REVERSE MATHEMATICS, ARCHIMEDEAN CLASSES, AND HAHN'S THEOREM

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Abstract. Archimedean classes and convex subgroups play important roles in the study of ordered groups. In this paper, we show that ACA_0 is equivalent to the existence of a set of representatives for the Archimedean classes of an ordered abelian group. Hahn's Theorem is the strongest known tool for classifying orders on abelian groups. It states that every ordered abelian group can be embedded into products of the additive group of the reals. We show that Hahn's Theorem is also equivalent to ACA_0.

§1. Introduction. The fundamental question in reverse mathematics is to determine which set existence axioms are required to prove particular theorems of ordinary mathematics. In this article, we consider Hahn's Theorem, one of the central results of ordered abelian group theory. This article is self-contained with respect to the material on ordered groups (see Section 2), but the reader who is unfamiliar with reverse mathematics is referred to [8] or [3] for more background in this area.

The work in this paper continues a line of inquiry into the computational and proof theoretic properties of ordered abelian groups started in [2], and continued in [5], [9], and [11]. Downey and Kurtz began this study in [2] by showing that the effective versions of certain classical theorems about ordered abelian groups failed. For example, while it is classically true that every torsion free abelian group is orderable, this result is not effectively true.

Theorem 1.1 (Downey, Kurtz). *There is a computable torsion free abelian group which does not admit a computable order.*

In [5], Hatzikiriakou and Simpson strengthen this result by determining the exact proof theoretic strength of the theorem that every torsion free abelian group is orderable.

Theorem 1.2 (Hatzikiriakou, Simpson). *(RCA_0) The following are equivalent.*

1. *WKL_0.*
2. *Every torsion free abelian group is orderable.*

In this article, we consider one of the deepest results in ordered abelian group theory. Hahn's Theorem states that every ordered abelian group can be embedded into a lexicographically ordered subgroup of some large sum of the additive

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reals. The background in ordered group theory necessary for a formal statement of this theorem is presented in Section 2.

One of the fundamental notions used in the proof of Hahn's Theorem is that of an Archimedean class. In Section 3, we show that the existence of a set of unique representatives for the Archimedean classes is equivalent to ACA_0. As a corollary, it follows that there is a computable torsion free abelian group for which any set of Archimedean representatives can Turing compute the halting problem. In Section 4, we give a formal statement of Hahn's Theorem in second order arithmetic and show that it implies ACA_0. Finally, in Section 5, we give a proof of Hahn's Theorem in ACA_0.

For the reverse mathematics, we will be concerned with two subsystems of second order arithmetic: RCA_0 and ACA_0. RCA_0 contains the ordered semiring axioms for the natural numbers, the Δ^0_1 comprehension scheme, and the Σ^0_1 formula induction scheme. We use $\mathbb{N}$ to denote the set defined by the formula $x = x$.

A model for RCA_0 is a two sorted first order structure $\mathfrak{A}$ which satisfies these axioms. If the first order part of $\mathfrak{A}$ is isomorphic to ω, then $\mathfrak{A}$ is called an ω-model. In this case, $\mathfrak{A}$ is often denoted by the subset of $\mathcal{P}(\omega)$ which specifies the second order part of the model.

The computable sets form the minimum ω-model of RCA_0, and any ω-model of RCA_0 is closed under both Turing reducibility and the Turing join. RCA_0 is strong enough to prove the existence of a set of unique codes for the finite sequences of elements from any set X. Such codes are used to formalize many of the arguments presented here.

ACA_0 consists of RCA_0 plus the arithmetical comprehension scheme. Every ω-model of ACA_0 is closed under the Turing jump, and the arithmetic sets form the minimum ω-model of ACA_0.

When proving reversals, we will use the following well-known result (see [8], Lemma III.1.3).

Theorem 1.3. *(RCA_0) The following are equivalent.*

1. *ACA_0.*
2. *The range of every one-to-one function exists.*

Given the characterizations of the ω-models of RCA_0 and ACA_0 in terms of Turing degrees, it is not surprising that equivalences in reverse mathematics have consequences in computable mathematics. We will state such consequences as corollaries to the work in reverse mathematics.

Our notation is standard. It follows both [8] and [3] for the reverse mathematics, and both [4] and [7] for the ordered groups.

§2. Ordered Group Theory.

