$\Pi^1_1 - CA_0$ and Order Types of Countable Ordered Groups

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1 Introduction

Reverse mathematics uses subsystems of second order arithmetic to determine which set existence axioms are required to prove particular theorems. Surprisingly, almost every theorem studied is either provable in RCA_0 or equivalent over RCA_0 to one of four other subsystems: WKL_0, ACA_0, ATR_0 or $\Pi^1_1 - CA_0$. Of these subsystems, $\Pi^1_1 - CA_0$ has the fewest known equivalences. This article presents a new equivalence of $\Pi^1_1 - CA_0$ which comes from ordered group theory.

One of the fundamental problems about ordered groups is to classify all possible orders for various classes of orderable groups. In general, this problem is extremely difficult to solve. Mal'tsev (1949) solved a related problem by showing that the order type of a countable ordered group is $\mathbb{Z}^\alpha\mathbb{Q}^\epsilon$ where $\mathbb{Z}$ is the order type of the integers, $\mathbb{Q}$ is the order type of the rationals, α is a countable ordinal, and ϵ is either 0 or 1. The goal of this article is to prove that this theorem is equivalent over RCA_0 to $\Pi^1_1 - CA_0$.

In Section 2, we give the basic definitions and notation for RCA_0, ACA_0 and $\Pi^1_1 - CA_0$ as well as for ordered groups. For more information on reverse mathematics, see Friedman et al. (1983) or Simpson (1999) and for ordered groups, see Kokorin and Kopytov (1974) or Fuchs (1963). Our notation will follow these sources. In Section 3, we show that $\Pi^1_1 - CA_0$ suffices to prove Mal'tsev's Theorem and the reversal is done over RCA_0 in Section 4.

2 Basic Definitions

We will be concerned with three subsystems of second order arithmetic: RCA_0, ACA_0 and $\Pi^1_1 - CA_0$. RCA_0 contains the ordered semiring axioms for the natural numbers plus Δ^0_1 comprehension, Σ^0_1 formula induction and the set induction axiom

$$\forall X\,\big(\big(0 \in X \,\wedge\, \forall n(n \in X \rightarrow n+1 \in X)\big) \,\rightarrow\, \forall n(n \in X)\big).$$

The Δ^0_1 comprehension scheme consists of all axioms of the form

$$\forall n\,\big(\varphi(n) \leftrightarrow \psi(n)\big) \rightarrow \exists X\,\forall n\,\big(n \in X \leftrightarrow \varphi(n)\big)$$

where φ is a Σ^0_1 formula, ψ is a Π^0_1 formula and X does not occur freely in either φ or ψ. We will use $\mathbb{N}$ to denote the set defined by the formula $x = x$. Notice that in the comprehension scheme φ may contain free set variables other than X as parameters.

The computable sets form the minimum ω-model of RCA_0 and any ω-model of RCA_0 is closed under Turing reducibility. RCA_0 is strong enough to prove the existence of a set of unique codes for the finite sequences of elements from any set X. We use Fin_X to denote this set of codes. Also, we use $\langle a, b\rangle$, or more generally $\langle x_0, \ldots, x_n\rangle$, to denote pairs, or longer sequences, of elements of $\mathbb{N}$. For any sequences σ and τ, we denote the length of σ by $\mathrm{lh}(\sigma)$, the k^{th} element of σ by $\sigma(k)$, and the concatenation of σ and τ by $\sigma * \tau$. The empty sequence is denoted by λ and has length 0. The i^{th} column of X is denoted X_i and consists of all n such that $\langle n, i\rangle \in X$.

Definition 2.1. (RCA_0) A **tree** is a set $T \subseteq \mathrm{Fin}_{\mathbb{N}}$ which is closed under initial segments. That is, for any $\sigma, \tau \in \mathrm{Fin}_{\mathbb{N}}$, if $\sigma \subseteq \tau$ and $\tau \in T$, then $\sigma \in T$. A **path** through T is a function f such that for all n, the sequence $f[n] = \langle f(0), \ldots, f(n-1)\rangle \in T$. A **sequence of trees** is a set A such that each column of A is a tree. A is frequently written as $\langle T_k | k \in \mathbb{N}\rangle$ to emphasize that we are thinking of it as a sequence of trees.

ACA_0 consists of the axioms of RCA_0 plus the scheme of arithmetic comprehension, $\exists X \forall n(n \in X \leftrightarrow \varphi(n))$ for all formulas φ which have no set quantifiers and in which X does not occur freely. All ω-models of ACA_0 are closed under the Turing jump and the arithmetic sets form the minimum ω-model.

$\Pi^1_1 - CA_0$ consists of the axioms of ACA_0 plus the scheme of Π^1_1 comprehension. An important property of $\Pi^1_1 - CA_0$ is that it is strong enough to define sets by the transfinite recursion of an arithmetic formula over any well ordered set. In the next section we will give formal definitions for a well order. The following theorem states an equivalence of this subsystem that can be proved in RCA_0. For a proof see Simpson (1999).

Theorem 2.2. *(RCA_0) The following are equivalent:*

1. $\Pi^1_1 - CA_0$
2. *For any sequence of trees $\langle T_k | k \in \mathbb{N}\rangle$, there exists a set X such that $k \in X$ if and only if T_k has a path.*

The main tools from ordered group theory needed to prove Mal'tsev's theorem are the induced order on quotient groups and the idea of an Archimedean class.

Definition 2.3. (RCA_0) A **linear order** is a set X together with a binary relation $\leq_X$ (formally a set of ordered pairs) which satisfy the usual axioms for a linear order. Similarly, a **group** is a set G together with a binary function $\cdot_G$ and a constant 1_G which satisfy the axioms for a group. An **ordered group** is a pair $(G, \leq_G)$ where G is a group, $\leq_G$ is a linear order on G and for all $a, b, g \in G$, if $a \leq b$ then $ag \leq bg$ and $ga \leq gb$.

Example 2.4. RCA_0 proves that $(\mathbb{Z}, +)$ and $(\mathbb{Q}, +)$ are ordered groups under the standard orders.

Example 2.5. Let G be the free abelian group with generators $A = \{a_i | i \in \mathbb{N}\}$ and let $\leq_A$ be any linear order on A. Elements of G have the form $\sum_{i \in I} c_i \cdot a_i$ where I is a finite set and each $c_i \in \mathbb{Z} \setminus \{0\}$. To compare this element with $\sum_{j \in J} d_j \cdot a_j$, let $K = I \cup J$ and define $c_j = 0$ for $j \in K \setminus I$ and $d_i = 0$ for $i \in K \setminus J$. The order on G is defined by:

$$\sum_{i \in I} c_i \cdot a_i \ < \ \sum_{j \in J} d_j \cdot a_j \ \Leftrightarrow \ c_k < d_k$$

where a_k is the $\leq_A$-greatest generator for which $c_k \neq d_k$. This order is called the **lexicographic order induced by** $\leq_A$ and its existence is provable in RCA_0.

The subscript on $\leq_G$ is frequently dropped and we refer to the ordered group as G, suppressing mention of the linear order. In the literature, these groups are often referred to as fully ordered groups to distinguish them from partially ordered groups. We will be concerned only with fully ordered groups and hence we use the term ordered group. Notice that any ordered group must be torsion free, but that being torsion free is not a sufficient condition to guarantee orderability.

