Π^0_1 classes and orderable groups

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Abstract

It is known that the spaces of orders on orderable computable fields can represent all Π^0_1 classes up to Turing degree. We show that the spaces of orders on orderable computable abelian and nilpotent groups cannot represent Π^0_1 classes in even a weak manner. Next, we consider presentations of ordered abelian groups, and we show that there is a computable ordered abelian group for which no computable presentation admits a computable set of representatives for its Archimedean classes.

1 Introduction

The methods of computability theory were originally developed to study computational properties of sets of natural numbers. Many of these techniques, however, can be applied in more general algebraic contexts, giving rise to the field of effective algebra. Our goal is to examine ordered groups from this perspective.

Definition 1.1. An **ordered group** is a pair $(G, \leq_G)$, such that G is a group and $\leq_G$ is a linear order on G which is preserved under multiplication on both sides. A group which admits an order is called an **orderable group**.

The simplest examples of ordered groups are the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ with the standard orders. We will be concerned only with abelian and nilpotent groups (defined below), since these groups are orderable if and only if they are torsion free. In general, orderability is not so easily characterized.

Definition 1.2. A countable structure $\mathcal{A}$ in a fixed countable signature is **computable** if the domain of $\mathcal{A}$ is a computable set and the interpretations of the function, relation, and constant symbols are uniformly computable.

For example, a computable group is given by a computable set $G \subset \omega$, a distinguished element $1_G \in G$, and a partial computable function $\cdot_G$ which satisfies the usual group axioms. A computable ordered group includes a partial computable relation $\leq_G$ which satisfies the additional axioms for an ordered group. These definitions relativize to any Turing degree $\mathbf{a}$, in which case we assume G, $\cdot_G$, and $\leq_G$ are all $\mathbf{a}$-computable. We drop the subscripts on $\cdot_G$ and $\leq_G$ unless they are needed to avoid confusion, and for abelian groups, we use $+$ and 0 in place of $\cdot$ and 1.

Note the following distinctions in terminology. A computable ordered group refers to a computable group with a fixed computable order. An orderable computable group is a computable group which admits an order, although this order need not be computable. On the other hand, a computably orderable computable group is a computable group which admits a computable order.

Considerable work has been done on orderable computable fields, and we will show that in some respects, groups behave quite differently from fields. The initial question to ask about orderable computable fields or groups is whether they always admit a computable order. In both cases, the answer is no.

Theorem 1.3 (Rabin (1960)). *There exists an orderable computable field with no computable order.*

Theorem 1.4 (Downey and Kurtz (1986)). *There exists an orderable computable abelian group with no computable order.*

We will use Π^0_1 classes to further analyze the computational properties of orders on groups. Given a fixed coding of $2^{<\omega}$, Π^0_1 classes are defined as follows.

Definition 1.5. A **computable binary branching tree** is a nonempty computable set $T \subseteq 2^{<\omega}$ which is closed under initial segments. The set of paths through a computable binary branching tree is called a $\boldsymbol{\Pi^0_1}$ **class**.

The Turing degrees of members of Π^0_1 classes have been intensely studied, and below are two examples of the known results. Recall that a set A has low Turing degree if $A' \equiv_T 0'$.

Theorem 1.6 (Jockusch Jr. and Soare (1972)). *Every nonempty Π^0_1 class has a member of low Turing degree.*

Theorem 1.7 (Jockusch Jr. and Soare (1972)). *There is an infinite Π^0_1 class C such that, for all $f, g \in C$, if $f \neq g$, then f and g have incomparable Turing degrees.*

Π^0_1 classes are powerful tools for studying theorems in combinatorics, algebra, and analysis (see Cenzer and Remmel (1999)). They arise naturally in mathematics, because many constructions can be viewed as taking place on a binary branching tree. The binary branching allows guessing in such a way that mistakes can be corrected at a finite level by terminating all branches extending an incorrect guess. Therefore, the infinite paths correspond exactly to the desired mathematical objects. The precise mechanics of such a construction are not relevant to this article, since we will prove only negative results concerning the representation of Π^0_1 classes. For the full details, see Cenzer and Remmel (1999), or for applications directly related to orderable groups, see Solomon (1998).

Metakides and Nerode used Π^0_1 classes to analyze the space of orders of computable fields. They showed that Π^0_1 classes exactly capture the effective content of these spaces. For a given field F, $\mathbb{X}(F)$ denotes the space of all orders on F.

Theorem 1.8 (Metakides and Nerode (1979)). *Let F be an orderable computable field. There is a Π^0_1 class C and a Turing degree preserving bijection $\varphi : \mathbb{X}(F) \to C$.*

Theorem 1.9 (Metakides and Nerode (1979)). *Let C be a Π^0_1 class. There is a computable field F and a Turing degree preserving bijection $\varphi : \mathbb{X}(F) \to C$.*

Theorems 1.8 and 1.9 allow the transfer of facts about Π^0_1 classes directly to the space of orders on a computable field. For example, Theorems 1.6 and 1.8 show that, although there are orderable computable fields which do not admit a computable order, every such field has an order of low Turing degree. Theorem 1.9 is more important, however, since it allows us to transfer results such as Theorem 1.7. Together, these theorems imply that there is an orderable computable field for which any two distinct orders have noncomparable Turing degrees.

It is part of the folklore that the analogue of Theorem 1.8 holds for orderable groups with essentially the same proof (for more details, see Solomon (1998)). Therefore, every orderable computable group must have an order of low Turing degree. Downey and Kurtz asked whether some analogue of Theorem 1.9 holds for computable torsion free abelian groups. In Section 2, we describe several variations of this question, and in Section 3, we show that none of these versions of Theorem 1.9 holds for computable torsion free abelian groups. In Section 4, we extend these negative results to the class of torsion free nilpotent groups.

The results of Sections 3 and 4 demonstrate an interesting phenomenon in reverse mathematics. In reverse mathematics, one studies the proof theoretic strength of theorems by proving they are equivalent to weak subsystems of second order arithmetic (see Simpson (1999)). The subsystem WKL_0 includes comprehension over Δ^0_1 formulas plus Weak König's Lemma, which states that every infinite binary branching tree has an infinite path.

Theorem 1.10 (Hatzikiriakou and Simpson (1990), Solomon (1998)). *The following are equivalent over RCA_0.*

1. *WKL_0.*
2. *Every torsion free abelian group is orderable.*
3. *Every torsion free nilpotent group is orderable.*

Typically, theorems which are equivalent to WKL_0 yield relatively strong representations of Π^0_1 classes. Sections 3 and 4 show that despite the equivalences of Theorem 1.10, the spaces of orders on computable torsion free abelian or nilpotent groups do not suffice to represent Π^0_1 classes in even a weak manner.

In the last two sections, we consider whether it is possible to avoid results like Theorem 1.4 by carefully choosing the presentation of a group.

Definition 1.11. A **computable presentation** of a group G is a computable group H such that $G \cong H$.

In Section 5, we show that although a particular computable torsion free abelian group G may not have a computable order, there is always a computable presentation of G which does admit a computable order.

In Section 6, we consider whether a similar situation holds for sets of representatives for the Archimedean classes of a computable ordered group (all terms defined below). In this case, it turns out that we cannot even guarantee that a computable ordered group will have a computable presentation which has a computable set of unique representatives for the Archimedean classes.

Our notation is standard. We follow Soare (1987) for the computability theory. Degrees will always refer to Turing degrees and the degree of a set A is denoted $\deg(A)$. For ordered groups, we follow the notation of Kokorin and Kopytov (1974) and Fuchs (1963).

2 Ordered groups

We begin this section with two useful examples of ordered abelian groups.

Example 2.1. If G is a free abelian group with generators a_i for $i \in \omega$, then the elements of G are formal sums $\sum_{i \in I} r_i a_i$ in which $I \subseteq \omega$ is a finite set, $r_i \in \mathbb{Z}$, and $r_i \neq 0$. To compare this element with $\sum_{j \in J} q_j a_j$, let $K = I \cup J$, $r_k = 0$ for $k \in K \setminus I$, and $q_k = 0$ for $k \in K \setminus J$. The lexicographic order is defined by $\sum_{i \in I} r_i a_i < \sum_{j \in J} q_j a_j$ if and only if $r_k < q_k$, where k is the largest element of K for which $r_k \neq q_k$.

Example 2.2. If $(G, \leq)$ is an ordered group, then so is $(G, \leq')$, where $\leq'$ is defined by $g \leq' h$ if and only if $h \leq g$. Notice that this construction of turning an order "upside down" does not work for ordered fields, because any field element with a square root must be positive in all orders.

It is frequently easier to consider sets of positive elements on a group rather than binary relations. The positive cone of an ordered group G is

$$P(G) \ = \ \{g \in G \mid 1_G \leq_G g\}.$$

The equivalence $ab^{-1} \in P(G) \leftrightarrow b \leq a$ defines $P(G)$ from the parameter $\leq$ and vice versa. It also shows that $\leq$ and $P(G, \leq)$ have the same Turing degree. For a proof of the following theorem see Kokorin and Kopytov (1974).