Definition 2.1. (RCA_0) An **abelian group** is a set $G \subseteq \mathbb{N}$ together with a constant, 0_G, and a function, $+_G$, which obeys the usual abelian group axioms. A **linear order** is a set X together with a binary relation $\leq_X$ on X which satisfies the standard axioms for a linear order. An **ordered abelian group** is a pair $(G, \leq_G)$, where G is an abelian group, $\leq_G$ is a linear order on the elements of G, and for any $a, b, c \in G$, if $a \leq_G b$ then $a +_G c \leq_G b +_G c$.

Except for cases when they are needed to avoid confusion, the subscripts on $+_G$ and $\leq_G$ are dropped.

EXAMPLE 2.2. The additive groups $(\mathbb{R}, +)$, $(\mathbb{Q}, +)$, and $(\mathbb{Z}, +)$ with the standard orders are ordered groups.

EXAMPLE 2.3. Let G be the free abelian group with generators x_i for $i \in \omega$. The elements of G are formal sums $\sum_{i \in I} a_i x_i$ where $I \subseteq \omega$ is finite, $a_i \in \mathbb{Z}$, and $a_i \neq 0$. To compare this element with $\sum_{j \in J} b_j x_j$, let $K = I \cup J$. For each $k \in K$, define $a_k = 0$ if $k \in J \setminus I$ and $b_k = 0$ is $k \in I \setminus J$. If k is the largest element of K such that $a_k \neq b_k$, then $\sum_{i \in I} a_i x_i \leq \sum_{j \in J} b_j x_j$ if and only if $a_k \leq b_k$. With this order, G is an ordered group, which can be formalized in RCA_0 using finite sequences.

Defining an order can sometimes be notationally complicated, so it is frequently easier to specify only the elements which are greater than 0_G. Such a specification uniquely determines the order.

DEFINITION 2.4. (RCA_0) For an ordered group G, the **positive cone** is defined by $P(G) = \{g \in G \mid 0_G \leq_G g\}$.

Because $P(G)$ has a Σ^0_0 definition, RCA_0 can prove its existence. Conversely, the order relationship between any two elements can be defined in RCA_0 using $P(G)$ as a parameter because $a \leq b$ if and only if $b - a \in P(G)$. Since subtraction is definable by a Δ^0_1 formula, RCA_0 suffices to prove that each positive cone uniquely determines an order on G.

There are classical algebraic conditions which determine if an arbitrary subset of an abelian group is the positive cone for some order. See [4] for a proof of Theorem 2.6.

DEFINITION 2.5. (RCA_0) If $X \subseteq G$, then $X^{-1} = \{-g \mid g \in X\}$. X is a **full subset** of G if $X \cup X^{-1} = G$ and X is a **pure subset** of G if $X \cap X^{-1} \subseteq \{0_G\}$.

THEOREM 2.6. *(RCA_0) A subset P of an abelian group G is the positive cone of some order on G if and only if P is a pure and full semigroup with identity.*

If H is a subgroup of G, then we choose unique representatives of each coset $g + H$ in G/H by picking the $\mathbb{N}$-least element of $g + H$. These choices can be made in RCA_0 because $a + H = b + H$ if and only if $b - a \in H$. Formally, G/H is defined by the set

$$\{a \mid a \in G \wedge \forall b <_{\mathbb{N}} a(b \notin G \vee b - a \notin H)\}$$

together with the operation $a +_{G/H} b = c$ if and only if $a, b, c \in G/H$ and $a +_G b -_G c \in H$. We denote elements of G/H by $a + H$.

For any ordered abelian group G and subgroup H, there is a natural induced order on the quotient G/H as long as H is convex.

DEFINITION 2.7. (RCA_0) A subset X of a linear order Y is **convex** if

$$\forall a, b, x \in Y\big((a, b \in X \wedge a \leq x \leq b) \to x \in X\big).$$

A subgroup H of an ordered group G is **convex** if it is convex as a subset of G.

DEFINITION 2.8. (RCA_0) Let G be an ordered group and H a convex normal subgroup. The **induced order**, $\leq_{G/H}$, on G/H is defined by $a \leq_{G/H} b$ if and only if $a+H = b+H$, or $a+H \neq b+H$ and $a <_G b$.

Since the condition in Definition 2.8 is Σ^0_0, RCA_0 suffices to prove that the induced order exists. Next, we define the notion of Archimedean equivalence.