The quotient group G/H is defined in RCA_0 by picking the $\mathbb{N}$-least representative of each coset, using the fact that $aH = bH$ if and only if $a^{-1}b \in H$. That is, $a \in G/H$ if and only if $a \in G$ and for all $x <_{\mathbb{N}} a$, either $x \notin G$ or $x^{-1}a \notin H$.

Definition 2.6. (RCA_0) A subgroup H of an ordered group G is **convex** if for all $a, b \in H$ and $g \in G$, if $a \leq g \leq b$ then $g \in H$. If H is a convex normal subgroup of G, then the induced order on G/H is defined by

$$aH \leq_{G/H} bH \ \Leftrightarrow \ (aH = bH) \vee (aH \neq bH \wedge a <_G b).$$

We need to know when we can combine an order $\leq_{G/H}$ on G/H and an order $\leq_H$ on H to get an order $\leq_G$ on G under which H is convex, $\leq_H$ is $\leq_G$ restricted to H and $\leq_{G/H}$ is the same as the induced order from $\leq_G$ on G/H. Notice that in an ordered group, the order is preserved under conjugation. $(H, \leq_H)$ is called **G-ordered** if the order on H is preserved not only under conjugation by elements of H, but under conjugation by any element of G. This condition turns out to be both necessary and sufficient. For a proof of the following theorem, see Kokorin and Kopytov (1974).

Theorem 2.7. *(RCA_0) Let H be a normal subgroup of G, $\leq_H$ be a G-order on H, and G/H be ordered by $\leq_{G/H}$. There exists an order $\leq_G$ on G such that H is convex, $\leq_H$ is $\leq_G$ restricted to H, and $\leq_{G/H}$ is the same as the induced order on G/H.*

Definition 2.8. (RCA_0) For an ordered group G, $|\,x\,| = x$ if $x > 1_G$ and $|\,x\,| = x^{-1}$ otherwise.

Definition 2.9. (RCA_0) If G is an ordered group, then $a \in G$ is **Archimedean less than** $b \in G$, denoted $a \ll b$, if $|\,a^n\,| < |\,b\,|$ for all $n \in \mathbb{N}$. If there exist $n, m \in \mathbb{N}$ such that $|\,a^n\,| \geq |\,b\,|$ and $|\,b^m\,| \geq |\,a\,|$, then a and b are **Archimedean equivalent**, denoted $a \approx b$. The notation $a \lessapprox b$ means $a \approx b \;\vee a \ll b$. G is an **Archimedean ordered group** if G is ordered and for all $a, b \neq 1_G$, $a \approx b$.

It is not hard to check that $\approx$ is an equivalence relation and that $\ll$ is transitive, antireflexive, and antisymmetric. The next lemma lists several other straightforward properties of $\approx$ and $\ll$. For proofs, see Fuchs (1963).

Lemma 2.10. *(RCA_0) If G is an ordered group, then the following conditions hold for all $a, b, c \in G$.*

1. *Exactly one of the following holds: $a \ll b$, $b \ll a$, or $a \approx b$.*
2. *$a \ll b$ implies that $xax^{-1} \ll xbx^{-1}$ for all $x \in G$.*
3. *$a \ll b$ and $a \approx c$ imply that $c \ll b$.*
4. *$a \ll b$ and $b \approx c$ imply that $a \ll c$.*

3 Order Type of a Group

Definition 3.1. (RCA_0) A linear order $(X, \leq_X)$ is a **well order** if there is no function $f : \mathbb{N} \to X$ such that $f(n+1) <_X f(n)$ for all n.

In keeping with the notation of set theory, we use α, β and γ to stand for well orders. It is useful to talk about the longest initial segment of a linear order which is well ordered.

Definition 3.2. (RCA_0) The **well ordered initial segment** of a linear order X is defined by

$$W(X) \;=\; \big\{x \in X \mid \neg\exists f\big(f : \mathbb{N} \to X \wedge f(0) = x \wedge \forall i(f(i+1) < f(i))\big)\big\}.$$

Notice that $W(X)$ need not exist in systems like RCA_0, but that $\Pi^1_1 - CA_0$ certainly suffices to prove its existence. It is clear from the definition that if $(W(X), \leq)$ does exist, then it is a well order and that if $y \in W(X)$ and $z \leq y$ then $z \in W(X)$.

Definition 3.3. (RCA_0) Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be linear orders. The product XY is the linear order $(Z, \leq_Z)$ where

$$Z = \{\, \langle x, y\rangle \,|\, x \in X \wedge y \in Y \,\}$$
$$\langle x_1, y_1\rangle \leq_Z \langle x_2, y_2\rangle \leftrightarrow y_1 <_Y y_2 \vee (y_1 = y_2 \wedge x_1 \leq_X x_2).$$

We also need a definition for $\mathbb{Z}^X$. In set theoretic terms, $\mathbb{Z}^X$ is the set of functions $f : X \to \mathbb{Z}$ with finite support. If $f \neq g$, then $f < g$ if and only if $f(x) <_{\mathbb{Z}} g(x)$ where x is the maximum value of X on which f and g disagree. To represent $\mathbb{Z}^X$ in second order arithmetic, we use finite sequences of pairs $\langle x, z\rangle$ with $x \in X$ and $z \in \mathbb{Z} \setminus 0$. To insure a unique normal form for the sequences, we require that the X-components in each sequence be in decreasing order. By convention, $\mathbb{Z}^{\emptyset}$ is the single element linear order. Recall that π_1 and π_2 are the projection functions for pairs.

Definition 3.4. (RCA_0) Let X be a nonempty linear order and $Y = X \times (\mathbb{Z} \setminus \{0\})$.

$$\mathbb{Z}^X = \{\, x \mid x \in \mathrm{Fin}_Y \wedge \forall i < (\mathrm{lh}(x) - 1) \big(\pi_1(x(i)) >_X \pi_1(x(i+1))\big) \,\}$$

Two elements x, y are equal if and only if they are identical as sequences. If $x \neq y$ and $\mathrm{lh}(x) \leq_{\mathbb{N}} \mathrm{lh}(y)$, then there are two cases to consider.

1. If $x \subset y$ then $x <_{\mathbb{Z}^X} y \leftrightarrow \pi_2\big(y(\mathrm{lh}(x))\big) >_{\mathbb{Z}} 0$.
2. If $x \not\subset y$, let i be the least number such that $x(i) \neq y(i)$ and suppose $x(i) = \langle x_i, u_i \rangle$ and $y(i) = \langle y_i, v_i \rangle$.
   (a) If $x_i <_X y_i$ then $x <_{\mathbb{Z}^X} y \leftrightarrow v_i >_{\mathbb{Z}} 0$.
   (b) If $x_i >_X y_i$ then $x <_{\mathbb{Z}^X} y \leftrightarrow u_i <_{\mathbb{Z}} 0$.
   (c) If $x_i = y_i$ then $x <_{\mathbb{Z}^X} y \leftrightarrow u_i <_{\mathbb{Z}} v_i$.