Theorem 2.3. *A subset P of a group G is the positive cone of an order on G if and only if P is a normal semigroup such that, for every $g \in G$, either g or g^{-1} is in P, and both g and g^{-1} are in P if and only if $g = 1_G$.*

Definition 2.4. The **space of orders** on an ordered group G is the set

$$\mathbb{X}(G) \ = \ \{\, P \subset G \,|\, P \text{ is the positive cone of an order on } G \,\}.$$

By Example 2.2, if P is a positive cone on G and $P^{-1} = \{ g^{-1} \mid g \in P \}$, then P^{-1} is also a positive cone on G. Since $P \equiv_T P^{-1}$, any Turing degree which contains a member of $\mathbb{X}(G)$ contains at least two members of $\mathbb{X}(G)$. However, by Theorem 1.7, there is a Π^0_1 class C such that for any $f, g \in C$, if $f \neq g$ then $\deg(f) \neq \deg(g)$. Therefore, there cannot be a degree preserving bijection between C and $\mathbb{X}(G)$ for any orderable computable group G.

We can now state several versions of Theorem 1.9 for groups. The strongest version (and the one in which Downey and Kurtz were interested) requires that for every Π^0_1 class C, there is a orderable computable group G and a two-to-one degree preserving map $\varphi : \mathbb{X}(G) \to C$. A weaker statement would require only that

$$\{ \deg(f) \mid f \in C \} \;=\; \{ \deg(P) \mid P \in \mathbb{X}(G) \}.$$

The weakest version we consider arises from the following special collection of Π^0_1 classes.

Definition 2.5. Let A and B be disjoint computably enumerable (c.e.) sets. A c.e. set D **separates** the pair A, B if $A \subset D$ and $D \cap B = \emptyset$. The collection of all such separating sets is called a $\mathbf{\Pi^0_1}$ **class of separating sets**.

Π^0_1 classes of separating sets are Π^0_1 classes, and they have the property that they are either finite or of cardinality 2^ω. The weakest form of representation considered here is whether, given a Π^0_1 class of separating sets C, there is a computable torsion free abelian group G such that

$$\{ \deg(f) | f \in C \} \;=\; \{ \deg(P) | P \in \mathbb{X}(G) \}.$$

In Section 3 we show that this last representation fails, which implies a negative answer to the previous questions.

3 Π^0_1 classes and orderable abelian groups

We begin this section with an outline of some abelian group theory.

Definition 3.1. An abelian group D is **divisible** if, for every $g \in D$ and every $n \geq 1$, there is a $d \in D$ such that $nd = g$ (here nd refers to d added to itself n times). Given an abelian group G, we say that D is a **divisible closure** of G if D is divisible and there is a monomorphism $\varphi : G \to D$ such that, for every $d \in D \setminus \{0_D\}$, there is a $g \in G \setminus \{0_G\}$ and an $n \in \mathbb{N}$ for which $nd = \varphi(g)$.

Definition 3.2. If G is a computable abelian group, we say (D, φ) is a **computable divisible closure** of G if D is a computable divisible abelian group and φ is a partial computable function with the properties listed above.

Smith (1981) proved that every computable abelian group has a computable divisible closure. The following two results connect the space of orders on an abelian group G with the space of orders on its divisible closure D.

Lemma 3.3. *If D is a divisible closure of G, then D is orderable if and only if G is orderable. Furthermore, each order on G extends uniquely to an order on D.*

Proof. To prove the first statement, notice that D is torsion free if and only if G is torsion free. For the second statement, suppose $\varphi : G \to D$ is the monomorphism from Definition 3.1 and P is the positive cone of an order on G. We need to show that $\varphi(P)$ extends uniquely to the positive cone of an order on D. For any $d \in D \setminus \{0_D\}$, there is an $n \geq 1$ and a $g \in G \setminus \{0_G\}$ such that $nd = \varphi(g)$. If $g \in P$, then $nd \in \varphi(P)$, so d must be positive, and if $g \notin P$, then $-nd \in \varphi(P)$, so d must be negative. Thus, there is at most one choice for an extension of $\varphi(P)$. Checking the appropriate properties shows that the following set

$$\{\, d \in D \mid d = 0_D \vee \big(d \neq 0_D \wedge \exists n \geq 1\, \exists g \in P \setminus \{0_G\}\, (nd = \varphi(g))\big)\,\}. \tag{1}$$

gives a well defined order on D. □ □

Lemma 3.4. *If G is a computable torsion free abelian group and D is a computable divisible closure of G, then there is a Turing degree preserving bijection $\psi : \mathbb{X}(G) \to \mathbb{X}(D)$.*

Proof. By Lemma 3.3, $\mathbb{X}(G)$ can be mapped bijectively onto $\mathbb{X}(D)$. Because there are n and g such that $nd = \varphi(g)$ (and these can be found by searching), Equation (1) shows that this correspondence preserves Turing degrees. □ □

These results show that without loss of generality, we may assume that G is divisible when we study the computable properties of $\mathbb{X}(G)$.

Definition 3.5. Let G be a torsion free abelian group. The elements $g_0, \ldots, g_n$ are **linearly independent** if for any $\alpha_0, \ldots, \alpha_n \in \mathbb{Z}$, the equality

$$\alpha_0 g_0 + \alpha_1 g_1 + \cdots + \alpha_n g_n \;=\; 0_G$$

implies $\alpha_i = 0$ for all i. An infinite set $B \subseteq G \setminus \{0_G\}$ is linearly independent if every finite subset of B is linearly independent. A maximal set of linearly independent elements is a **basis** for G, and the cardinality of any basis is the **rank** of G.

For example, G has rank 1 if and only if G is isomorphic to a subgroup of $\mathbb{Q}$. If G is a torsion free divisible abelian group, then G can be viewed as a vector space over $\mathbb{Q}$. In this case, the definitions of linear independence, basis, and rank for G as a group agree with the definitions of the same terms for G as a vector space.

The concept of a basis plays a central role in the structure theory of infinite abelian groups. The corresponding structural object in the study of ordered group is the collection of Archimedean classes.

Definition 3.6. Let $(G, \leq)$ be an ordered group (not necessarily abelian) and let $a, b \in G$. We use $|a|$ to denote whichever of a or a^{-1} is positive. We say a is **Archimedean less than** b, if $|a^n| < |b|$ for every $n \geq 1$. If there are $n, m \geq 1$ such that $|b| \leq |a^n|$ and $|a| \leq |b^m|$, then we say a and b are **Archimedean equivalent**, denoted $a \approx b$. If any two nonidentity elements of G are Archimedean equivalent, then $(G, \leq)$ is called an **Archimedean group** and $\leq$ is called an **Archimedean order** on G.

It is easy to verify that $\approx$ is an equivalence relation on G and that the equivalence classes of the positive cone of G are linearly ordered by $\leq_G$ with least element 1_G. Typically, we only consider Archimedean classes of elements other than 1_G.

Lemma 3.7. *Let G be a computable torsion free abelian group. If B is a basis for G, then G has an Archimedean order computable from B.*

Proof. Without loss of generality, assume that G is divisible. Let $e_0, e_1, \ldots, e_n, \ldots$ be a (possibly finite) one-to-one enumeration of the elements of B. Effectively in B, each element of G can be uniquely assigned to a sum of the form $g = \sum_{i \in I} q_i e_i$, where $I \subset \omega$ is finite, $q_i \in \mathbb{Q}$, and each $q_i \neq 0$. Let p_i be an enumeration of the primes in increasing order. G can be embedded into $\mathbb{R}$ by sending $e_0 \mapsto 1$, each $e_i \mapsto \sqrt{p_i}$ for $i \geq 1$, and extending the map linearly. $\mathbb{R}$ induces an order on G which is computable in B since each $\sqrt{p_i}$ is a computable real. □ □

Our study of computable torsion free abelian groups breaks into three cases: groups of rank 1, groups of finite rank > 1, and groups of infinite rank. The following lemmas describe part of the computational content of the algebraic classification of the orders on abelian groups of finite rank (see Teh (1960)).

Lemma 3.8. *If G is a computable torsion free abelian group of rank 1, then G has exactly two orders both of which are computable.*

Proof. Any divisible closure of G is computably isomorphic to $\mathbb{Q}$. Since $\mathbb{Q}$ has exactly two orders both of which are computable, G has exactly two orders both of which are computable. □ □

Lemma 3.9. *If G is a computable torsion free abelian group with finite rank strictly greater than 1, then G has orders of every Turing degree. Furthermore, these orders can be assumed to have at most two nontrivial Archimedean classes.*

Proof. Without loss of generality, assume G is divisible. We first consider the case when the rank of G is 2. Let $\{a, b\}$ be a basis for G. Any element of G can be written as $p_1 a + p_2 b$, for some $p_1, p_2 \in \mathbb{Q}$. By Lemma 3.7, G admits a computable Archimedean order.