DEFINITION 2.9. (RCA_0) For an ordered group G, the **absolute value** is defined by

$$|x| = \begin{cases} x & \text{if } x \in P(G) \\ -x & \text{otherwise.} \end{cases}$$

For $n \in \mathbb{N}$ and $g \in G$, we let ng denote the result of adding g to itself n times (formally defined by primitive recursion).

DEFINITION 2.10. (RCA_0) Let G be an ordered group with $a, b \in G$. We say that a is **Archimedean less than** b, denoted $a \ll b$, if $|na| < |b|$ for all $n \in \mathbb{N}$. If there are $n, m \in \mathbb{N}$ such that $|na| \geq |b|$ and $|mb| \geq |a|$, then a and b are **Archimedean equivalent**, denoted $a \approx b$. The notation $a \lessapprox b$ means $a \approx b \vee a \ll b$. G is an **Archimedean ordered group** if G is an ordered group and for all $a, b \neq 0_G$, $a \approx b$.

It is not hard to check in RCA_0 that $\approx$ is an equivalence relation and that $\ll$ is transitive, antireflexive, and antisymmetric.

Hölder's Theorem states that every ordered Archimedean group can be embedded in the naturally ordered additive group of the reals. To state Hölder's Theorem in second order arithmetic, recall that real numbers are defined by functions from $\mathbb{N}$ to $\mathbb{Q}$ with appropriate convergence properties. See [9] for a proof of Hölder's Theorem in RCA_0. The formal definitions are as follows.

DEFINITION 2.11. (RCA_0) A **real number** is a function $f : \mathbb{N} \to \mathbb{Q}$, usually denoted by $\langle q_k | k \in \mathbb{N} \rangle$, such that for all k and i, $|q_k - q_{k+i}| \leq 2^{-k}$. Two real numbers $x = \langle q_k | k \in \mathbb{N} \rangle$ and $y = \langle q'_k | k \in \mathbb{N} \rangle$ are **equal** if for all k, $|q_k - q'_k| \leq 2^{-k+1}$. The **sum** $x + y$ is the real number $\langle q_{k+1} + q'_{k+1} | k \in \mathbb{N} \rangle$.

DEFINITION 2.12. (RCA_0) An **ordered subgroup of** $(\mathbb{R}, +_{\mathbb{R}})$ indexed by the set X is a sequence of distinct reals $A = \langle r_n \mid n \in X \rangle$ together with a function $+_A : X \times X \to X$, a binary relation $\leq_A$ on X, and a distinguished number $i \in X$ such that

1. $r_i = 0_{\mathbb{R}}$,
2. $n +_A m = p$ if and only if $r_n +_{\mathbb{R}} r_m = r_p$,
3. $n \leq_A m$ if and only if $r_n \leq_{\mathbb{R}} r_m$, and
4. $(X, +_A, \leq_A)$ satisfies the ordered abelian group axioms with i as the identity element.

HÖLDER'S THEOREM. *(RCA_0) Every Archimedean ordered group G is isomorphic to an ordered subgroup of $(\mathbb{R}, +)$ indexed by G for which $+_A$ and $\leq_A$ are $+_G$ and $\leq_G$.*

It is frequently convenient to work with divisible abelian groups. Often, this can be done without loss of generality by passing from an abelian group to its divisible closure.

DEFINITION 2.13. (RCA_0) An abelian group D is **divisible** if for all $d \in D$ and all $n \geq 1$ there exists a $c \in D$ such that $nc = d$. If G is an abelian group, then a **divisible closure** of G is a divisible group D, together with a monomorphism $h : G \to D$ such that for all $d \in D, d \neq 0_D$, there exists $n \in \mathbb{N}$ with $nd = h(g)$ for some $g \in G, g \neq 0_G$.

The range of h need not exist in RCA_0. However, since we will only consider divisible closures in ACA_0, we will always be able to assume that G is isomorphic to a subgroup of D. The property of divisible groups which will be important to us is given in the following theorem.

THEOREM 2.14 ([8], Lemma VI.4.2). *(ACA_0) If D is a divisible subgroup of an abelian group G, then $G = D + A$ for some subgroup A. Here, $D + A$ denotes the direct sum of D and A.*

Although we will not use the following theorem, it fits nicely with the preceding discussion and shows that there are natural theorems from the theory of abelian groups which require relatively strong axioms.