To see why this definition captures the set theoretic notion, think of each sequence $x \in \mathbb{Z}^X$ as representing the function that sends $\pi_1(x(i))$ to $\pi_2(x(i))$ for each $i < \mathrm{lh}(x)$ and sends all other values in X to 0. In Case 1 of the definition, $\pi_1(y(\mathrm{lh}(x)))$ represents the largest value of X on which the functions associated to x and y differ. The function for x sends this element to 0, so $x < y$ if and only if y maps this element to something greater than 0. The other cases have similar explanations.

Notice that instead of representing elements of $\mathbb{Z}^X$ as sequences, we could represent them as finite formal sums. In this notation, the sum $\sum_{i \in I} r_i x_i$, with $r_i \in \mathbb{Z} \setminus \{0\}$, would represent the function mapping x_i to r_i for $i \in I$ and all other elements of X to 0. We can move back and forth between these representations in RCA_0 and we will use whichever is the most convenient.

Definition 3.5. (RCA_0) If G is an ordered group and X is a linear order, then X is the **order type** of G if there is an order preserving bijection $f : G \to X$.

If $f : G \to X$ and $g : G \to Y$ are two order types of G, then the map $g \circ f^{-1} : X \to Y$ is an order preserving bijection between X and Y. So, in RCA_0 the order type is unique up to order preserving bijection.

If G, A, $\leq_A$ and $\leq_G$ are as in Example 2.5, then the order type of G is $\mathbb{Z}^A$. This fact is most easily proved in RCA_0 using the formal sum notation for elements of $\mathbb{Z}^A$. For more details on this correspondence, see the discussion following Proposition 4.4

To clear up a possibly confusing point of terminology, an order preserving bijection is a bijection between linearly ordered structures that preserves the order, but ignores any other structure they might have. On the other hand, an order isomorphism, or o-isomorphism, is a group isomorphism that preserves order. We can now state the main theorem.

Theorem 3.6. *(RCA_0) The following are equivalent:*

1. $\Pi^1_1 - CA_0$
2. *Mal'tsev's Theorem: Let G be a countable ordered group. There is a well order α and $\epsilon \in \{0, 1\}$ such that $\mathbb{Z}^\alpha \mathbb{Q}^\epsilon$ is the order type of G.*
3. *Let G be a countable ordered abelian group. There is a well order α and $\epsilon \in \{0, 1\}$ such that $\mathbb{Z}^\alpha \mathbb{Q}^\epsilon$ is the order type of G.*

In this section, we prove that (1) implies (2). The idea of the proof is that if G is an ordered group, then either G has a least strictly positive element or it does not. If G does not have such an element, then it has order type $\mathbb{Q}$. If G does have a least strictly positive element a, then the order type of G is the product of $\mathbb{Z}$ and $G/\langle a \rangle$, where $\langle a \rangle$ is the convex normal subgroup generated by a. This process is repeated with $G/\langle a \rangle$ and continues to be repeated until we have either used up all of G or found a quotient of G which has order type $\mathbb{Q}$. The recursion can be done in ATR_0 (a subsystem strictly between ACA_0 and $\Pi^1_1 - CA_0$), but $\Pi^1_1 - CA_0$ is required to prove that the process eventually terminates.

Lemma 3.7. *(RCA_0) Let G be an ordered group and H a convex normal subgroup. If X is the order type of H and Y is the order type of the induced order on G/H, then XY is the order type of G.*

Proof. Since G/H is a set of representatives for the cosets, each element of G can be uniquely written as ah where $a \in G/H$ and $h \in H$. If $g_1 \neq g_2$, $g_1 = a_1h_1$ and $g_2 = a_2h_2$, then by the definition of the induced order, $g_1 <_G g_2$ if and only if $a_1 <_{G/H} a_2$ or $a_1 = a_2$ and $h_1 <_G h_2$. Suppose $f_H : H \to X$ and $f_{G/H} : G/H \to Y$ are the order preserving bijections. Define $f : G \to XY$ by $g \mapsto \langle f_H(h), f_{G/H}(a)\rangle$, where $g = ah$ is the decomposition of g given above. This map is the desired order preserving bijection. □

Definition 3.8. (RCA_0) Let G be an ordered group. The set Arch(G) is a set of unique representatives of the Archimedean classes of G.

$$\mathrm{Arch}(G) \;=\; \big\{g \in G \mid \forall h \in G\,\big(h <_{\mathbb{N}} g \to \neg(h \approx g)\big)\big\}$$

Arch(G) is ordered by taking $x < y$ if and only if $x \ll y$. In general, RCA_0 is not strong enough to prove the existence of Arch(G). However, if Arch(G) exists, then we can define a function $f : G \to \mathrm{Arch}(G)$ such that $f(g) = d$ if and only if $d \in \mathrm{Arch}(G)$ and $g \approx d$. If $d \in \mathrm{Arch}(G)$, then we can express $g \approx d$ by both a Σ^0_1 and a Π^0_1 formula.

$$\begin{aligned} g \approx d \;&\Leftrightarrow\; \exists n \exists m\,(|g^m| > |d| \wedge |d^n| > |g|) \\ &\Leftrightarrow\; \forall y \in \mathrm{Arch}(G)\,\forall n \forall m\,\big(y \neq d \to (|y^m| < |g| \vee |g^n| < |y|)\big) \end{aligned}$$

Since $f(g)$ has a Δ^0_1 definition with Arch(G) as a parameter, it is definable in RCA_0 from Arch(G).

Lemma 3.9. *(RCA_0) Let G be an ordered group. Suppose Arch(G), $Y = W(\mathrm{Arch}(G))$ and $H = \{g \in G | \exists y \in Y (g \ll y \vee g \approx y)\}$ exist and $Y \neq \mathrm{Arch}(G)$. Then H is a convex normal subgroup of G and G/H has order type $\mathbb{Q}$.*

Proof. H is clearly a convex subgroup. To show H is normal, suppose $h \in H$, $g \in G$ and $ghg^{-1} \notin H$. Let $a \in \mathrm{Arch}(G)$ be such that $a \approx ghg^{-1}$ and notice that $g^{-1}ag \approx h$. Because $ghg^{-1} \notin H$, it follows that $a \notin Y$ and so there is an infinite descending chain $f : \mathbb{N} \to \mathrm{Arch}(G)$ with $f(0) = a$. By Lemma 2.10, $f(n+1) \ll f(n)$ implies $g^{-1}f(n+1)g \ll g^{-1}f(n)g$. Define $\hat{f} : \mathbb{N} \to \mathrm{Arch}(G)$ by setting $\hat{f}(n)$ to be the element of Arch(G) which is Archimedean equivalent to $g^{-1}f(n)g$. $\hat{f}$ is an infinite descending chain below $\hat{f}(0)$, and so $\hat{f}(0)$ is not in Y. However, $\hat{f}(0) \approx g^{-1}ag \approx h$ which contradicts the fact that $h \in H$.