To see that G admits orders of every noncomputable degree, let $\mathbf{a}$ be an arbitrary noncomputable degree and fix a set $A \in \mathbf{a}$ with $0 \notin A$. Let χ_A denote the characteristic function of A, and define $r = \sum_{i=1}^{\omega} 2^{-i} \chi_A(i)$. Notice that $r \in \mathbb{R}$ is irrational, strictly between 0 and 1, and $\deg(r) = \mathbf{a}$. Let $f_r : G \to \mathbb{R}$ denote the map that sends $p_1 a + p_2 b$ to $p_1 + p_2 r$. Because r is irrational, this map is an isomorphism between G and a subgroup of $\mathbb{R}$. The structure of $\mathbb{R}$ defines an Archimedean order $\leq_r$ on G by

$$g \leq_r h \ \Leftrightarrow \ f_r(g) \leq_{\mathbb{R}} f_r(h).$$

It remains to show that $\deg(r) = \deg(\leq_r)$, for then we have $\deg(\leq_r) = \mathbf{a}$.

Claim. $\deg(r) \leq_T \deg(\leq_r)$

To compute the coefficients $a_i \in \{0,1\}$ in $r = \sum_{i=1}^{\omega} a_i/2^i$, notice that

$$r >_{\mathbb{R}} \frac{1}{2} \Leftrightarrow 2r >_{\mathbb{R}} 1 \Leftrightarrow 2b >_r a$$
$$\text{and } r <_{\mathbb{R}} \frac{1}{2} \Leftrightarrow 2r <_{\mathbb{R}} 1 \Leftrightarrow 2b <_r a.$$

Using $\leq_r$, we can tell whether $2b >_r a$ or $2b <_r a$, so we can determine a_1. To compute a_{n+1}, assume by induction that $a_1, \ldots, a_n$ are such that

$$\sum_{i=1}^{n} \frac{a_i}{2^i} \; <_{\mathbb{R}} \; r \; <_{\mathbb{R}} \; \sum_{i=1}^{n} \frac{a_i}{2^i} \; + \; \frac{1}{2^n}.$$

To find a_{n+1}, we need to check if the following inequality holds:

$$r \; <_{\mathbb{R}} \; \sum_{i=1}^{n} \frac{a_i}{2^i} \; + \; \frac{1}{2^{n+1}}.$$

If this inequality holds, then $a_{n+1} = 0$, and otherwise $a_{n+1} = 1$. However, the definition of $\leq_r$ implies that

$$r \; <_{\mathbb{R}} \; \sum_{i=1}^{n} \frac{a_i}{2^i} \; + \; \frac{1}{2^{n+1}} \; \Leftrightarrow \; b \; <_r \; \left(\sum_{i=1}^{n} \frac{a_i}{2^i} \right) a \; + \; \frac{a}{2^{n+1}}.$$

The required induction hypothesis holds once we set a_{n+1} to the correct value, so the claim is proved.

Claim. $\deg(\leq_r) \leq_T \deg(r)$

Assume we know the coefficients in $r = \sum_{i=1}^{\omega} a_i/2^i$. We need to compare elements of G of the form $p_1 a + q_1 b$ and $p_2 a + q_2 b$, with coefficients from $\mathbb{Q}$. Multiplying by the denominators of p_1, p_2, q_1, and q_2, shows that it suffices to compare elements of the form $n_1 a + m_1 b$ and $n_2 a + m_2 b$ with coefficients from $\mathbb{Z}$. Moreover, collecting terms shows that it suffices to compare na and mb for $n, m \in \mathbb{Z}$. We will further assume that $n, m > 0$, since the other cases are either easy (i.e. $n = 0$ and $m > 0$) or reduce to this case (i.e. $n < 0$ and $m < 0$).

Because $f_r(na) = n$ and $f_r(mb) = mr$, it suffices to compare n and mr as elements of $\mathbb{R}$. Our strategy for doing this is to compute a rational approximation of mr and an error bound for this approximation, such that there are no integers within the error bounds. From here, we can tell if $n < mr$ by comparing n with the approximation. To compute such an approximation, notice that for any k,

$$m \cdot \sum_{i=1}^{k} \frac{a_i}{2^i} \; <_{\mathbb{R}} \; m \cdot r \; <_{\mathbb{R}} \; m \cdot \left(\sum_{i=1}^{k} \frac{a_i}{2^i} \; + \; \frac{1}{2^k} \right).$$

We need to find a k such that there are no integers in the interval

$$\left[m \cdot \sum_{i=1}^{k} \frac{a_i}{2^i} \; , \; m \cdot \left(\sum_{i=1}^{k} \frac{a_i}{2^i} \; + \; \frac{1}{2^k} \right) \right].$$

Let $d > 0$ be the distance from mr to the nearest integer, and let k be such that $m/2^k < d$. It follows that

$$m \cdot \left(r - \sum_{i=1}^{k} \frac{a_i}{2^i} \right) < \frac{m}{2^k} < d \text{ and } m \cdot \left(\sum_{i=1}^{k} \frac{a_i}{2^i} + \frac{1}{2^k} - r \right) < \frac{m}{2^k} < d.$$

Thus, the required k exists, and we can find it by searching. We compare na and mb by

$$na <_r mb \;\Leftrightarrow\; n <_{\mathbb{R}} mr \;\Leftrightarrow\; n <_{\mathbb{R}} m \cdot \sum_{i=1}^{k} \frac{a_i}{2^i}.$$

This completes both the claim and the proof for abelian groups of rank 2.

Assume G is a computable torsion free abelian group of finite rank $n > 2$. By Lemma 3.7, G has a computable Archimedean order. Let $\{e_0, \ldots, e_{n-1}\}$ be a basis for G, $\mathbf{a}$ be an arbitrary Turing degree with $A \in \mathbf{a}$ and $0 \notin A$. G is the direct product of G_1, the computable subgroup generated by $\{e_0, e_1\}$, and G_2, the computable subgroup generated by $\{e_2, \ldots, e_{n-1}\}$. By Lemma 3.7, there is a computable Archimedean order on G_2, and by the argument for groups of rank 2, there is an Archimedean order of degree $\mathbf{a}$ on G_1. Combining these orders lexicographically yields an order of degree $\mathbf{a}$ on G. □ □

Lemma 3.10. *If G is a computable torsion free abelian group with infinite rank, then G has orders of every degree $\mathbf{a} \geq_T 0'$. Furthermore, these orders can be assumed to have at most two nontrivial Archimedean classes.*

Proof. Assume G is divisible and fix a degree $\mathbf{a} \geq_T 0'$. Unlike the finite rank case, G need not have a computable basis. However, we show that G must have a basis computable in $0'$, and hence computable in $\mathbf{a}$. Let g_i, for $i \in \omega$, be an enumeration of G. Define $X \leq_T 0'$ by

$$X = \{ \langle g_0, \ldots, g_{n+1} \rangle \mid \exists \langle q_0, \ldots, q_n \rangle \in \mathbb{Q}^{n+1} (g_{n+1} = q_0 g_0 + \cdots + q_n g_n) \}.$$

Using X, we define a sequence e_i, for $i \in \omega$. Let e_0 be the ω-least nonidentity element of G and let e_{n+1} be the ω-least element of G such that $\langle e_0, \ldots, e_n, e_{n+1} \rangle \notin X$. The set $B = \{e_i | i \in \omega\}$ is a basis for G which is computable in X, and hence computable in $0'$.

As before, every $g \in G$ can be uniquely written (effectively in B) as a sum $g = \sum_{i \in I} q_i e_i$, where I is a finite set, $q_i \in \mathbb{Q}$, and $q_i \neq 0$. Split G into the direct product of G_1, the B-computable subgroup generated by $\{e_0, e_1\}$, and G_2, the B-computable subgroup generated by $\{e_i | i \geq 2\}$. By Lemma 3.7, there is an Archimedean order on G_2 which is computable from B, and hence from $\mathbf{a}$. Furthermore, by relativizing Lemma 3.9, G_1 has an order $\leq_1$ of degree $\mathbf{a}$. Let $\leq_G$ be the result of combining these orders lexicographically with the elements of G_2 Archimedean less that those of G_1. Notice that since the elements of G can be written (effectively in B) as sums of rational multiples of elements of B, this order is computable in $\mathbf{a}$.

To show that this order has degree $\mathbf{a}$, it suffices to show that G_1 is computable from $\leq_G$. For then, from $\leq_G$, we can compute $\leq_1$, which has degree $\mathbf{a}$. To compute G_1, notice that every element of G can be uniquely written as $q_0e_0 + q_1e_1 + h$, where $q_0, q_1 \in \mathbb{Q}$ and $h \in G_2$. Furthermore, since G_2 consists of the smallest Archimedean class under $\leq_G$, we have that $h \in G_2$ if and only if $|h| \leq ne_2$ for some n. Therefore, G_2 is c.e. relative to $\leq_G$ and by searching, we can write each element of G in the form $q_0e_0 + q_1e_1 + h$ effectively in $\leq_G$. Finally, notice that an element of this form is in G_1 if and only if $h = 0_G$. □ □

Theorem 3.11. *There is a Π^0_1 class C such that for any computable torsion free abelian group G*

$$\{\, deg(f) \mid f \in C \,\} \;\neq\; \{\, deg(p) \mid P \in \mathbb{X}(G) \,\}.$$

Proof. Let C be the Π^0_1 class from Theorem 1.7, and let G be any computable torsion free abelian group. By Lemmas 3.8, 3.9, and 3.10, we know that G has either only computable orders, orders of every degree, or orders of every degree above $0'$. In each of these cases, it is impossible for the set of degrees of elements of $\mathbb{X}(G)$ to be equal to the set of degrees of elements of C. □ □

We can extend this negative result to Π^0_1 classes of separating sets using the following theorem.