THEOREM 2.15 ([8], Theorem VI.4.1). *(RCA_0) The following are equivalent.*

1. $\Pi^1_1 - CA_0$.
2. *Every countable abelian group is the direct sum of a divisible group and a reduced group.*

Hahn's Theorem is an extension of Hölder's Theorem. Let T be a linear order and consider the unrestricted sum $\sum_T \mathbb{R}$. The elements of this space are functions $f : T \to \mathbb{R}$. We view this function space as an abelian group, with $(f+g)(t) = f(t)+g(t)$, $(-f)(t) = -(f(t))$, and $f(t) = 0$ as the identity element.

DEFINITION 2.16. Classically, a **Hahn group** is a subgroup $G \subset \sum_T \mathbb{R}$ such that for every function $f \in G$, the set $\{t \in T | f(t) \neq 0\}$ is well ordered. The order on the Hahn group is defined by $f \leq g$ if and only if $f(t_0) \leq g(t_0)$, where t_0 is the T-least element of $\{t \in T | (f-g)(t) \neq 0\}$.

HAHN'S THEOREM. *Every ordered abelian group can be embedded in a Hahn group.*

In second order arithmetic, we will state a slightly different, but equivalent, formulation of Hahn's Theorem to avoid dealing with subgroups of the reals.

§3. Archimedean classes. In this section, we define the notion of a set of Archimedean representatives for an ordered group and show that the existence of such a set is equivalent to ACA_0.

DEFINITION 3.1. (ACA_0) Let G be an ordered abelian group. A subset $X \subset G$ is a **set of Archimedean representatives** for G if the following two conditions hold.

1. For all $g \in G$, there is an $x \in X$ such that $x \approx g$.
2. For all $x \neq y \in X$, $x \not\approx y$.

There are several points worth noting about this definition. First, ACA_0 suffices to prove the existence of the following set of Archimedean representatives.

$$\{g \in G | \forall n <_{\mathbb{N}} g(n \notin G \vee n \not\approx g)\}$$

Second, if X is a set of Archimedean representatives, then so is the set X' consisting of $|x|$ for all $x \in X$. Hence, we can assume without loss of generality (in RCA_0) that the elements of any particular set of Archimedean representatives are positive.

Third, if X is a set of Archimedean representatives, then we can define a function $\mathrm{Rep} : G \to X$ which assigns each $g \in G$ to the unique element of X such that $g \approx x$. The definition of Rep is Δ^0_1 with X as a parameter.

$$\mathrm{Rep}(g) = x \Leftrightarrow x \in X \wedge \exists n, m > 0(|g| \leq_G |nx| \wedge |x| \leq_G |mg|)$$

$$\Leftrightarrow x \in X \wedge \forall y \in X\big(y \neq x \to (\forall m > 0(|g| \leq_G |my|) \vee \forall m > 0(|y| \leq_G |mg|))\big)$$

Therefore, once we know X exists, RCA_0 is strong enough to prove the existence of the set of all pairs $\langle g, h\rangle$ such that $g \approx h$. Conversely, if the set of all pairs such that $g \approx h$ exists, then RCA_0 can prove the existence of a set of Archimedean representatives by choosing the $\mathbb{N}$-least element of each Archimedean class. We summarize this discussion in the following lemma.

LEMMA 3.2. *(RCA_0) For any ordered abelian group G, the following are equivalent.*

1. *There exists a set of Archimedean representatives for G.*
2. *The set $\{\langle g, h\rangle | g \approx h\}$ exists.*

We turn to the main theorem of this section.

THEOREM 3.3. *(RCA_0) For any ordered abelian group G, the following are equivalent.*

1. *ACA_0.*
2. *There exists a set of Archimedean representatives for G.*

PROOF. *Case.* (1) $\Rightarrow$ (2).
Above, we gave an explicit arithmetical (in fact, Π^0_1) definition for a set of Archimedean representatives.

Case. (2) $\Rightarrow$ (1).

Let $f : \mathbb{N} \to \mathbb{N}$ be a one-to-one function. By Theorem 1.3, it suffices to show that the range of f exists. Let p_s be an enumeration of the odd primes. G will be the abelian group defined by the generators x_n, for $n \in \mathbb{N}$, and the relations $p_s x_{2n} = x_{2n+1}$ if $f(s) = n$.