To finish the proof, it suffices to show that the induced order on G/H is dense with no endpoints. The key fact is that for any $b \in \mathrm{Arch}(G) \setminus Y$ there exists $c \in \mathrm{Arch}(G) \setminus Y$ such that $c \ll b$. Consider the case when $1_GH < g_1H < g_2H$, then there are $b, c \in \mathrm{Arch}(G) \setminus Y$ such that $g_1 \approx b$ and $c \ll g_1$. Since $h \ll c$ for all $h \in H$ and $g_1 \cdot |c| \approx g_1 \ll g_2$, it follows that $(g_1 \cdot |c|)H$ is strictly between g_1H and g_2H. The other cases showing that G/H is dense and has no endpoints are similar. □

Lemma 3.10. *(ACA_0) Let G be an ordered group. If G has a least strictly positive element x then $\forall g \in G(gx = xg)$ and the subgroup generated by x is convex and isomorphic to $\mathbb{Z}$.*

Proof. Suppose there exists g such that $gx \neq xg$. Without loss of generality, assume that $gx < xg$ and hence $gxg^{-1} < x$. Since $1_G < x$, we have $1_G < gxg^{-1} < x$ which contradicts the fact that x is the least strictly positive element.

The subgroup generated by x has a Σ^0_1 definition, so its existence can be proved in ACA_0. Because G is torsion free, the elements of this subgroup have the form x^n for $n \in \mathbb{Z}$. Suppose there is an $n \in \mathbb{Z}$ and a $c \in G$ such that $x^n < c < x^{n+1}$. It follows that $1_G < cx^{-n} < x$ which contradicts the hypothesis. □

Lemma 3.11. *(RCA_0) Let G be an ordered group. If G contains elements a_1, a_2 such that $a_1 < a_2$ and $\forall g\big(a_1 \leq g \leq a_2 \to (a_1 = g \vee a_2 = g)\big)$, then G has a least strictly positive element.*

Proof. Let $x = a_2a_1^{-1}$. Because $a_1 < a_2$, we have $1_G < x$ and if $1_G < b < x$, then $a_1 < ba_1 < a_2$ which contradicts the hypothesis. □

We are ready to prove that (1) implies (2) in Theorem 3.6.

Proof. Let G be an ordered group. $\Pi^1_1 - CA_0$ suffices to prove that $W(\mathrm{Arch}(G))$ exists. Let $X = W(\mathrm{Arch}(G))$. We use 0 to denote the least element of X, $\beta+1$ to denote the successor of β in X, and we say γ is a limit if γ has no immediate predecessor in X. For any $\beta \in X$, let $\hat{\beta} = \{ y \in X \,|\, y < \beta \}$.

The strategy is to construct a chain of convex normal subgroups $A_\beta \subseteq G$ for $\beta \in X$. At each step, we prove that the order type of A_β is $\mathbb{Z}^{\hat{\beta}}$ and that unless we have reached the end of X, A_β is strictly contained in G. If we reach a step where A_β cannot be extended to $A_{\beta+1}$, the construction terminates early. Otherwise the construction terminates at the end of X.

Construction: Define $A_0 = \{1_G\}$.

Successor Step: Assume A_β is a convex normal subgroup, $A_\beta \neq G$ and the order type of A_β is $\mathbb{Z}^{\hat{\beta}}$. G/A_β is given the induced order. There are two cases to consider:

1. If G/A_β has no least strictly positive element, then terminate the construction early at β. By Lemma 3.11, G/A_β has order type $\mathbb{Q}$.

2. If G/A_β has a least strictly positive element, let $a_{\beta+1} \in G$ represent this least positive coset. Define $A_{\beta+1}$ to be the subgroup generated by A_β and $a_{\beta+1}$.

Limit Step: If λ is a limit ordinal in X, $A_\lambda = \bigcup_{\beta<\lambda} A_\beta$.

End of Construction

We need to verify that at each step of the construction, A_β is a convex normal subgroup with order type $\mathbb{Z}^{\hat{\beta}}$. Notice that proving the order type is $\mathbb{Z}^{\hat{\beta}}$ requires transfinite induction along $W(\mathrm{Arch}(G))$ with a Σ^1_1 formula. This induction is possible because $\Pi^1_1 - CA_0$ proves the general scheme of Σ^1_1 transfinite induction (see Simpson (1999)).

First, consider the successor step $\beta+1$. Since $a_{\beta+1}A_\beta$ is the least positive element of G/A_β, Lemma 3.10 says that $a_{\beta+1}A_\beta$ is in the center of G/A_β and the subgroup it generates is convex. This fact means that $a_{\beta+1}$ commutes with elements of G modulo A_β. That is, for every g there is an $a \in A_\beta$ such that $ga_{\beta+1} = a_{\beta+1}ga$. Thus any element of $A_{\beta+1}$ can be written in the form $a^n_{\beta+1}b$ for some $n \in \mathbb{Z}$ and $b \in A_\beta$. Also, since A_β is convex and $a_{\beta+1} \notin A_\beta$, $a_{\beta+1}$ is Archimedean greater than all the elements of A_β. We can now verify the following claims.

Claim. (1) $A_{\beta+1}$ is normal.

Let $x = a^n_{\beta+1}b$. Because $a_{\beta+1}$ commutes with elements of G modulo A_β, there is a $\tilde{b} \in A_\beta$ such that $ga^n_{\beta+1}bg^{-1} = a^n_{\beta+1}g\tilde{b}bg^{-1}$. The fact that A_β is normal implies that $g\tilde{b}bg^{-1} \in A_\beta$ and therefore that $gxg^{-1} \in A_{\beta+1}$.

Claim. (2) $A_{\beta+1}$ is convex.

If $a^n_{\beta+1}b < z < a^m_{\beta+1}\tilde{b}$ then $a^n_{\beta+1}A_\beta \leq zA_\beta \leq a^m_{\beta+1}A_\beta$ in G/A_β. Since the subgroup of G/A_β generated by $a_{\beta+1}A_\beta$ is convex, $zA_\beta = a^p_{\beta+1}A_\beta$ for some p. It follows that $z = a^p_{\beta+1}c$ for some $c \in A_\beta$, so $z \in A_{\beta+1}$.

Claim. (3) The order type of $A_{\beta+1}/A_\beta$ is $\mathbb{Z}$.

Elements of $A_{\beta+1}/A_\beta$ are of the form $a^n_{\beta+1}A_\beta$. Since $b \ll a_{\beta+1}$ for all $b \in A_\beta$, it follows that $a^n_{\beta+1} \neq a^m_{\beta+1}$ modulo A_β if $n \neq m$.

Claim. (4) For all $b \in A_{\beta+1}$, either $b \ll a_{\beta+1}$ or $b \approx a_{\beta+1}$.

If $a_{\beta+1} \ll b$, then $a \ll b$ for all $a \in A_\beta$ and so b is not in the subgroup generated by $a_{\beta+1}$ and A_β.

Claim (4) shows that unless $\beta+1$ is the maximum element of X, $A_{\beta+1} \neq G$. Claim (3), together with Lemma 3.7 and the induction hypothesis, shows that the order type of $A_{\beta+1}$ is $\mathbb{Z}^{\widehat{\beta+1}}$.