Theorem 3.12 (Jockusch Jr. and Soare (1972)). *There are disjoint c.e. sets A and B such that $A \cup B$ is coinfinite and, for any pair of separating sets D and E, either $D \equiv_T E$, or D and E have incomparable Turing degrees.*

Theorem 3.13. *There is a Π^0_1 class of separating sets C such that for any computable torsion free abelian group G*

$$\{\, deg(f) \mid f \in C \,\} \;\neq\; \{\, deg(p) \mid P \in \mathbb{X}(G) \,\}.$$

Proof. Let C be the Π^0_1 class from Theorem 3.12, and let G be any computable torsion free abelian group. Just as in Theorem 3.11, it is impossible for the set of degrees of elements of $\mathbb{X}(G)$ to be equal to the set of degrees of elements of C. □ □

4 Extension to nilpotent groups

In this section, we show that the negative results of Section 3 also hold for torsion free nilpotent groups.

Definition 4.1. The **center** of a group G is defined by

$$C(G) \;=\; \{\, g \in G \mid \forall x \in G\, (gx = xg) \,\}.$$

Definition 4.2. Let G be a group. The **upper central series** of G is the sequence of subgroups

$$\zeta_0(G) \leq \zeta_1(G) \leq \zeta_2(G) \leq \cdots$$

defined by $\zeta_0(G) = \langle 1_G \rangle$, $\zeta_1(G) = C(G)$, and $\zeta_{i+1}(G) = \pi_i^{-1}\big(C(G/\zeta_i(G))\big)$, where π_i is the projection function from G to $G/\zeta_i(G)$. In particular, $\zeta_{i+1}(G)/\zeta_i(G) \cong C(G/\zeta_i(G))$.

Definition 4.3. G is **nilpotent** if $\zeta_n(G) = G$ for some n. G is **properly n-step nilpotent** if $\zeta_n(G) = G$, but $\zeta_{n-1}(G) \neq G$.

A computable nilpotent group is a computable group which happens to be classically nilpotent. This condition does not imply that the terms in the upper central series are computable, as shown in the following theorem.

Theorem 4.4 (Solomon (1998)). *There is a computable 2-step nilpotent group G for which $C(G) \equiv_T 0'$.*

To prove the main result of this section, we describe a general method for building orders on a torsion free nilpotent group G. Initially, we ignore the questions of effectiveness, although we will eventually return to these questions.

Our general method works by constructing an order on each term in the upper central series for G. To define the order on $\zeta_{i+1}(G)$, we combine an order on $\zeta_i(G)$ with an order on $\zeta_{i+1}(G)/\zeta_i(G)$. Such orders can be combined only under special circumstances, which are described below. Lemma 4.5 plays a key role in this construction (see Robinson (1982)).

Lemma 4.5 (Mal'cev). *If G is an n-step torsion free nilpotent group, then for each $0 \leq i < n$, $\zeta_{i+1}(G)/\zeta_i(G)$ is a torsion free abelian group.*

Definition 4.6. Let N be a normal subgroup of G. An order $\leq_N$ on N is called a **G-order** if, for any $a, b \in N$ and $g \in G$, $a \leq_N b$ implies $gag^{-1} \leq_N gbg^{-1}$.

Any order $\leq_N$ on N must be preserved under conjugation by elements of N, but to be a G-order, $\leq_N$ must be preserved under conjugation by arbitrary elements of G. Let $P(N)$ be the positive cone of a G-order on N and $\leq_{G/N}$ be an order on the quotient group. These orders induce an order on G defined by

$$g \leq_G h \ \Leftrightarrow\ (gN <_{G/N} hN) \vee (gN = hN \wedge g^{-1}h \in P(N)). \tag{2}$$

Lemma 4.7. *If $\leq_N$ is an Archimedean order, then the normal subgroup N forms the least nontrivial Archimedean class under the induced order $\leq_G$ defined by Equation (2).*

Proof. Fix $c \neq 1_G \in P(N)$ and let $g \in G$ be arbitrary. Since the order on N is Archimedean, if $g \in N$, the there is an n such that $|g| \leq_G c^n$. On the other hand, if there is an n such that $|g| \leq_G c^n$, then by the definition of $\leq_G$, we have $|g|N \leq_{G/N} c^nN$. But $c^nN = N$, so $g \in N$. □ □

Fix a torsion free nilpotent group G, and let $\leq_{\zeta_1}$ be an order on $\zeta_1(G)$. Because $\zeta_1(G)$ is the center of G, $\leq_{\zeta_1}$ is a G-order. Next, consider $\zeta_2(G)/\zeta_1(G)$. By Lemma 4.5, this quotient group is torsion free and abelian, so it is orderable. Fix an order $\leq_{\zeta_2/\zeta_1}$ on this quotient. Because $\leq_{\zeta_1}$ is a G-order, and hence a $\zeta_2(G)$-order, we can combine the orders on $\zeta_2(G)/\zeta_1(G)$ and $\zeta_1(G)$ to get an order, $\leq_{\zeta_2}$, on $\zeta_2(G)$.

The key fact is that $\leq_{\zeta_2}$ is a G-order. To verify this statement, suppose $a <_{\zeta_2} b$. There are two cases to consider. If $a\zeta_1(G) \neq b\zeta_1(G)$, then $a\zeta_1(G) <_{\zeta_2/\zeta_1} b\zeta_1(G)$. Because $\zeta_2(G)/\zeta_1(G)$ is the center of $G/\zeta_1(G)$,

$$gag^{-1}\zeta_1(G) = a\zeta_1(G) <_{\zeta_1/\zeta_2} b\zeta_1(G) = gbg^{-1}\zeta_1(G).$$

Therefore, by the definition of $\leq_{\zeta_2}$, $gag^{-1} <_{\zeta_2} gbg^{-1}$. The second case is when $a\zeta_1(G) = b\zeta_1(G)$. In this case, $gag^{-1}\zeta_1(G) = gbg^{-1}\zeta_1(G)$. By the definition of $\leq_{\zeta_2}$, the order on gag^{-1} and gbg^{-1} is determined by $\leq_{\zeta_1}$, which we already know is a G-order.

Now, we repeat this process with $\zeta_2(G)$. By Lemma 4.5, we can fix an order $\leq_{\zeta_3/\zeta_2}$ on $\zeta_3(G)/\zeta_2(G)$. Combining this order with $\leq_{\zeta_2}$ yields a G-order on $\zeta_3(G)$. Continuing up the upper central series, we eventually construct an order on G. This method does not construct every order on G, but it does build enough orders to prove the main theorem.

Lemma 4.8. *Let G be a torsion free properly 2-step nilpotent group, and let C be the center of G. The rank of G/C is greater than or equal to 2.*

Proof. For a contradiction, suppose that the rank of G/C is 1, or equivalently, that G/C is isomorphic to a subgroup of $\mathbb{Q}$. Since G is properly 2-step nilpotent, and hence not abelian, there must be elements $a, b \in G$ such that $ab \neq ba$. Thus, neither a nor b is in C, so $aC \neq 1_G C$ and $bC \neq 1_G C$.

Since G/C is a subgroup of $\mathbb{Q}$, there must be integers $p, q \neq 0$ such that $a^p C = b^q C$. This equality implies that there is a $c \in C$ such that $a^p = b^q c$, which in turn, implies that a^p and b commute:

$$a^p b = b^q cb = b^q bc = bb^q c = ba^p.$$

Recall from group theory that the commutator $[x, y]$ is defined by $x^{-1}y^{-1}xy$. Because a^p commutes with b, we know that $[a^p, b] = 1_G$, and because a and b do not commute, we know that $[a, b] \neq 1_G$.

The following well known commutator identities, which hold for any elements x, y, z of any group, can be verified by direct calculation:

$$[xy, z] = [x, z] \cdot [[x, z], y] \cdot [y, z] \text{ and } [x, yz] = [x, z] \cdot [x, y] \cdot [[x, y], z].$$

2-step nilpotent groups have the property that all commutators are in the center. Therefore, in the context of 2-step nilpotent groups, these commutator identities become $[xy, z] = [x, z] \cdot [y, z]$ and $[x, yz] = [x, z] \cdot [x, y]$. By induction, these equations imply that $[x^n, z] = [x, z]^n$ and $[x, z^n] = [x, z]^n$ for all $n \in \omega$. Combining these equalities with the identity $[x^{-1}, z] = [z, x]$ (which holds in any group), shows that $[x^p, z] = [x, z]^p$ for any $x, z \in G$ and any $p \in \mathbb{Z}$.