To present the elements of G formally, we consider sums $\sum_{i\in I} a_i x_i$ where $I \subset \mathbb{N}$ is finite. Such a sum is called reduced if $a_i \neq 0$ for all $i \in I$ and, for each even number $2j \in I$, there is no prime $p_s < |2a_{2j}|$ such that $f(s) = j$. The elements of G are these reduced sums.

There is a standard method for reducing an arbitrary finite formal sum to a reduced sum by applying the reduction relations. Given an arbitrary finite sum $\sum_J b_j x_j$, let $2n_0, 2n_1, \ldots , 2n_l$ be the even numbers in J. For each $2n$ in this collection, we check if there is a prime $p_s < |2b_{2n}|$ for which $f(s) = n$. If not, no reduction rule applies to $b_{2n}x_{2n}$. If so, then we let $b_{2n} = cp_s + r$ where $c \in \mathbb{Z}$ and $|r| < p_s/2$. The reduced coefficient for x_{2n} is r, and the reduced coefficient for x_{2n+1} is either $b_{2n+1} + c$ (if $2n+1 \in J$) or c (if $2n+1 \notin J$). The maximum element of J provides a bound on the number of reduction relations we need to apply. RCA_0 is strong enough formalize this recursive process and prove that it terminates in a reduced sum, once any extra terms with coefficients of 0 are removed.

To add two reduced sums $g = \sum_I a_i x_i$ and $h = \sum_J b_j x_j$, we let $K = I \cup J$ and extend the coefficients in g and h by setting $a_k = 0$ for all $k \in K \setminus I$ and $b_k = 0$ for all $k \in K \setminus J$. We define $g + h$ to be the sum formed by reducing $\sum_K (a_k + b_k)x_k$. Because the sums for g and h are reduced, if we apply the reduction relation $p_s x_{2n} = x_{2n+1}$ to $g + h$, then we have $a_{2n} + b_{2n} = cp_s + r$, with c either 1 or -1. Therefore, the coefficient for any odd indexed generator x_{2n+1} in the unreduced form of $g + h$ changes by at most 1 when we apply the reduction process. Finally, the identity element, which we denote by 0_G, is the sum in which $I = \emptyset$.

To define the order on G, we specify the positive cone.

$$P = \{0_G\} \cup \left\{ \sum_I a_i x_i \in G | a_n > 0 \text{ where } n = \max(I) \right\}$$

Since $P \subset G$, it contains only reduced sums. We must show that P is a pure and full semigroup with identity. By definition, P contains 0_G. Notice that if $g = \sum_I a_i x_i$ is a reduced sum, then $-g = \sum_I -a_i x_i$. From here it is clear that P is pure and full.

It remains to show that P is closed under addition. Let $g = \sum_I a_i x_i$ and $h = \sum_J b_j x_j$ be two nonidentity elements of P (with reduced sums), and let $n = \max(I)$ and $m = \max(J)$. We know that $a_n > 0$ and $b_m > 0$. To check that the reduced sum $g + h = \sum_K c_k x_k$ is in P, we consider several cases.

First, suppose that $n < m$. If $n = 2d$ and $m = 2d+1$ for some d, then we may have reduced $a_n + b_n$ using a relation $p_s x_n = x_m$. However, because $a_n > 0$, if we reduced this sum, it was because $a_n + b_n = p_s + r$ for some $|r| < p_s/2$. In this case, we have $m \in K$, $m = \max(K)$ and the coefficient c_m of x_m in $g+h$ is $b_m + 1 > 0$. Therefore, $g + h \in P$. If we did not need to reduce $a_n + b_n x_n$, or if n and m are not consecutive even and odd numbers, then $m = \max(K)$ and the coefficient c_m of x_m in $g + h$ is $b_m > 0$, so $g + h \in P$. The case for $m < n$ is analogous.

Second, suppose that $n = m$ and n is odd. We know the coefficient c_n in $g+h$ differs from $a_n + b_n$ by at most 1, so either $c_n = a_n + b_n + 1$ or $c_n = a_n + b_m - 1$. In either case, since $a_n > 0$ and $b_n > 0$, c_n remains positive, and $g + h \in P$.