To check the properties at a limit step, assume λ is a limit in X. From the construction it is clear that A_λ is a convex normal subgroup and that unless λ marks the end of X, there are elements of $\mathrm{Arch}(G)$ above A_λ, and so $A_\lambda \neq G$. For $\beta < \lambda$ assume $f_\beta : A_\beta \to \mathbb{Z}^{\hat{\beta}}$ is an order preserving bijection. Define $f_\lambda : A_\lambda \to \mathbb{Z}^{\hat{\lambda}}$ by $a \mapsto f_\beta(a)$ where β is the least element of X such that $a \in A_\beta$. Notice that $\mathbb{Z}^{\hat{\beta}} \subset \mathbb{Z}^{\hat{\lambda}}$, so we can view $f_\beta(a)$ as an element of $\mathbb{Z}^{\hat{\lambda}}$. This map is an order preserving bijection, so A_λ has the desired order type.

Since the construction may have terminated early and $W(\text{Arch}(G))$ may or may not be $\text{Arch}(G)$, there are four cases to consider to finish the proof. First, if $W(\text{Arch}(G)) = \text{Arch}(G)$, then

- If the construction terminates early at β, then A_β has order type $\mathbb{Z}^{\hat{\beta}}$, G/A_β has order type $\mathbb{Q}$, and so by lemma 3.7 G has order type $\mathbb{Z}^{\hat{\beta}}\mathbb{Q}$.
- If the construction completes and β is the maximum element of X, then $G = A_\beta$ and so G has order type $\mathbb{Z}^{\hat{\beta}}$.

Second, if $\text{Arch}(G)$ is not well ordered, then

- If the construction terminates early at β, then as in the first case, G has order type $\mathbb{Z}^{\hat{\beta}}\mathbb{Q}$.
- If the construction is completed and β is the maximum element of X, then G/A_β has order type $\mathbb{Q}$ by Lemma 3.9 and G has order type $\mathbb{Z}^{\hat{\beta}}\mathbb{Q}$.

□

4 The Reversal

The goal of this section is to show that RCA_0 suffices to prove that (3) implies (1) in Theorem 3.6. The proof takes place in two steps. First, we show that RCA_0 plus statement (3) in Theorem 3.6 suffices to prove the well ordered initial segment of every linear order exists. Second, we use this fact plus some properties of the Kleene-Brouwer order on trees to prove that (3) implies (1).

Definition 4.1. (RCA_0) For a linear order X, $U \subseteq X$ is **dense** if U has at least two elements and for every $u, v \in U$, if $u <_X v$ then there is a $w \in U$ such that $u <_X w <_X v$.

The next two lemmas show that there are no dense subsets of $\mathbb{Z}^X$ for a well order X. The arguments are longer than one might expected because of the restriction to RCA_0. Recall that $u \in \mathbb{Z}^X$ is a finite sequence of pairs $\langle x, z\rangle$ with $x \in X$ and $z \in \mathbb{Z} \setminus \{0\}$ such that $\pi_1(u(i)) > \pi_1(u(i+1))$. u is best thought of as representing the function that maps each $\pi_1(u(i))$ to $\pi_2(u(i))$ and maps all other elements of X to 0.

Lemma 4.2. *(RCA_0) Let X be a well order and U be a dense subset of $\mathbb{Z}^X$. There are sequences of elements of U, $u_0, u_1, \ldots$ and $v_0, v_1, \ldots$ such that for each $n \in \mathbb{N}$*

1. $u_n < u_{n+1} < v_{n+1} < v_n$,
2. $lh(u_n) > n$ *and* $lh(v_n) > n$, *and*
3. $u_n(0) = v_n(0), u_n(1) = v_n(1), \ldots, u_n(n) = v_n(n)$.

Proof. If $X = \emptyset$, then $\mathbb{Z}^X$ has only one element and hence has no dense subsets. Assume that $X \neq \emptyset$ and $U \subseteq \mathbb{Z}^X$ is dense. We define the sequences by induction starting with u_0 and v_0.
Claim. There are $u \neq v$ in U such that $\pi_1(u(0)) = \pi_1(v(0))$.

Suppose there are no such u and v. We will contradict the fact that X is a well order. Since U is infinite, we can pick u and v such that either both $\pi_2(u(0))$ and $\pi_2(v(0))$ are positive or both are negative. Without loss of generality, assume they are both positive and $u <_{\mathbb{Z}^X} v$. Define a function $g : \mathbb{N} \to \mathbb{Z}^X$ such that $g(0) = v$ and $g(i+1)$ is the $\mathbb{N}$-least element of U strictly between u and $g(i)$. The density of U insures that $g(i+1)$ is defined. For any $i \in \mathbb{N}$ we have

$$u <_{\mathbb{Z}^X} g(i+1) <_{\mathbb{Z}^X} g(i) <_{\mathbb{Z}^X} v.$$

We verify that $\pi_1(u(0)) <_X \pi_1(g(i+1)(0))$. Assume that this inequality does not hold. By assumption, $\pi_1(u(0)) \neq \pi_1(g(i+1)(0))$, so we must have $\pi_1(g(i+1)(0)) <_X \pi_1(u(0))$. However, by the definition of $\leq_{\mathbb{Z}^X}$ and because $\pi_2(u(0)) > 0$, this inequality implies that $g(i+1) <_{\mathbb{Z}^X} u$, which is a contradiction.

Because $\pi_1(u(0)) <_X \pi_1(g(i+1)(0))$ and $u <_{\mathbb{Z}^X} g(i+1)$, the definition of $\leq_{\mathbb{Z}^X}$ implies that $\pi_2(g(i+1)(0)) > 0$. Therefore, we can apply the reasoning of the previous paragraph to $g(i+1) <_{\mathbb{Z}^X} g(i)$ and conclude that $\pi_1(g(i+1)(0)) <_X \pi_1(g(i)(0))$. Define the function $h : \mathbb{N} \to X$ by $h(i) = \pi_1(g(i)(0))$. The properties of g imply that $h(i+1) <_X h(i)$ for all i, which contradicts the fact that X is a well order and proves the claim.

Let $u, v \in U$ be such that $u < v$ and $\pi_1(u(0)) = \pi_1(v(0))$. Let $U_1 = \{x \in U | u \leq x \leq v\}$. U_1 is also a dense subset of X and for any $x \in U_1$, $\pi_1(x(0)) = \pi_1(u(0))$. To finish the $n = 0$ case, it suffices to find $r, s \in U_1$ such that $r \neq s$ and $\pi_2(r(0)) = \pi_2(s(0))$. Suppose there are no such elements. If $r \neq s \in U_1$, then

$$r <_{\mathbb{Z}^X} s \;\Leftrightarrow\; \pi_2(r(0)) < \pi_2(s(0)).$$

However, if $r \in U_1$, then $\pi_2(r(0))$ is between $\pi_2(u(0))$ and $\pi_2(v(0))$. Thus there are a finite number of elements in U_1, which contradicts the density of U_1. Let $\langle u_0, v_0 \rangle$ be the $\mathbb{N}$-least pair of elements of U such that $u_0 < v_0$ and $u_0(0) = v_0(0)$.