Returning to the context of our proof, we know that $[a^p, b] = 1_G$, and hence, $[a, b]^p = 1_G$. However, because G is torsion free, it follows that $[a, b] = 1_G$, which contradicts our choice of a and b as noncommuting elements. □ □

We can now turn to questions about the effectiveness of this procedure. First, we must be more specific about our method of representing quotient groups. Let G be a computable group and H be a computable normal subgroup of G. The elements of G/H are the ω-least representatives of each coset. Since $gH = xH$ if and only if $g^{-1}x \in H$,

$$G/H \ = \ \{ \, g \in G \,|\, \forall x <_\omega g\big(x \notin G \vee g^{-1}x \in H \, \}.$$

Multiplication in G/H is defined by first multiplying in G, and then picking the least representative for the resulting coset. It is clear from this definition that G/H is a computable group. Furthermore, if $\mathbf{a}$ is any Turing degree and G and H are $\mathbf{a}$-presented groups, then G/H is $\mathbf{a}$-presented as well.

Definition 4.9. A subgroup H of G is **convex** under the order $\leq_G$ if for all $a, b \in H$ and all $g \in G$, $a \leq_G g \leq_G b$ implies $g \in H$.

If H is a convex normal subgroup of $(G, \leq_G)$, then the induced order $\leq_{G/H}$ on G/H is defined by

$$aH \leq_{G/H} bH \leftrightarrow aH = bH \vee (aH \neq bH \wedge a <_G b).$$

If G is a computable group, then this definition shows that the induced order $\leq_{G/H}$ is computable from H and $\leq_G$.

We now have two types of induced orders: those induced by $\leq_G$ on G/H when H is a convex normal subgroup and those induced by $\leq_H$ and $\leq_{G/H}$ on G when $\leq_H$ is a G-order. These induced orders are inverses in the sense that if $\leq_G$ is induced by $\leq_H$ and $\leq_{G/H}$, then H is a convex subgroup of G and the order induced by $\leq_G$ on G/H is exactly $\leq_{G/H}$.

Lemma 4.10. *If G is a computable torsion free properly 2-step nilpotent group, then G has orders of every Turing degree above $0''$.*

Proof. Fix such a group G, and let C denote the center of G. Since C is definable from G by a Π^0_1 formula, $C \leq_T 0'$. Therefore, C has a basis computable in $0''$ and, by Lemma 3.7, C has an Archimedean order $\leq_C$ (with positive cone $P(C)$) which is computable in $0''$. Since C is the center, $\leq_C$ is a G-order. By Lemmas 4.5 and 4.8, G/C is a $0'$-presented torsion free abelian group with $\text{rank}(G/C) \geq 2$. Relativizing Lemmas 3.9 and 3.10, we know that G/C has orders of every degree $\mathbf{a} \geq_T 0''$. Fix $\mathbf{a} \geq_T 0''$, let $\leq_{G/C}$ be an order of this degree, and let $\leq_G$ be the order induced by $\leq_C$ and $\leq_{G/C}$ as defined by Equation (2).

The fact that $\deg(\leq_G) \leq_T \mathbf{a}$ is clear. It remains to show that $\mathbf{a} \leq_T \deg(\leq_G)$. By Lemma 4.7, we know that C forms the least nontrivial Archimedean class under $\leq_G$. Hence, C is Σ_1 relative to the order $\leq_G$. Also, since C is the center of G, C is Π_1 definable. Therefore, C is computable in $\leq_G$, and so are both G/C and the induced order $\leq_{G/C}$ on the this quotient. By assumption, $\deg(\leq_{G/C}) = \mathbf{a}$, which completes the proof of this lemma. □ □

Lemma 4.11 will allow us to extend this result to all torsion free nilpotent groups. For a proof of this lemma, see Robinson (1982).

Lemma 4.11. *If $m \geq 3$ and G is a properly m-step nilpotent group, then $G/\zeta_{m-2}(G)$ is a properly 2-step nilpotent group.*

Lemma 4.12. *If $m \geq 2$ and G is a computable torsion free properly m-step nilpotent group, then G has orders of every degree above $0^{(m)}$. ($0^{(m)}$ denotes the m^{th} Turing jump of the empty set).*

Proof. The case for $m = 2$ was handled in Lemma 4.10, so we assume $m \geq 3$. As above, $\zeta_1(G)$ is a $0'$-presented group. For any $1 \leq i < m$, we have

$$\zeta_{i+1}(G) \;=\; \{\, g \in G \mid \forall h \in G\big(gh\zeta_i(G) = hg\zeta_i(G)\big) \,\}.$$

Equality between these cosets is equivalent to $(gh)^{-1}hg \in \zeta_i(G)$, so $\zeta_{i+1}(G)$ is definable from $\zeta_i(G)$ by a Π^0_1 formula. By induction, it follows that $\zeta_{i+1}(G) \leq_T 0^{(i+1)}$. Therefore, each $\zeta_{i+1}(G)/\zeta_i(G)$ is a $0^{(i+1)}$-presented torsion free abelian group and each has a basis computable in $0^{(i+2)}$. By Lemma 3.7, each $\zeta_{i+1}(G)/\zeta_i(G)$ admits an Archimedean order computable in $0^{(i+2)}$.

We apply our general method to construct an order on G. Fix Archimedean orders $\leq_{\zeta_1}$ and $\leq_{\zeta_2/\zeta_1}$ on $\zeta_1(G)$ and $\zeta_2(G)/\zeta_1(G)$ which are computable in $0^{(2)}$ and $0^{(3)}$, respectively. Combining these orders, as in Equation (2), yields a G-order $\leq_{\zeta_2}$ on $\zeta_2(G)$, which is computable in $0^{(3)}$ and under which $\zeta_1(G)$ is convex. Continuing this process for each $\zeta_i(G)$ up to $\zeta_{m-2}(G)$, we define a G-order $\leq_{\zeta_i}$ which is computable in $0^{(i+1)}$ and under which $\zeta_{i-1}(G)$ is convex.

We now have a G-order $\leq_{\zeta_{m-2}}$ on $\zeta_{m-2}(G)$ which is computable from $0^{(m-1)}$. Also, we know that $\zeta_{m-2}(G)$ is $0^{(m-2)}$-presented, and so $G/\zeta_{m-2}(G)$ is a $0^{(m-2)}$-presented torsion free properly 2-step nilpotent group. Relativizing Lemma 4.10 to $0^{(m-2)}$, we see that $G/\zeta_{m-2}(G)$ has orders of every degree $\mathbf{a} \geq 0^{(m)}$. Fix such a degree $\mathbf{a}$, and let $\leq_{G/\zeta_{m-2}}$ be an order of that degree. Let $\leq_G$ be the order on G formed by combining $\leq_{\zeta_{m-2}}$ with $\leq_{G/\zeta_{m-2}}$ as in Equation (2).

Notice that for each $i \leq m-2$, $\zeta_i(G)$ is a convex subgroup of G. Therefore, $C(G/\zeta_i(G))$, which is isomorphic to $\zeta_{i+1}(G)/\zeta_i(G)$, is a convex subgroup of $G/\zeta_i(G)$. Furthermore, the restriction of the induced order $\leq_{G/\zeta_i}$ to $\zeta_{i+1}(G)/\zeta_i(G)$ is equal to $\leq_{\zeta_{i+1}/\zeta_i}$, and hence $C(G/\zeta_i(G))$ consists of the least nontrivial Archimedean class of $G/\zeta_i(G)$.

The fact that $\deg(\leq_G) \leq_T \mathbf{a}$ is clear. It remains to show that $\mathbf{a} \leq_T \deg(\leq_G)$. We proceed as in Lemma 4.10. $\zeta_1(G)$ is both the center of G and the least Archimedean class under $\leq_G$, and hence is both Π_1 and Σ_1 in $\leq_G$. Therefore, $\zeta_1(G)$ is computable from $\leq_G$ and so are both $G/\zeta_1(G)$ and the induced order $\leq_{G/\zeta_1}$ on this quotient group. Furthermore, $g \in \zeta_2(G)$ if and only if $g \in C(G/\zeta_1(G))$. Therefore $\zeta_2(G)$ is Π_1 in $\zeta_1(G)$, and hence is Π_1 in $\leq_G$. Since $C(G/\zeta_1(G))$ is the least Archimedean class under $\leq_{G/\zeta_1}$, it is Σ_1 in $\leq_{G/\zeta_1}$, and hence is Σ_1 in $\leq_G$. Together, these facts imply that $C(G/\zeta_1(G))$ (and hence $\zeta_2(G)$) are computable from $\leq_G$. By induction, it is clear that $\zeta_{m-2}(G)$ and $\leq_{G/\zeta_{m-2}}$ are computable from $\leq_G$. However, $\deg(\leq_{G/\zeta_{m-2}}) = \mathbf{a}$, which completes the proof of this lemma. □ □

Theorem 4.13. *There is a Π^0_1 class of separating sets C such that for any torsion free nilpotent group G*

$$\{\, deg(f) \mid f \in C \,\} \;\neq\; \{\, deg(p) \mid P \in \mathbb{X}(G) \,\}.$$

Proof. If G is abelian, the theorem follows from Theorem 3.13. Otherwise, G must be properly n-step nilpotent for some $n > 1$. The theorem now follows from Lemmas 4.10 and 4.12 and the proof of Theorem 3.13. □ □

5 Computable presentations

In this section, we turn our attention to the question of whether every orderable computable group is classically isomorphic to a computably orderable computable group. In other words, we ask whether every orderable computable group has a computable presentation which admits a computable order. Downey and Kurtz originally asked this question because in their example of a computable torsion free abelian group G with no computable order, G is classically isomorphic to $\Sigma_{i=1}^{\omega}\mathbb{Z}_i$ (see Theorem 1.4). With the right presentation, this group obviously has a computable order. Downey and Kurtz were interested in the answer to this question for groups in general, not just abelian groups. The general question is still open, but, in this section, we point out a simple answer for the special case of abelian groups.

