24
$(6, 4)$	24

The maximum occurs at both $(0, 8)$ and $(6, 4)$ and is equal to 24. In retrospect, that's not surprising, given the first constraint.

8. Set up the initial simplex tableau for the problem.

Solution: $\begin{pmatrix} 2 & 3 & 1 & 0 & 24 \\ 1 & 1 & 0 & 1 & 10 \\ -2 & -3 & 0 & 0 & 0 \end{pmatrix}$

9. Solve the problem using the Simplex Method.

Solution: Since $|-3| > |-2|$, we pivot using the second column and since $\frac{24}{3} < \frac{10}{1}$, we pivot on the first row.

$R_1/3 \to R_1$:

$$\begin{pmatrix} 2/3 & 1 & 1/3 & 0 & 8 \\ 1 & 1 & 0 & 1 & 10 \\ -2 & -3 & 0 & 0 & 0 \end{pmatrix}$$

$R_2 - R_1 \to R_2$, $R_3 + 3R_1 \to R_3$:

$$\begin{pmatrix} 2/3 & 1 & 1/3 & 0 & 8 \\ 1/3 & 0 & -1/3 & 1 & 2 \\ 0 & 0 & 1 & 0 & 24 \end{pmatrix}$$

Since there are no negative numbers in the bottom row, we do not have to pivot further and find the maximum value of the objective function is 24 and occurs when $x = 0$, $y = 8$.

10. Find the mathematical expectation $E(X)$ for a random variable X with the following probability distribution:

k	P(X=k)
0	0.3
5	0.2
7	0.4
8	0.1

Solution: $E(X) = 0 \cdot 0.3 + 5 \cdot 0.2 + 7 \cdot 0.4 + 8 \cdot 0.1 = 4.6$

11. List all combinations of three elements chosen from the set {business, english, history, math, sports}.

Solution:
{business, english, history}
{business, english, math}
{business, english, sports}
{business, history, math}
{business, history, sports}
{business, math, sports}
{english, history, math}
{english, history, sports}
{english, math, sports}
{history, math, sports}

12. How many ten card gin rummy hands are possible? *A standard deck has fifty-two cards.*

    Solution: $C(52, 10)$ *This comes out to* $\frac{52!}{10!42!} = 15,820,024,220$*, but it was unnecessary to do the calculation.*

13. Calculate $C(7, 2)$.

    Solution: $C(7, 2) = \frac{7!}{2!5!} = \frac{7 \cdot 6}{2 \cdot 1} = 21.$

14. Calculate $P(7, 2)$.

    Solution: $P(7, 2) = 7 \cdot 6 = 42.$