Mathematics of Finance

Exponents, Radicals and Logarithms

Definition 1. $x^n = x \cdot x \cdots x$ *for* n *a positive integer.*

Definition 2. $x^{-n} = \dfrac{1}{x^n}$

Definition 3. $\sqrt[n]{x}$ *is the number whose* n^{th} *power is* x*.*

Definition 4. $x^{1/n} = \sqrt[n]{x}$

Definition 5. $x^{m/n} = \sqrt[n]{x^m}$

Definition 6. $\log_b x$ *is the power* b *must be raised to in order to obtain* x*.*

Consequences: $\log_b(b^x) = x$, $b^{\log_b x} = x$.

Common Logs – Base 10

Natural Logs – Base e, denoted by ln

Properties of Logarithms

1. $\log(\alpha\beta) = \log\alpha + \log\beta$ – *The log of a product is the sum of the logs.*

2. $\log(\alpha/\beta) = \log\alpha - \log\beta$ – *The log of a quotient is the difference of the logs.*

3. $\log(\alpha^\beta) = \beta\log\alpha$ – *The log of a number raised to a power is the power times the log.*

Logarithms are useful in solving *exponential equations*. The key is to isolate the term with the variable in the exponent and then take logarithims of both sides.

Example: Solve $18 + 3^{5x} = 93$.

Solution: $3^{5x} = 75$, $\ln(3^{5x}) = \ln 75$, $5x\ln 3 = \ln 75$, $x = \dfrac{\ln 75}{5\ln 3}$.

Simple Interest

Notation:

$P =$ principal or present value

$r =$ annual interest rate

$I =$ amount of interest

$t =$ time (generally in years)

$F =$ balance or future value after t years

With *simple interest*, the amount of interest in one year is the product of the principal and the interest rate.

More generally, the amount of *simple interest* earned in a period of time is equal to the product of the principle, the interest rate and the amount of time.

$$I =$$

Compound Interest

Interest is computed periodically, added to the balance, and future interest is computed based on the updated balance. If interest is compounded n times per year, each time the amount of interest compounded will equal $\frac{1}{n}$ times the amount that would be compounded for an entire year. In other words, the amount of interest will be $\frac{1}{n} \cdot Pr$ and the new balance will be $P + \frac{1}{n} \cdot Pr = P\left(1 + \frac{r}{n}\right)$.

Effectively, the old balance is multiplied by $1 + \frac{r}{n}$ to get the new balance.

In t years, the original balance will be multiplied by $1+\frac{r}{n}$ nt times, once for every interest period. Effectively, it will be multiplied by $\left(1 + \frac{r}{n}\right)^{nt}$, so the future balance will be $F = P\left(1 + \frac{r}{n}\right)^{nt}$.

Compound Interest Formula

$$F = P\left(1+\frac{r}{n}\right)^{nt}$$

We may look at this as an equation involving five different variables, F, P, r, n, t, any of which may be found if the values of the others are known.

At different times, we may use this formula to find the future value, the original balance, the amount of time it will take for the balance to reach a certain amount or the annual interest rate.

Continuous Interest

If interest is compounded very frequently, which corresponds to n being very large, the balance one obtains does not change very much. One may see this by taking the *Compound Interest Formula* and manipulating it as follows:

$$F = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$F = P\left(1 + \frac{1}{n/r}\right)^{\frac{n}{r}\cdot rt}$$

$$F = P\left[\left(1 + \frac{1}{n/r}\right)^{\frac{n}{r}}\right]^{rt}$$

We may write this as

$$F = P\left[\left(1 + \frac{1}{N}\right)^{N}\right]^{rt}, \text{ where } N = \frac{n}{r}.$$

When n is very large, so is N, and $\left(1 + \frac{1}{N}\right)^{N}$ gets very close to the mathematical constant

e, which is an irrational number approximately 2.71828.

We thus find F will be close to Pe^{rt}.

The formula $F = Pe^{rt}$ is known as the *Continuous Interest Formula*.

Annual Percentage Yield (APY) or Effective Annual Yield

Definition 7 (Annual Percentage Yield). *The annual percentage yield is the annual interest rate which would be needed to obtain the same future balance in one year if interest was compounded annually.*

Suppose the annual interest rate is r. The balance after a year will be $F = P\left(1 + \frac{r}{n}\right)^{n \cdot 1} = P\left(1 + \frac{r}{n}\right)^{n}$. This must equal $P(1 + APY)$, the balance the account would have if the annual rate was equal to APY and compounded once per year.