Lemma 5.1. *Every computable torsion free abelian group of finite rank has a computable order.*

Proof. This theorem follows immediately from Lemma 3.7. □ □

We know that Theorem 5.1 is not true for all groups of infinite rank. However, the answer to our question about presentations follows directly from Lemma 3.7 and the following result.

Theorem 5.2 (Dobritsa (1983)). *Let G be a computable torsion free abelian group. There is a computable group H which is classically isomorphic to G and has a computable basis.*

Theorem 5.3. *Every computable torsion free abelian group is classically isomorphic to a computable group with a computable order.*

Proof. Let G be a computable torsion free abelian group. Lemma 5.1 has already handled the case when G has finite rank. If G has infinite rank, then let H be as in Theorem 5.2. By Lemma 3.7, H admits a computable order. □ □

6 Representatives for Archimedean classes

In this section, we show that the analogue of Theorem 5.2 for the Archimedean classes of an ordered group fails.

Definition 6.1. Let G be an ordered group. The subset $U \subseteq G$ is a **set of unique Archimedean representatives** for G if, for any $u \neq v \in U$, $u \not\approx v$, and, for every $g \in G$, there is a $u \in U$ with $u \approx g$.

For any computable ordered group G and any set U of unique Archimedean representatives for G, the set $\{\langle g, h\rangle | g \approx h\}$ is computable in U. Also, for any such G and U, there is a set V of unique Archimedean representatives such that $V \equiv_T U$ and $V \subseteq P(G)$. Therefore, up to Turing degree, we can always assume that our Archimedean representatives are positive.

The goal of this section is to show that not every computable ordered group has a computable presentation which admits a computable set of unique Archimedean representatives. The proof requires some background on the presentations of linear orders.

A computable presentation of a linear order L is an isomorphic copy of L in which both the domain and the ordering are computable. To generalize this definition, we allow the underlying equality in the presentation to be Σ^0_1 rather than computable.

Definition 6.2. Let L be a countable linear order. A $\mathbf{\Sigma^0_1}$**-presentation** of L is given by a computably enumerable set $\leq_Z \subset \omega \times \omega$ such that the relation $\approx_Z$, defined by $n \approx_Z m$ if and only if $n \leq_Z m$ and $m \leq_Z n$, is an equivalence relation, and L is isomorphic to $\langle \omega / \approx_Z, \leq_Z \rangle$.

Definition 6.3. L is **computably presentable** ($\mathbf{\Sigma^0_1}$**-presentable**) if there is a computable presentation (Σ^0_1-presentation, respectively) of L.

For a survey of the known results about presentations of linear orders, see Downey (1999). For our purposes, the most important result is Feiner's Theorem.

Theorem 6.4 (Feiner (1970)). *There is a Σ^0_1-presented order L which is not computably presentable.*

The intuition for a Σ^0_1-presented linear order L is that the elements of L are enumerated along with their ordering relations, but occasionally two apparently distinct elements collapse to a single point. Stated slightly more formally, given any $n, m \in \omega$, we can enumerate $\leq_L$ until we see either $n \leq_L m$ or $m \leq_L n$. What we cannot determine computably is whether n and m are equal in L. If they are equal, this information will eventually show up, but we cannot wait for it.

For our purposes, it will be better not to think of collapsing the elements of L as we discover they are equal, but rather, just to group the equal elements together in equivalence classes. To distinguish natural numbers from elements of L, we will use l_i to denote the elements of L. We think of L as enumerated in stages. At stage 0, we have l_0. Assume that at stage t we have the elements $l_0, \ldots l_t$ ordered (and grouped according to our current knowledge of equalities) as

$$l_{i_0} \approx_L l_{i_1} \approx_L \cdots \approx_L l_{i_{j_0}} \leq_L l_{i_{j_0+1}} \approx_L l_{i_{j_0+2}} \approx_L \cdots \approx_L l_{i_{j_1}} \leq_L \cdots$$
$$\cdots \leq_L l_{i_{j_{k-1}}} \approx_L \cdots \approx_L l_{i_{j_k}}.$$

The particular order in which we write each set of L-equivalent elements will be important. Consider the element l_{t+1}. By enumerating $\leq_L$, we can figure out where l_{t+1} fits in this order, but in doing so, we may collapse some of the inequalities. If we discover a new equality among $l_0, \ldots, l_t$, then we replace the appropriate $\leq_L$ by $\approx_L$, leaving the elements within the

L-equivalence classes in the same order. Our only stipulation about placing l_{t+1} in this order is that we do not place it in the middle of a string of L-equivalent elements. If l_{t+1} belongs between $l_n \approx_L l_m$, then we put l_{t+1} at the left end of the collection of elements which we currently know are L-equivalent to l_n.

The are two important features of this method of enumerating L. First, we determine a fixed order among the L-equivalent elements, and second, once we know $l_n \approx_L l_m$, we never place an L-equivalent element between them.

Theorem 6.5. *There is a computable ordered abelian group $(G, \leq_G)$ such that no computable presentation of $(G, \leq_G)$ admits a computable set of unique representatives for its Archimedean classes.*

The remainder of this section is devoted to the proof of Theorem 6.5. We let L be the Σ^0_1-presented order from Feiner's Theorem, and as above, we use l_i to denote the elements of L. Our strategy is to build a computable ordered group G such that, for any set $U \subseteq P(G)$ of unique Archimedean representatives, $\langle U \setminus \{0_G\}, \leq_G\rangle$ is isomorphic (as a linear order) to L. If such a set U were computable in any computable presentation of G, then we would have a contradiction to Feiner's Theorem.

We build G from an infinite set of generators, a_i for $i \in \omega$, using a computable set of relations of the form $p_k a_i = a_j$ for some primes p_k. The elements of G are formal sums $\sum_{I\subset\omega} c_i a_i$, where I is a finite set, $c_i \in \mathbb{Z}\setminus\{0\}$, and none of our relations can be used to reduce the sum (this definition is given formally below).

The intuition is that each a_i starts out representing its own Archimedean class, which is intended to correspond to $l_i \in L$. If at some some stage, we discover that $l_i \approx l_j$, then we collapse the Archimedean classes of a_i and a_j by picking a large number n and adding the relation $na_i = a_j$ to our group. This relation forces a_i and a_j into the same Archimedean class.

Construction of the set of relations

Stage 0: We start with one generator a_0 and no relations.

Stage $\mathbf{t+1}$: Assume that we have the generators $a_0, \ldots, a_t$, with the corresponding elements $l_0, \ldots l_t$ of L ordered as

$$l_{i_0} \approx_L l_{i_1} \approx_L \cdots \approx_L l_{i_{j_0}} \leq_L l_{i_{j_0+1}} \approx_L l_{i_{j_0+2}} \approx_L \cdots \approx_L l_{i_{j_1}} \leq_L \cdots$$
$$\cdots \leq_L l_{i_{j_{k-1}}} \approx_L \cdots \approx_L l_{i_{j_k}}.$$

First, add l_{t+1} to the order on the elements of L considered so far, as described above. Second, check the enumeration of $\leq_L$ to see if there are any new equalities between the elements $l_0, \ldots, l_{t+1}$. If not, proceed to stage $t+2$.

For each new equality $l_{i_k} \approx_L l_{i_{k+1}}$ between adjacent elements in our fixed order on $l_0, \ldots, l_{t+1}$, we need to force the corresponding generators a_{i_k} and $a_{i_{k+1}}$ into the same Archimedean class. Let p be the least odd prime greater than $2t$ which has not been used, and add $pa_{i_k} = a_{i_{k+1}}$ to our set of relations. Proceed to stage $t+2$.

End of construction

Lemma 6.6. *The set of relations is computable.*

Proof. To decide if $pa_i = a_j$ is in our set of relations, go to a stage $t+1$ such that $i, j \leq t+1$ and $p \leq 2t$. If this relation has not appeared by this stage, then it is not in our set. □ □

Definition 6.7. A **reduced sum** is a formal sum $\sum_I c_i a_i$, where $I \subset \omega$ is finite, each c_i is in $\mathbb{Z} \setminus \{0\}$, and there is no $i \in I$, $k \in \omega$, and prime $p < 2|c_i|$ for which $pa_i = a_k$ holds.

If $pa_i = a_k$ is in our set of relations and $p < 2|c_i|$, then this relation must enter by stage $|c_i|$ of the construction. Therefore, $k \leq |c_i|$. This bound on k implies that the set of reduced sums is computable. Furthermore, there is an obvious procedure for reducing any formal finite sum to a reduced sum by applying a sequence of our relations.