The argument for the induction step is similar. Assume we have u_n and v_n. Consider the set V of elements $x \in U$ with $u_n \leq_{\mathbb{Z}^X} x \leq_{\mathbb{Z}^X} v_n$. For any $x \in V$, $x(i) = u_n(i)$ for $0 \leq i \leq n$. By a notationally cumbersome, but similar argument, we can find $r, s \in V$ such that $r \neq s$ and $r(n+1) = s(n+1)$. Let $\langle u_{n+1}, v_{n+1} \rangle$ be the $\mathbb{N}$-least pair in V such that $u_{n+1} < v_{n+1}$ and $u_{n+1}(n+1) = v_{n+1}(n+1)$. □

Lemma 4.3. *(RCA_0) If X is a well order, then there are no dense subsets of $\mathbb{Z}^X$.*

Proof. Suppose X is a well order and U is a dense subset of $\mathbb{Z}^X$. Let $u_0, u_1, \ldots$ and $v_0, v_1, \ldots$ be the sequences from Lemma 4.2. Define $F : \mathbb{N} \to X$ by $F(n) = \pi_1(u_n(n))$. F is an infinite descending chain which contradicts the fact that X is a well order. □

Proposition 4.4. *(RCA_0) (1) implies (2) where*

1. *For any countable ordered abelian group A, there is a well order α and $\epsilon \in \{0,1\}$ such that $\mathbb{Z}^\alpha \mathbb{Q}^\epsilon$ is the order type of A.*
2. *The well ordered initial segment $W(X)$ exists for all linear orders X.*

The proof of Proposition 4.4 follows from the next two lemmas. Let $X = \{x_0, x_1, \ldots\}$ be an infinite linear order and G be the free abelian group on the generators $\{a_0, a_1, \ldots\}$. Elements of G are represented by finite sums, $\sum_{i \in I} r_i a_i$ where I is a finite set and $r_i \in \mathbb{Z} \setminus \{0\}$. Rather that writing the elements of $\mathbb{Z}^X$ as finite sequences, we will write them here as finite sums, $\sum_{i \in I} r_i x_i$ where I is a finite set and $r_i \in \mathbb{Z} \setminus \{0\}$. This sum represents the function that sends each x_i to r_i and all other elements of X to 0. When G and $\mathbb{Z}^X$ are presented this way, there is a natural bijection between them that sends $\sum_{i \in I} r_i a_i$ to $\sum_{i \in I} r_i x_i$.

X is used to define an order on G, just as in Example 2.5. To compare two distinct elements of G, $\sum_{i \in I} r_i a_i$ and $\sum_{j \in J} s_j a_j$, let $K = I \cup J$, $r_k = 0$ for $k \in J \setminus I$ and $s_k = 0$ for $k \in I \setminus J$. Let n be such that x_n is X-maximal in $\{x_k | k \in K \wedge r_k \neq s_k\}$. The order is defined by

$$\sum_{i \in I} r_i a_i \;<\; \sum_{j \in J} s_j a_j \;\leftrightarrow\; r_n < s_n.$$

Under this order, the bijection from G to $\mathbb{Z}^X$ is order preserving. Statement (1) in Proposition 4.4 guarantees an order preserving bijection from G to $\mathbb{Z}^\alpha \mathbb{Q}^\epsilon$ for some well order α and $\epsilon = 0$ or 1. We continue to use the sequence notation for elements of $\mathbb{Z}^\alpha \mathbb{Q}^\epsilon$.

Lemma 4.5. *(RCA_0) X is a well order if and only if $\epsilon = 0$.*

Proof.

Case. (⇒)

Suppose X is a well order. Because there are order preserving bijections between G and $\mathbb{Z}^X$ and between G and $\mathbb{Z}^\alpha\mathbb{Q}^\epsilon$, it follows that there is an order preserving bijection between $\mathbb{Z}^X$ and $\mathbb{Z}^\alpha\mathbb{Q}^\epsilon$. If $\epsilon = 1$, then the set $\{\langle\lambda, q\rangle \mid q \in \mathbb{Q}\}$ is dense in $\mathbb{Z}^\alpha\mathbb{Q}$ (recall that λ denotes the empty sequence). Because RCA_0 proves that the image of any subset of the domain of a bijection exists, the image of this set exists and is dense in $\mathbb{Z}^X$. This statement contradicts Lemma 4.3.

Case. (⇐)

Suppose X is not a well ordering and $g : \mathbb{N} \to \mathbb{N}$ is such that $x_{g(0)} > x_{g(1)} > \cdots$ forms an infinite descending chain in X. There is a corresponding infinite descending chain of generators in G, $a_{g(0)} \gg a_{g(1)} \gg \cdots$. Let H be the subgroup of G generated by $p_n a_{g(n)}$ where p_n is the n^{th} prime.

$$\sum_{i\in I} r_i a_i \in H \ \leftrightarrow\ \forall i \in I\, \exists p_n \le |r_i|\, (g(n) = i \ \wedge\ p_n \text{ divides } r_i)$$

Because the quantification is bounded, this condition is Σ^0_0. It suffices to show that H is dense, for in that case, the image of H in $\mathbb{Z}^\alpha\mathbb{Q}^\epsilon$ is dense and by Lemma 4.3, ϵ must be 1.

To show H is dense, consider two elements of H, $\sum_{i\in I} r_i a_i \ <\ \sum_{j\in J} s_j a_j$. Define the coefficients in the sums for all the elements of $K = I \cup J$ by setting $r_k = 0$ for $k \in K \setminus I$ and $s_k = 0$ for $k \in K \setminus J$. Let n be such that x_n is X-maximal in $\{x_k | k \in K \wedge s_k \ne r_k\}$. n is in the range of g, so there is an m with $g(m) = n$. The element $\sum_{i\in I} r_i a_i \ +\ p_{m+1} a_{g(m+1)}$ lies in H and is strictly between the two elements given above. □

If X is not well ordered, then by Lemma 4.5, G has order type $\mathbb{Z}^\alpha\mathbb{Q}$. Let $f : G \to \mathbb{Z}^\alpha\mathbb{Q}$ be the order preserving bijection. For any $x_k \in X$, there is an associated generator a_k in G. $f(a_k)$ has two components, one from $\mathbb{Z}^\alpha$ and one from $\mathbb{Q}$. The second component is the key to defining $W(X)$.

Lemma 4.6. *(RCA_0) If X is not well ordered, then $x_k \in X$ is in $W(X)$ if and only if $\pi_2(f(a_k)) = \pi_2(f(1_G))$.*

Proof.

Case. (⇒)

Suppose $x_k \in W(X)$. Let H be the subgroup generated by a_i for $x_i \le_X x_k$. H exists since its elements are exactly those of the form $\sum_{i\in I} r_i a_i$ where $\forall i \in I(x_i \le_X x_k)$.

Claim. H is convex.

Let $\sum_{i\in I} r_i a_i \ <_G\ \sum_{j\in J} s_j a_j$ be two elements in H and let $g \in G$ lie strictly between them. $\sum_{i\in I} r_i a_i <_G g$ implies that for all $x_n >_X x_k$, the coefficient of a_n in g is greater than or equal to 0. On the other hand, $g <_G \sum_{j\in J} s_j a_j$ implies that for all $x_n >_X x_k$, the coefficient of a_n in g is less than or equal to 0. Hence, $g \in H$ as required.

