Combinatorics

Definition 1 (Combinatorics). *Combinatorics is the science of counting.*

Theorem 1 (Fundamental Principle of Counting). *If a sequence of choices are made and the first choice can be made in n_1 ways, the second in n_2 ways, the third in n_3 ways, and so on, then the entire sequence of choices can be made in $n_1 \cdot n_2 \cdot n_3 \ldots$ ways.*

Definition 2 (Permutation). *A* permutation *is a arrangement or list.*

Definition 3 (Combination). *A* combination *is a set.*

Given a combination, there may be several permutations of the same elements.

Example: The set $\{a, b, c\}$ yields the following permuations:

a,b,c
a,c,b
b,a,c
b,c,a
c,a,b
c,b,a

We may refer to permutations with or *without* replacement. In a permutation *with replacement*, the same element may appear more than once. In a permuation *without replacement*, no element may appear more than once.

Theorem 2. *The number of permutations* with replacement *of length r of elements chosen from a set of size n is n^r.*

This theorem is almost obvious from the *Fundamental Principle of Counting*.

Notation. *The number of permutations* without replacement *of length r of elements chosen from a set of size n is denoted by $P(n, r)$.*

Alternate notations which may be found in other sources: $P_{n,r}$, ${}_nP_r$.

Theorem 3. $P(n, r) = n(n-1)(n-2)\cdots(n-[r-1]) = \dfrac{n!}{(n-r)!}$.

Here, $k!$, read k *factorial*, means $k(k-1)(k-2)\cdots 3 \cdot 2 \cdot 1$. In other words, $k!$ is the product of all the positive integers between 1 and k. k *must be an integer!*

Notation (Combinations). *The number of combinations of r elements chosen from a set of size n is denoted by $C(n, r)$.*

Alternate notations which may be found in other sources: $C_{n,r}$, ${}_nC_r$, $\binom{n}{r}$.

We sometimes refer to $C(n, r)$ as n *choose* r, since it may be thought of as the number of ways of choosing r objects from a set of size n.

Theorem 4. $C(n, r) = \dfrac{n!}{r!(n-r)!}$.

Proof. Each combination of r elements gives rise to $P(r, r)$ di□erent permutations of the same elements. Thus, the number of permutations of size r is $P(r, r)$ times the number of combinations of the same size.

It follows that $P(n, r) = C(n, r)P(r, r)$. Since $P(n, r) = \dfrac{n!}{(n-r)!}$ and $P(r, r) = \dfrac{r!}{0!} = r!$, it follows that $C(n, r) = \dfrac{n!/(n-r)!}{r!} = \dfrac{n!}{r!(n-r)!}$. □

Example: The number of Gin Rummy hands is $C(52, 10) = \dfrac{52!}{10!42!}$.