The elements of G are exactly the reduced sums, and addition in G is defined by adding the sums componentwise and reducing the result. The identity element is the reduced sum in which $I = \emptyset$. To show that G is a group, it suffices to show that no two reduced sums are equivalent by a sequence of applications of our relations.

Lemma 6.8. *If $x = \sum_I c_i a_i$ and $y = \sum_J d_j a_j$ are distinct reduced sums, then there is no sequence of applications of our relations which takes x to y.*

Proof. At stage $t+1$, we do not place l_{t+1} between two elements we know are equal. Therefore, each a_i is involved in at most two relations, $pa_j = a_i$ and $qa_i = a_k$, where p and q may or may not be the same prime. To prove the lemma, assume that there is a sequence of reduction relations taking x to y. We derive a contradiction by induction on the number of applications of our relations in such a sequence.

The base case is when the sequence of reduction relations has length one. Without loss of generality, suppose the only rule applied is $pa_j = a_i$. Let c_j denote the coefficient of a_j in x if $j \in I$ and $c_j = 0$ otherwise. Since x is in reduced form, $|c_j| < p/2$. After applying this relation, we see that the coefficient d_j of a_j in y is either $c_j + p$ or $c_j - p$. In either case, $j \in J$ and $|d_j| > p/2$, which contradicts the fact that y is a reduced sum.

For the induction case, assume that x is not equivalent to y via any sequence of reduction relations of length n. We need to show that no such sequence of length $n+1$ suffices. Without loss of generality, assume the first relation applied is $pa_{j_0} = a_{i_0}$, and split into the following three subcases.

For the first subcase, assume this relation is applied again in the reduction in the opposite direction. That is, if the relation is originally applied to change c_{j_0} to $c_{j_0} + p$, then later it is applied to alter the coefficient of a_{j_0} by subtracting p (or vice versa). In this case, we can remove both occurrences of this relation to obtain a reduction sequence from x to y of length $n-1$. This sequence violates the induction hypothesis.

For the second subcase, assume that we are not in the first subcase, and that the first relation is applied to change the coefficient of a_{j_0} to $c_{j_0} - p$. The coefficient of a_{j_0} is no longer in reduced form. Since we are not in the first subcase and since a_{j_0} is involved in only two relations, there is a unique relation of the form $qa_{j_1} = a_{j_0}$ that can be applied to fix the coefficient of a_{j_0}. Furthermore, this relation must be applied to increase the coefficient of a_{j_0}, and hence subtract q from the coefficient of a_{j_1}. However, after applying this relation, the coefficient of a_{j_1} is not reduced. Either we are in the first subcase with respect to a_{j_1}, or we are in this second subcase with respect to a_{j_1}. If the former occurs, we can remove the two occurrences of the relation $qa_{j_1} = a_{j_0}$ as above, contradicting the induction hypothesis. If the latter occurs, then we repeat the argument in this subcase. Either we find a relation with respect to which we are in the first subcase, or else, after applying our sequence of relations, the sum is not in reduced form.

For the third subcase, assume we are not in either of the first two subcases. Then, the first relation must change the coefficient of a_{j_0} from c_{j_0} to $c_{j_0} + p$. The coefficient of a_{j_0} is too large to be in reduced form, so there is a unique relation of the form $qa_{j_1} = a_{j_0}$ that can be used to fix it. We now repeat the argument from the second subcase. Either we are in the first subcase with respect to a_{j_1} (in which case we get a contradiction as above), or we are in the third subcase with respect to a_{j_1}. We repeat this process, and either find a relation with respect to which we are in the first subcase, or after applying our sequence of relations, the sum is not in reduced form. □ □

We still need to specify the order on G. Let $x = \sum_I c_i a_i$ and $y = \sum_J d_j a_j$ be distinct reduced sums. Let $K = I \cup J$, $c_k = 0$ for $k \in K \setminus I$, $d_k = 0$ for $k \in K \setminus J$, and let t be the ω-largest element of K. At stage t, we have an order on $l_0, \ldots, l_t$ as

$$l_{i_0} \leq_L l_{i_1} \leq_L \cdots \leq_L l_{i_t},$$

where some of the $\leq_L$ may actually be $\approx_L$. Let i_k be such that l_{i_k} is the right most spot in this order at which $c_{i_k} \neq d_{i_k}$. Define $x <_G y$ if and only if $c_{i_k} <_{\mathbb{Z}} d_{i_k}$. The following two lemmas are useful for showing that $\leq_G$ defines an order on G.

Lemma 6.9. *Let $p_1, \ldots, p_n$ be odd primes, and let $c_1, \ldots c_n, d_1, \ldots, d_n$ be integers such that $|c_i| < p_i/2$ and $|d_i| < p_i/2$ for each i. If $c_n < d_n$, then $C < D$, where*

$$C = c_1 + c_2 p_1 + c_3 p_1 p_2 + \cdots + c_n \prod_{i=1}^{n-1} p_i$$

$$\text{and} \quad D = d_1 + d_2 p_1 + d_3 p_1 p_2 + \cdots + d_n \prod_{i=1}^{n-1} p_i.$$

Proof. The bounds $|c_i| < p_i/2$ imply that

$$\left| c_1 + c_2 p_1 + c_3 p_1 p_2 + \cdots + c_{n-1} \prod_{i=1}^{n-2} p_i \right| \leq$$

$$\frac{p_1 - 1}{2} + \frac{p_2 - 1}{2} p_1 + \frac{p_3 - 1}{2} p_1 p_2 + \cdots + \frac{p_{n-1} - 1}{2} \prod_{i=1}^{n-2} p_i.$$

The right hand side of this inequality is a telescoping sum which adds to $(\prod_{i=1}^{n-1} p_i)/2 - 1/2$. The corresponding sum with the d_i coefficients is similarly bounded. These bounds yield the following two strict inequalities:

$$C \le c_n \left(\prod_{i=1}^{n-1} p_i\right) + \left(\frac{\prod_{i=1}^{n-1} p_i}{2} - \frac{1}{2}\right) < \left(\prod_{i=1}^{n-1} p_i\right)(c_n + 1/2)$$

$$D \ge d_n \left(\prod_{i=1}^{n-1} p_i\right) - \left(\frac{\prod_{i=1}^{n-1} p_i}{2} - \frac{1}{2}\right) > \left(\prod_{i=1}^{n-1} p_i\right)(d_n - 1/2).$$

Since $c_n, d_n \in \mathbb{Z}$ and $c_n < d_n$, we have $c_n + 1/2 \le d_n - 1/2$, so the lemma follows. □ □

We say that a_i and a_j are related by a product of primes if for some $n > 0$, either $a_i = na_j$ or $a_j = na_i$ is a consequence of our set of relations.

Lemma 6.10. *If $l_i \approx l_j$, then a_i and a_j are related by some product of primes.*

Proof. Assume $l_i \approx_L l_j$. Since L is Σ^0_1-presented, we discover this fact at some finite stage $t \ge i, j$. At this stage, we have an order on $l_0, \ldots, l_t$, and we assume without loss of generality that l_i is to the left of l_j. If l_i and l_j are adjacent elements, then by construction, we include a relation of the form $pa_i = a_j$. If l_i and l_j are not adjacent elements, then we know that the finite set of elements between l_i and l_j are all L-equivalent. Consider an arbitrary adjacent pair l_n and l_m between l_i and l_j. Either we previously knew that $l_n \approx_L l_m$, in which case we have a relation of the form $pa_n = a_m$ in our set, or we have just learned that $l_n \approx_L l_m$, in which case we put a relation of the form $pa_n = a_m$ in our set. It follows that for every adjacent pair l_n and l_m between l_i and l_j, the generators a_n and a_m are related by some prime. Therefore, a_i and a_j are related by the product of these primes. □ □

Lemma 6.11. *$\le_G$ is an order on G.*

Proof. To make the notation simpler for this proof, we denote l_i by just i, but continue to use a_i to designate the corresponding generator in G. For $k \in \omega$, let $[k]$ denote the ω-least element of the equivalence class of k under $\approx_L$, and let $[L]$ denote the set of all $[k]$. Thus, $[L] \cong L/\approx_L$, with the order induced by $\le_L$.

Consider the ordered group $\sum_{[L]} \mathbb{Q}$. The elements of $\sum_{[L]} \mathbb{Q}$ are functions from $[L]$ into $\mathbb{Q}$ which are nonzero at only finitely many places. These functions are ordered by comparison at the $\le_L$-greatest element of $[L]$ at which they differ. To prove the lemma, it suffices to define a group homomorphism from G into $\sum_{[L]} \mathbb{Q}$ and to show that this homomorphism preserves $<_G$. The appropriate properties of $\le_G$ then follow from the corresponding properties in $\sum_{[L]} \mathbb{Q}$.