$f(H)$ exists because f is a bijection, and $f(H)$ is convex because f is order preserving. Suppose $\pi_2(f(a_k)) \ne \pi_2(f(1_G))$. The contradiction we will derive is that $f(H)$ has a dense suborder while H does not.

Define the well order $\hat{X} = \{y \in X | y \le x_k\}$. Since H is the free abelian group on the generators a_n with $x_n \in \hat{X}$, it follows from the definition of $\le_G$ that the order type of H is $\mathbb{Z}^{\hat{X}}$. Lemma 4.3 shows that H has no dense suborders.

As for $f(H)$, since $\pi_2(f(a_k)) \ne \pi_2(f(1_G))$ and $1_G \le_G a_k$, it follows that $\pi_2(f(1_G)) <_{\mathbb{Q}} \pi_2(f(a_k))$. Thus, for any $q \in \mathbb{Q}$ strictly between these values, $\langle\lambda, q\rangle$ is strictly between $f(1_G)$ and $f(a_k)$. Since $f(H)$ is convex, the set of such points is in $f(H)$. Thus $f(H)$ has a dense suborder.

Case. (⇐)

This case is similar to the proof of Lemma 4.5. Suppose $x_k \notin W(X)$. Let $g : \mathbb{N} \to \mathbb{N}$ be such that $x_k = x_{g(0)} > x_{g(1)} > \cdots$ is an infinite descending chain. This function also gives a descending chain of generators with $a_{g(0)} = a_k$ and $a_{g(i)} \gg a_{g(i+1)}$ for all i. Let H be the subgroup of G which is generated by the elements of the form $p_n a_{g(n)}$ for $n \ge 1$, where p_n is the n^{th} prime. Let $P = \{g | g \ge 1_G\}$. By an argument similar to the one in Lemma 4.5, $P \cap H$ is a dense suborder of G. Now, suppose that $\pi_2(f(a_k)) = \pi_2(f(1_G))$. To complete the proof, we show that $f(P \cap H)$ does not have a dense suborder.

Claim. For any $y \in f(P \cap H)$, $\pi_2(y) = \pi_2(f(1_G))$.

Suppose not. If $\pi_2(y) <_{\mathbb{Q}} \pi_2(f(1_G))$ then $y <_{\mathbb{Z}^\alpha\mathbb{Q}} f(1_G)$ and $f^{-1}(y) <_G 1_G$. This contradicts the fact that 1_G is the least element of $f(P \cap H)$. If $\pi_2(f(1_G)) <_{\mathbb{Q}} \pi_2(y)$, then by similar reasoning and the fact that $\pi_2(f(1_G)) = \pi_2(f(a_k))$ we have that $a_k <_G f^{-1}(y)$. However, since H is generated by $p_n a_{g(n)}$ for $n \geq 1$, any element of H is below a_k.

To show that $f(P \cap H)$ has no dense suborders, let $\tilde{f} : f(P \cap H) \to \mathbb{Z}^\alpha$ be the map that takes y to $\pi_1(y)$. $\tilde{f}$ is order preserving and one-to-one, but is not necessarily a bijection. If $U \subseteq f(P \cap H)$ then $\tilde{f}(U)$ is definable by a Σ^0_0 condition.

$$\tilde{f}(U) \;=\; \{z \in \mathbb{Z}^\alpha | \langle z, \pi_2(f(1_G))\rangle \;\in U\}$$

If $U \subseteq f(P \cap H)$ is dense, so is $\tilde{f}(U) \subseteq \mathbb{Z}^\alpha$. Hence, by Lemma 4.3, there are no dense subsets of $f(P \cap H)$. □

We can now give a proof of Proposition 4.4.

Proof. If X is well ordered, then $W(X) = X$. Otherwise, by Lemmas 4.6,

$$W(X) = \{x_k \in X \mid \pi_2(f(a_k)) = \pi_2(f(1_G))\},$$

which exists by the recursive comprehension axiom. □

To finish the proof of the reversal we need a certain property of the Kleene Brouwer order (defined below) which requires ACA_0. Therefore, to make the reversal work over RCA_0, we use Proposition 4.4 to show that Mal'tsev's Theorem implies ACA_0 over RCA_0. If α is a well order, then we use $\alpha + 1$ to denote the well order obtained by adding a new largest element onto the end of α.

Definition 4.7 (Friedman and Hirst (1990)). (RCA_0) Let α and β be well orders. We say α is strongly less than β, denoted $\alpha \leq_s \beta$, if there is a 1-1 order preserving map from α onto an initial segment of β.

Theorem 4.8 (Hirst (1999)). *(RCA_0) The following are equivalent:*

1. *ACA_0.*

2. *If α and β are well orders such that $\alpha \leq_s \beta$ and $\beta \not\leq_s \alpha$, then $\alpha + 1 \leq_s \beta$.*

Lemma 4.9. *(RCA_0) (1) implies (2) where*

1. *For any countable abelian ordered group A, there is a well order α and $\epsilon \in \{0, 1\}$ such that $\mathbb{Z}^\alpha\mathbb{Q}^\epsilon$ is the order type of A.*

2. *ACA_0.*

Proof. By Theorem 4.8, it suffices to show that given well orders α and β such that $\alpha \leq_s \beta$ and $\beta \not\leq_s \alpha$, we have that $\alpha + 1 \leq_s \beta$. Let $f : \alpha \to \beta$ be an order preserving injection onto an initial segment of β. Notice that $\beta \not\leq_s \alpha$ implies that the range of f is not all of β. Define a new linear order X by setting $X = \beta \times \mathbb{N}$ with the strict inequality given by $(b_1, m) < (b_2, n)$ if and only if either

- $b_1 < b_2$ or
- $b_1 = b_2$, $m < n$, and $\exists t < n(f(t) = b_1)$ or
- $b_1 = b_2$, $m > n$, and $\forall t < m(f(t) \neq b_1)$.

To get a picture for X, think of X as replacing each point $y \in \beta$ by the collection of points (y, m) for $m \in \mathbb{N}$. First, if $y \in \text{range}(f)$, say $y = f(n)$, then X replaces y by a copy of $\mathbb{N}$.

$$(y, n) <_X (y, n-1) <_X \cdots <_X (y, 0) <_X (y, n+1) <_X (y, n+2) <_X \cdots \quad (1)$$

In particular, if $x \in X$ satisfies $x <_X (y, n)$, then $x = (b, m)$ for some $b <_\beta y$. Second, if $y \notin \text{range}(f)$, then X replaces y by an infinite descending chain.

$$\cdots <_X (y, 2) <_X (y, 1) <_X (y, 0) \quad (2)$$

In particular, if $y \notin \text{range}(f)$, then $(y, 0)$ is not in the well ordered initial segment of X.

By Proposition 4.4, we know that Mal'tsev's Theorem implies that $W(X)$ exists. Hence, RCA_0 suffices to prove the existence of

$$Y \;=\; \{\, b \in \beta \,|\, (b, 0) \notin W(X) \,\}.$$

Y is not empty since it contains all pairs $(b, 0)$ for which b is not in the range of f. Since β is a well order, Y has a $\leq_\beta$-least element, which we call b_0. We can now extend the map f to have domain $\alpha + 1$. Let a denote the largest element of $\alpha + 1$ and define $f(a) = b_0$.

