Mathematics 103 <u>Solutions</u>
Professor Alan H. Stein
June 9, 2004

Solutions are given for the objective questions involving computations. The other questions do not have one single correct conclusion so solutions are not given for them.

(1) In the election for First Selectman of Canterbury November 4, 2003, decided by the plurality method, the vote totals were as follows:

Neil Dupont	Republican	716
Robert J. Droesch	Democrat	308
Paul R. Santoro	PC	858

(a) Who won?
Solution: Paul R. Santoro

(b) Suppose the registrar had used preference rankings to determine the winner through a runoff between the top two finishers. How many and what percentage of Droesch's supporters would have had to have ranked Dupont second for Dupont to win the runoff?
Solution: There are a total of 1882 voters. To win the runoff, Dupont would have had to get a majority. Since $\frac{1}{2} \cdot 1882 = 941$, Dupont would have needed a total of 942 votes. Since he had 716 votes already, he would have needed an additional 226 votes from Droesh's supporters. This represents 73.3766% of Droesh's supporters.

(2) Consider an election among four candidates: Gail, Ricco, Shawn and Twanda. Suppose 1500 voters cast ballots on which they recorded their preferences as shown in the table below.

Place	390	360	300	450
1st	Shawn	Gail	Gail	Ricco
2nd	Twanda	Twanda	Twanda	Twanda
3rd	Ricco	Ricco	Shawn	Shawn
4th	Gail	Shawn	Ricco	Gail

(a) Who would win if the votes were counted using the plurality method?
Solution: Gail would win with 660 votes to 390 for Shawn and 450 for Ricco.

(b) Who would win if the votes were counted using the plurality method with a runoff, where after each round the candidate with the fewest votes was dropped until one candidate received a majority?
Solution: Twanda would have been dropped after the first ballot. Since nobody chose her first, the second ballot would also have come out with Gail getting 660 votes, Ricco coming in second with 450 votes and Shawn coming in third with 390 votes. Shawn would be dropped for the third ballot, leaving a runoff between Gail and Ricco.
Ricco would then get 390 votes that originally were cast for Shawn, giving him a total of 840 to Gail's 660, so Ricco would win.

(c) Who would win if the votes were counted using the plurality method with a runoff between the top two votegetters if no candidate received a majority?
Solution: This would come out the same way as if the candidate with the fewest votes was dropped after each ballot and Ricco would win.

(d) Who would win if Borda's System was used, with the first choice for each voter receiving 4 points, the second choice receiving 3 points, the third choice receiving 2 points and the last choice receiving 1 point?
Solution: The Borda Count would come out as follows:
Shawn: $390 \cdot 4 + 360 \cdot 1 + 300 \cdot 2 + 450 \cdot 2 = 3420$
Twanda: $1500 \cdot 3 = 4500$
Ricco: $390 \cdot 2 + 360 \cdot 2 + 300 \cdot 1 + 450 \cdot 4 = 3600$
Gail: $390 \cdot 1 + 360 \cdot 4 + 300 \cdot 4 + 450 \cdot 1 = 3480$
In this case, Twanda would win!

(e) Is there a Condorcet winner? If so, who is the Condorcet winner?
Solution: Twanda beats Gail 840 to 660, Shawn 1110 to 390 and Ricco 1050 to 450, so Twanda is a Condorcet winner.

(f) Which of the methods do you feel best reflects the will of the electorate? Explain why.

(3) Consider an election among three candidates, Adams, Barnes and Collins, where the winner is determined using the approval method. Suppose 50 voters approve of both Adams and Collins, 60 voters approve of both Adams and Barnes, 45 voters approve only of Adams, 55 voters approve of both Barnes and Collins, 40 voters approve only of Barnes and 50 voters approve only of Collins. Who wins the election?
Solution: Adams is approved by $40+60+45 = 145$ voters, Barnes by $60+55+40 = 155$ voters and Collins by $50+55+50 = 155$ voters, so we actually have a tie between Barnes and Collins!

(4) Recall that the Pareto Optimality Property requires that if all the voters prefer Candidate A to Candidate B, then the group choice should not prefer Candidate B to Candidate A. Explain why Borda's Method satisfies the Pareto Optimality Property.
Solution: Suppose we number the voters $1, 2, 3, \ldots, n$, let a_k correspond to the points Candidate A gets from the k^{th} voter and let

Congress consisting of 50 seats, and each of the 5 planets will be entitled to a number of seats that is proportional to its population. The population of each planet is:

Planet	Alanos	Betta	Conii	Dugos	Ellisium	Total
Pop (billions)	150	78	173	204	295	900

(a) Determine the *natural divisor*.
Solution: The natural divisor is the quotient of the population divided by the number of seats. This comes out to 900/50 = 18.

(b) Determine the *natural quota* for each state. *For these and other calculations, round each answer to four places after the decimal point.*
Solution: The natural quota for each state is the quotient of its population divided by the natural divisor. This comes out as follows:
Alamos: 150/18 = 8.3333
Betta: 78/18 = 4.3333
Conii: 173/18 = 9.6111
Dugos: 204/18 = 11.3333
Ellisium: 295/18 = 16.3889

(c) What goes wrong if we use simple rounding to determine the number of seats for each planet?
Solution: Alamos gets 8 seats, Betta 4, Conii 10, Dugos 11 and Ellisium 16, for a total of 49 seats, one seat short.

(d) Determine the apportionment using *Hamilton's Method*, where each state's natural quota is rounded down and then the leftover seats are assigned to the planets whose natural quotas have the largest fractional parts.
Solution: If the seats are rounded down, Alamos starts with 8 seats, Betta 4, Conii 9, Dugos 11 and Ellisium 16, for a total of 48, two seats short of 50. Conii and Ellisium have the largest fractional parts, so they each get another seat and the apportionment comes out: Alamos 8, Betta 4 Conii 10, Dugos 11, Ellisium 17.

(e) Determine the number of seats each planet would have if its natural quota was rounded down. How many seats short would the Intergalactic Congress be?
Solution: As shown above, there would be a total of 48 seats, two short of the desired number.

(f) Recall that in *Jefferson's Method*, a modified divisor is obtained so that when the modified quota for each *state* is rounded down, the *house* will have the desired number of seats. Find the *threshold divisor* needed to increase Conii's modified quota to 10 under *Jefferson's Method*.
Solution: $\frac{173}{10} = 17.3$.

(g) Determine the apportionment of seats using the *threshold divisor* determined in the previous part. Does this lead to the correct number of seats?
Solution: The apportionment comes out as follows:
Alamos: 150/17.3 = 8.6705, so Alamos gets 8 seats.
Betta: 78/17.3 = 4.5087, so Betta gets 4 seats.
Conii: 173/17.3 = 10, so Conii gets 10 seats.

Dugos: 204/17.3 = 11.7919, so Dugos gets 11 seats.
Ellisium: 295/17.3 = 17.0520, so Ellisium gets 17 seats.
This adds up to 50 seats, which is the correct number. *Indeed, this is the same as the apportionment comes out under Hamilton's Method.*

(h) Recall that in *Webster's Method*, a modified divisor is obtained so that when the modified quota for each *state* is rounded in the natural way, the correct number of seats is allocated. Find the *threshold divisor* needed to increase Ellisium's modified quota to 17 under *Webster's Method.*

Solution: $\frac{295}{16.5} = 17.8787$

Note: If we compute more places, we get 17.87787878788*, which rounds off to* 17.8788*, but we always need to have the modified divisor smaller than the threshold divisor.*

(7) Among the different apportionment methods discussed in class, which do you believe is the fairest? Explain why.