By definition, $i \approx_L [i]$ for each $i \in L$, so it follows from Lemma 6.10 that a_i and $a_{[i]}$ are related by some product of primes. That is, either $n_i a_{[i]} = a_i$ or $a_{[i]} = n_i a_i$ for some $n_i \in \omega$. Let $\varphi : G \to \sum_{[L]} \mathbb{Q}$ be the map defined as follows. For $k \in [L]$, φ sends a_k to the function $f_k : [L] \to \mathbb{Q}$ such that $f_k(k) = 1$ and $f_k(n) = 0$ for all $n \in [L]$ with $n \neq k$. For a_i with $i \notin [L]$, φ sends a_i to the function f_i such that $f_i(n) = 0$ for $n \in [L]$ with $n \neq [i]$, $f([i]) = n_i$

if $a_i = n_i a_{[i]}$, and $f([i]) = 1/n_i$ if $n_i a_i = a_{[i]}$. Notice that with these definitions, φ respects the relations between the generators of G. Therefore, φ can be extended to a homomorphism from G to $\sum_{[L]} \mathbb{Q}$.

It remains to show that $g <_G h$ implies $\varphi(g) < \varphi(h)$ under the order on $\sum_{[L]} \mathbb{Q}$. Consider the reduced sums $g = \sum_I c_i a_i$ and $h = \sum_J d_j a_j$. Let t be the ω-largest element of $I \cup J$. For all $s \leq t$, set $c_s = 0$ if $s \notin I$ and set $d_s = 0$ if $s \notin J$. At stage t, there is an ordering on $0, 1, \ldots, t$ as elements of L, $i_0 \leq_L i_1 \leq_L \cdots \leq_L i_t$, where, as usual, some of the $\leq_L$ relations may be $\approx_L$. Assume that in the definition of $\leq_G$, we compare g and h at i_k. This implies that $c_{i_k} < d_{i_k}$ and $c_{i_s} = d_{i_s}$ for all $k < s \leq t$.

By the definition of φ, the order relationship between $\varphi(g)$ and $\varphi(h)$ is determined by their values in the $[i_k]$ component of $\sum_{[L]} \mathbb{Q}$. Therefore, we need to consider the indices $i_{k-n}, \ldots, i_k$ which are equal in L in the end. That is, we need to look at more than the just the indices which are equal at stage t. It suffices to look at these indices, because for any other index, either the coefficients in g and h are equal, or the index corresponds to a generator which is mapped to an Archimedean smaller component in $\sum_{[L]} \mathbb{Q}$. Assume we have

$$i_{k-n} \approx_L i_{k-n+1} \approx_L \cdots \approx_L i_k,$$

and $i_{k-n-1} \not\approx_L i_{k-n}$. To show $\varphi(g) < \varphi(h)$, it suffices to show the $[i_k]$ component of $\varphi(g')$ is less than the $[i_k]$ component of $\varphi(h')$ where

$$\begin{aligned} g' &= c_{i_{k-n}} a_{i_{k-n}} + c_{i_{k-n+1}} a_{i_{k-n+1}} + \cdots + c_{i_k} a_{i_k} \\ \text{and} \quad h' &= d_{i_{k-n}} a_{i_{k-n}} + d_{i_{k-n+1}} a_{i_{k-n+1}} + \cdots + d_{i_k} a_{i_k}. \end{aligned}$$

Since $i_{k-n}, \ldots, i_k$ are equal in L, the corresponding generators are related by certain primes:

$$p_{j_{k-n}} a_{i_{k-n}} = a_{i_{k-n+1}}, \; p_{j_{k-n+1}} a_{i_{k-n+1}} = a_{i_{k-n+2}}, \ldots, p_{j_{k-1}} a_{i_{k-1}} = a_{i_k}.$$

The fact that g and h are represented by reduced sums implies that $|c_{i_{k-x}}| < p_{j_{k-x}}/2$ and $|d_{i_{k-x}}| < p_{j_{k-x}}/2$ for all $1 \leq x \leq n$. Assume without loss of generality that the $[i_k]$ component of $\varphi(a_{i_{k-n}})$ is 1. (The general case follows by multiplying or dividing by the actual value of this component.) The $[i_k]$ components of $\varphi(g')$ and $\varphi(h')$ are, respectively,

$$\begin{aligned} & c_{i_{k-n}} + c_{i_{k-n+1}} p_{j_{k-n}} + c_{i_{k-n+2}} p_{j_{k-n}} p_{j_{k-n+1}} + \cdots + c_{i_k} \prod_{x=1}^{n} p_{j_{k-x}} \\ \text{and} \quad & d_{i_{k-n}} + d_{i_{k-n+1}} p_{j_{k-n}} + d_{i_{k-n+2}} p_{j_{k-n}} p_{j_{k-n+1}} + \cdots + d_{i_k} \prod_{x=1}^{n} p_{j_{k-x}} \end{aligned}$$

Since $|c_{i_{k-x}}| < p_{j_{k-x}}/2$ and $|d_{i_{k-x}}| < p_{j_{k-x}}/2$ for all $1 \leq x \leq n$, and also $c_{i_k} < d_{i_k}$, Lemma 6.9 can be applied with the primes $p_{j_{k-n}}, \ldots, p_{j_{k-1}}$. This application of Lemma 6.9 shows that the $[i_k]$ component of $\varphi(g)$ is less than the $[i_k]$ component of $\varphi(h)$, which completes the proof of the lemma. □ □

Lemma 6.12. *For every i and j, $a_i \approx a_j$ if and only if $l_i \approx_L l_j$.*

Proof. The fact that $l_i \approx_L l_j$ implies $a_i \approx a_j$ follows directly from Lemma 6.10. For the other direction, assume that $a_i \approx a_j$, and let φ be as in Lemma 6.11. From $a_i \approx a_j$, it follows that $a_{[i]} \approx a_{[j]}$. Since φ is order preserving, $\varphi(a_{[i]}) \approx \varphi(a_{[j]})$. From the definition of φ and the order on $\Sigma_{[L]}\mathbb{Q}$, it follows that for any n and m, $\varphi(a_n) \approx \varphi(a_m)$ if and only if $l_n \approx_L l_m$. Therefore, $l_{[i]} \approx_L l_{[j]}$, which implies $[i] = [j]$ and $l_i \approx_L l_j$. □ □

Lemma 6.13. *For any ordered group $H \cong G$ and any set of unique Archimedean representatives $U \subset P(H)$, $\langle U \setminus \{0_H\}, \leq_H \rangle$ is isomorphic to L as a linear order.*

Proof. This lemma follows immediately from Lemma 6.12. It also completes the proof of Theorem 6.5, since if H and U were both computable, then $\langle U \setminus \{0_H\}, \leq_H \rangle$ would be a computable presentation on L, contradicting Feiner's Theorem. □ □

References

Cenzer, D. and J. Remmel (1999): "Π^0_1 classes in mathematics," in Y. Ershov, S. Goncharov, A. Nerode and J. Remmel, eds., *Handbook of recursive mathematics, Volume 2*, pp. 623–821, North Holland.

Dobritsa, V. (1983): "Some constructivizations of abelian groups," *Siberian Math. J.*, 24, no. 2: pp. 167–173.

Downey, R. (1999): "Computability theory and linear orderings," in Y. Ershov, S. Goncharov, A. Nerode and J. Remmel, eds., *Handbook of recursive mathematics, Volume 2*, pp. 823–976, North Holland.

Downey, R. and S. A. Kurtz (1986): "Recursion theory and ordered groups," *Ann. Pure Appl. Logic*, 32: pp. 137–151.

Feiner, L. (1970): "Hierarchies of Boolean algebras," *J. Symbolic Logic*, 35: pp. 365–374.

Fuchs, L. (1963): *Partially ordered algebraic systems*, Pergamon Press.

Hatzikiriakou, K. and S. G. Simpson (1990): "WKL_0 and orderings of countable abelian groups," *Contemporary Mathematics*, 106: pp. 177–180.

Jockusch Jr., C. G. and R. I. Soare (1972): "Π^0_1 classes and degrees of theories," *Trans. Amer. Math. Soc*, 173: pp. 33–56.

Kokorin, A. and V. Kopytov (1974): *Fully ordered groups*, Halsted Press.

Metakides, G. and A. Nerode (1979): "Effective content of field thoery," *Ann. Math. Logic*, 17: pp. 289–320.

Rabin, M. (1960): "Computable algebra, general theory and theory of computable fields," *Trans. Amer. Math. Soc.*, 95: pp. 341–360.

Robinson, D. J. (1982): *A course in the theory of groups*, Springer–Verlag.

Simpson, S. G. (1999): *Subsystems of second order arithmetic*, Springer–Verlag.

Smith, R. L. (1981): "Two theorems on autostability in p–groups," in A. Dold and B. Eckmann, eds., *Logic Year 1979–80*, vol. 859 of *Lecture Notes In Mathematics*, pp. 302–311, Springer–Verlag.

Soare, R. I. (1987): *Recursively enumerable sets and degrees*, Springer–Verlag.

Solomon, R. (1998): "Reverse mathematics and fully ordered groups," *Notre Dame J. Formal Logic*, 39, no. 2: pp. 157–189.

Teh, H. (1960): "Construction of orders in abelian groups," *Cambridge Philosophical Society Proceedings*, 57: pp. 476–482.