We verify that f now witnesses $\alpha + 1 \leq_s \beta$. First notice that if $b_1 <_\beta b_0$, then b_1 is in the range of f, for otherwise $(b_1, 0) \notin W(X)$ which contradicts the choice of b_0 as the least such element. All that remains to show is that b_0 is not itself in the range of f. Suppose for a contradiction that $f(x) = b_0$ for some $x \in \alpha$. Let $g : \mathbb{N} \to X$ be an infinite strictly decreasing chain below $(b_0, 0)$. Since g is strictly decreasing, we have by Equation (1) that $g(x+1) <_X (b_0, x)$ and so $g(x+1) = (b', n)$ for some $b' <_\beta b_0$. Either $(b', n) < (b', 0)$, which implies $g(x+1) <_X (b', 0)$, or $(b', 0) <_X (b', n)$, which implies $g(x+1+n) <_X (b', 0)$ by Equations (1) and (2). In either case, $(b', 0) \notin W(X)$, contradicting out choice of b_0. □

We can now continue the proof of the reversal working in ACA_0.

Definition 4.10. (RCA_0) The **Kleene-Brouwer order**, **KB**, on $\text{Fin}_\mathbb{N}$ is defined by: $\sigma \leq_{KB} \tau$ if and only if $\sigma \supseteq \tau$ or there is a $j < \min(\text{lh}(\sigma), \text{lh}(\tau))$ with $\sigma(j) < \tau(j)$ and $\sigma(i) = \tau(i)$ for all $i < j$. If T is a tree, then $KB(T)$, the Kleene-Brouwer order of T, is $KB \cap (T \times T)$.

Lemma 4.11 (Friedman et al. (1983)). *(ACA_0) A tree T has a path if and only if $KB(T)$ is not a well order.*

Lemma 4.12. *(ACA_0) In Theorem 3.6, (3) implies (1).*

Proof. Assume (3) and let $\langle T_i \mid i \in \mathbb{N} \rangle$ be a sequence of trees. By Theorem 2.2, it suffices to construct the set $\{i | T_i \text{ has a path}\}$. If none of the T_i's has a path, then $X = \emptyset$. Therefore, assume at least one T_i has a path.

First we define a tree T that contains each T_i as a subtree.

$$T \;=\; \{\lambda\} \cup \{i * \tau | \tau \in T_i\}$$

The set T is ordered by the Kleene-Brouwer order, $KB(T)$. Since at least one T_i has a path, T has a path and $KB(T)$ is not a well order. Define A_i by

$$A_i \;=\; \{\, \tau \in T \,|\, \langle i-1 \rangle <_{KB} \tau \leq_{KB} \langle i \rangle \,\}.$$

The map which sends $\tau \in T_i$ to $i * \tau \in T$ is one-to-one, preserves the tree structure and has image A_i. Thus, while A_i is technically not a tree (since $\lambda \notin A_i$), it can be viewed as an isomorphic copy of T_i with the property that $KB(T_i)$ and $KB(T) \cap (A_i \times A_i)$ are isomorphic as linear orders.

By Lemma 4.11, T_i has a path if and only if $KB(T_i)$ is not well ordered, which holds if and only if $KB(T) \cap (A_i \times A_i)$ is not well ordered. Thus it suffices to form the set

$$\{\, i \,|\, KB(T) \cap (A_i \times A_i) \text{ is not well ordered} \,\}.$$

Let G be the free abelian group on the generators a_τ, for $\tau \in T$. Order the generators by $a_\tau \ll a_\gamma$ if and only if $\tau <_{KB} \gamma$. As in Example 2.5, this order of the generators determines a lexicographic order on G with order type $\mathbb{Z}^{KB(T)}$ (see also the discussion following Proposition 4.4). By statement (3) in Theorem 3.6, there is an order preserving bijection $f : G \to \mathbb{Z}^\alpha\mathbb{Q}^\epsilon$. Because $KB(T)$ is not a well order, it follows from the proof of Lemma 4.5 that $\epsilon = 1$.

Claim. If $Y \subseteq T$ then there is an order preserving bijection between $\mathbb{Z}^Y$ and the subgroup generated by a_τ with $\tau \in Y$, denoted $\langle a_\tau | \tau \in Y\rangle$.

As shown above, when the elements of $\langle a_\tau | \tau \in Y\rangle$ and $\mathbb{Z}^Y$ are represented as finite sums, there is a natural order preserving bijection between them.

Claim. For each i, $\langle a_\tau | \tau \in A_i\rangle$ has a dense suborder if and only if A_i is not well ordered by $KB(T) \cap (A_i \times A_i)$.

If A_i is well ordered, then by the first claim and Lemma 4.3 it follows that $\langle a_\tau | \tau \in A_i\rangle$ has no dense suborders. If A_i is not well ordered, then we can use a descending chain to build a dense suborder as in Lemma 4.5.

Claim. For each $i > 0$, the subgroup $\langle a_\tau | \tau \in A_i\rangle$ has a dense suborder if and only if $\pi_2(f(a_{\langle i-1\rangle})) \neq \pi_2(f(a_{\langle i\rangle}))$.

Suppose $\pi_2(f(a_{\langle i-1\rangle})) \neq \pi_2(f(a_{\langle i\rangle}))$. Then the set

$$U \;=\; \{\, \gamma \in T \mid \pi_2(f(a_{\langle i-1\rangle})) < \pi_2(f(a_\gamma)) < \pi_2(f(a_{\langle i\rangle}))\,\}$$

is contained in A_i by the definition of the Kleene-Brouwer order. Since the elements of T are all actually elements of $\mathbb{N}$ we can trim this set down farther. Let V be the set defined by

$$V = \{\, \tau \in U \mid \forall \gamma \in U(\gamma <_{\mathbb{N}} \tau \to \pi_2(f(a_\gamma)) \neq \pi_2(f(a_\tau))\,\}.$$

Because f maps G onto $\mathbb{Z}^\alpha\mathbb{Q}^\epsilon$, $V \subseteq A_i$ is dense which implies that A_i is not well ordered. It follows by the second claim that $\langle a_\tau | \tau \in A_i\rangle$ has a dense suborder. To prove the other direction, suppose $\pi_2(f(a_{\langle i-1\rangle})) = \pi_2(f(a_{\langle i\rangle})) = q$. The image of $\langle a_\tau | \tau \in A_i\rangle$ in $\mathbb{Z}^\alpha\mathbb{Q}$ is contained in $\mathbb{Z}^\alpha \times \{q\}$ which has no dense suborders by Lemma 4.3.

Similarly, we can show $\langle\, a_\tau | \tau \leq \langle 0\rangle \,\rangle$ has a dense suborder if and only if $\pi_2(f(a_{\langle 0\rangle})) \neq \pi_2(f(1_G))$. This fact together with the last claim give us a Σ^0_0 definition for the set

$$\{\, i \mid KB(T) \cap (A_i \times A_i) \text{ is not well ordered}\,\},$$

showing that RCA_0 suffices to prove its existence. □

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