Apportionment

Quota Methods

Terminology:

Seats – Items to be apportioned

House Size – Number of Seats

States – Parties to whom seats are apportioned

Population – Measurement of state's size

Total Population – Sum of populations of all states

$$\text{Natural Divisor} = \frac{\text{Total Population}}{\text{House Size}}$$

$$\text{State's Natural Quota} = \frac{\text{State's Population}}{\text{Natural Divisor}}$$

Alternative:

$$\text{State's Natural Quota} = \frac{\text{State's Population}}{\text{Total Population}} \cdot (\text{House Size})$$

Obvious Solution: Simple Rounding – Round each state's natural quota to nearest integer.

Problem: House Size may not come out correctly.

Hamilton's Method – Method of Largest Fractional Parts

- Calculate the natural quota for each state.
- Round each state's natural quota down to the nearest integer that does not exceed the quota.
- Assign extra seats to the states whose quotas have the largest fractional parts.

Lowndes' Method – Method of Largest Relative Fractions

- Calculate the natural quota for each state.
- Round each state's natural quota down to the nearest integer that does not exce ed the quota.
- Assign extra seats to the states whose quotas have the largest relative fractional parts.

The relative fractional part is the fractional part of the natural quota divided by its integral part.

Definition 1 (Quota Property). *An apportionment is said to satisfy the Quota Property if each state's allocation di ers from its natural quota by less than one.*

Early Divisor Methods

Definition 2 (Jefferson's Method). *Find a divisor such that the correct number of seats is allotted when all the resulting modified quotas are rounded down to the nearest whole number.*

To find a divisor that would make a state's modified quota a certain value, divide its population by its desired quota. In other words,

threshold divisor for n seats = $\frac{\text{population}}{n}$.

Possible Problem: Violation of the Quota Property. *All divisor methods violate the Quota Property.*

Smaller divisors result in larger modified quotas and larger divisors result in smaller modified quotas.

Je erson's Method has a bias in favor of the larger states.

Definition 3 (Webster's Method). *Find a divisor such that the correct number of seats is allocated when the resulting modified quotas are rounded in the natural way.*

$$\text{threshold divisor for } n \text{ seats} = \frac{\text{population}}{n - 0.5}.$$

Apportionment in Today's House of Representatives

Definition 4 (Hill-Huntington Method). *Find a divisor such that the correct number of seats is allocated when the resulting modified quotas are rounded using the geometric mean* $\sqrt{(n-1)n}$ *for rounding between* $n-1$ *and* n*.*

threshold divisor for n seats = $\frac{\text{population}}{\sqrt{(n-1)n}}$

Alternatives

- Adam's Method – Quota's are rounded up from the modified quota. *Bias in favor of smaller states.*

- Dean's Method – Cuto for rounding is $\frac{(n-1)n}{n-0.5}$.

Note: $n-1 < (n-1)\cdot\frac{n}{n-0.5} = \frac{(n-1)n}{n-0.5} = n\cdot\frac{n-1}{n-0.5} < n.$

- Condorcet's Method – Quotas are rounded up at 0.4.

Search for an Ideal Apportionment Method

Definition 5 (Quota Property). *An apportionment is said to satisfy the Quota Property if each state's allocation di ers from its natural quota by less than one.*

Theorem 1. *No divisor method satisfies the Quota Property.*

Definition 6 (House Size Property). *No state ever loses a seat when the size of the house increases.*

Theorem 2. *Every divisor method satisfies the House Size Property.*

Definition 7 (Population Property – Population Monotonicity). *For a fixed house size, no state whose population increases ever loses a seat to another state whose population decreases.*

Theorem 3. *An apportionment method satisfies the Population Property if and only if it is a "generalized" divisor method, where the rounding method may depend on the number of states and the house size. Consequently, no apportionment method satisfies both the Quota Property and the Population Property.*