It follows that we must have:

$$P\left(1 + \frac{r}{n}\right)^{n} = P(1 + APY)$$

$$\left(1 + \frac{r}{n}\right)^{n} = 1 + APY$$

$$APY = \left(1 + \frac{r}{n}\right)^{n} - 1$$

If interest is compounded continuously, the corresponding formula is

$$APY = e^r - 1.$$

Systematic Savings

Systematic Savings Plans: A deposit of amount D is made at the end of each of n interest periods each year for a period of t years.

To find the balance after t years, we may treat each deposit as if it were placed in a separate account.

After t years . . .

the first deposit would have grown to
$D\left(1+\frac{r}{n}\right)^{nt-1}$

the second deposit would have grown to
$D\left(1+\frac{r}{n}\right)^{nt-2}$

the third deposit would have grown to
$D\left(1+\frac{r}{n}\right)^{nt-3}$

. . .

the final deposit would not have collected interest yet and would have *grown* to D.

The total balance after the t years of all the deposits would be

$$F = D + D\left(1+\frac{r}{n}\right) + D\left(1+\frac{r}{n}\right)^2 + \ldots D\left(1+\frac{r}{n}\right)^{nt-1}.$$

Factoring out the common factor of D, we may write

$$F = D\left(1+\left(1+\frac{r}{n}\right)+\left(1+\frac{r}{n}\right)^2+\ldots\left(1+\frac{r}{n}\right)^{nt-1}\right).$$

The sum in parentheses is in the form

$1+a+a^2+\ldots a^{k-1}$, where $a = 1+\frac{r}{n}$ and $k = nt$

There is a formula for such sums, which are called *geometric series*.

From the tedious but relatively routine calculation

$(1-a)\left(1+a+a^2+\ldots a^{k-1}\right) = 1-a^k$, which may also be looked at as a factorization formula, one obtains the formula

$$1+a+a^2+\ldots a^{k-1} = \frac{1-a^k}{1-a}.$$

Using this formula in the formula for the balance, we obtain

$$F = D\left(\frac{1-\left(1+\dfrac{r}{n}\right)^{nt}}{1-\left(1+\dfrac{r}{n}\right)}\right)$$

$$F = D\left(\frac{\left(1+\dfrac{r}{n}\right)^{nt}-1}{\dfrac{r}{n}}\right)$$

This can be used to find the future balance as well as to find the size of the periodic deposits needed to obtain a given future balance.

To find the periodic deposits needed, we may use the formula as is or solve it for D as follows:

$$D\left(\left(1+\frac{r}{n}\right)^{nt}-1\right)=F\cdot\tfrac{r}{n}$$

$$D=\frac{F\cdot\frac{r}{n}}{\left(1+\frac{r}{n}\right)^{nt}-1}.$$

Amortized Loans

In a typical loan, the borrower is lent an amount P and makes periodic repayments R to reduce the balance owed until the balance is reduced to 0. This may be analyzed directly in a manner similar to the analysis of periodic savings.

We may cut the process short by viewing the repayments as periodic deposits made by the borrower designed so that they would grow, at the end of the repayment period, to an amount equal to what the original loan amount would grow to if it were invested at an interest rate equal to the rate charged.

Using the formula obtained for systematic savings, the periodic payments would grow to a balance

$$F = R\left(\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right).$$

A loan in the amount P would grow to a future value $P\left(1+\frac{r}{n}\right)^{nt}$.

We must therefore have

$$P\left(1+\frac{r}{n}\right)^{nt} = R\left(\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right).$$

$$P = \frac{1}{\left(1+\frac{r}{n}\right)^{nt}} \cdot R\left(\frac{\left(1+\frac{r}{n}\right)^{nt}-1}{\frac{r}{n}}\right)$$

$$P = R\left(\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right)$$

This could be used to determine the maximum loan someone could afford if the maximum size of the monthly payment they could make was R.

One may take the same formula and solve as follows for R to obtain the monthly payment necessary on a loan of a certain size.

$$P = R\left(\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right)$$

$$R = \frac{P}{\left(\frac{1-\left(1+\frac{r}{n}\right)^{-nt}}{\frac{r}{n}}\right)}$$

$$R = \frac{P \cdot \frac{r}{n}}{1-\left(1+\frac{r}{n}\right)^{-nt}}$$

Note the numerator $P \cdot \frac{r}{n}$ is a month's worth of interest on the original balance. Obviously, the monthly payment must be greater than that since otherwise the outstanding balance would keep increasing.

Definition 8 (Amortization Schedule). *An amortization schedule is a list of payments to be made on a loan which breaks down each payment into principal and interest.*

An amortization schedule can be set up fairly easily using a spreadsheet.