Third, suppose $n = m$ and n is even. In this case, we may have reduced $(a_n + b_n)x_n$ using a relation $p_s x_n = x_{n+1}$. However, if $a_n + b_n = cp_s + r$ as above, then since $a_n > 0$ and $b_n > 0$, we know $c = 1$. Therefore, $n + 1 = \max(K)$ and $c_{n+1} = 1$, so $g + h \in P$. If we did not have to apply a reduction relation, then $n = \max(K)$ and $c_n = b_n + a_n > 0$, so $g + h \in P$. In all cases, we showed that $g + h \in P$. Therefore, P is closed under addition and defines an order on G.

By assumption, there is a set of Archimedean representatives X for G, and by Lemma 3.2, the set $\{\langle g, h\rangle | g \approx h\}$ exists. The following claim finishes the proof.

Claim. $n \in \text{range}(f) \Leftrightarrow x_{2n} \approx x_{2n+1}$

The implication ($\Rightarrow$) is clear since $f(s) = n$ implies that $p_s x_{2n} = x_{2n+1}$. To prove the implication ($\Leftarrow$), notice that if n is not in the range of f, then for every positive $m \in \mathbb{N}$, mx_{2n} is a reduced sum and $mx_{2n} < x_{2n+1}$. Therefore, $x_{2n} \ll x_{2n+1}$. ⊣

COROLLARY 3.4. *There is a computable ordered abelian group for which the Turing degree of every set of Archimedean representatives is above* $0'$.

§4. Statement of Hahn's Theorem and Reversal. In this section, we give a statement of Hahn's Theorem in second order arithmetic and prove that it implies ACA_0. Throughout this section, G will be an ordered abelian group. Classically, Hahn's Theorem says that G can be embedded into a lexicographically ordered subgroup of the unrestricted sum $\sum_T \mathbb{R}$. Rather than working with sums of $\mathbb{R}$ (which are notationally cumbersome in second order arithmetic), we will work with sequences of Archimedean ordered groups. Of course, since the proof of Hölder's Theorem is constructive, we could apply Hölder's Theorem to construct a sequence of isomorphic subgroups of $\mathbb{R}$.

DEFINITION 4.1. (RCA_0) Let $(T, \leq_T)$ be a linear order, and let K_t, for $t \in T$, be a sequence of Archimedean ordered abelian groups. A **subgroup of** $\mathbf{\sum_{t \in T} K_t}$, indexed by I, is a sequence of functions $f_i : T \to \cup_T K_t$, for $i \in I$ which meet the following conditions. (To cut down on notation, i, j, k range over I and t ranges over T.)

1. $\forall i \forall t (f_i(t) \in K_t)$.
2. $\exists i \forall t (f_i(t) = 0_{K_t})$.
3. $\forall i, j \exists k \forall t (f_k(t) = f_i(t) + f_j(t))$. We denote this situation by $f_k = f_i + f_j$.
4. $\forall i \exists j \forall t (f_j(t) = -f_i(t))$. We denote this situation by $f_j = -f_i$.
5. $\forall i, j (i \neq j \to \exists t (f_i(t) \neq f_j(t)))$.

Let F denote this sequence of functions f_i, $i \in I$. For every $t_0 \in T$, we associate a **cut** C which is defined by

$$Cf_i(t) = \begin{cases} f_i(t) & \text{if } t <_T t_0 \\ 0_{K_t} & \text{otherwise} \end{cases}$$

The subgroup F has the **cut property** if for every cut C and every $i \in I$, there is a $j \in I$ such that $f_j = Cf_i$. Furthermore, if $\{t | f_i(t) \neq 0_{K_t}\}$ is a well ordered subset of T for every $i \in I$, then we define an order (in fact, the reverse lexicographic order) by $f_i <_F f_j$ if and only if $f_i(t_0) <_{K_{t_0}} f_j(t_0)$, where t_0 is the T-minimal element of $\{t | (f_i - f_j)(t) \neq 0_{K_t}\}$. If this order obeys the ordered group axioms, in the sense that $f_i <_F f_j$ implies that $f_i + f_k <_F f_j + f_k$ for every $i, j, k \in I$, and the subgroup has the cut property, then we say the sequence f_i is a **Hahn subgroup of** $\mathbf{\sum_T K_t}$.

Much of our notation for ordered groups can be used in the context of Hahn subgroups. We define $(mf_i)(t) = m(f_i(t))$ and $|f_i|(t) = |f_i(t)|$. We say $f_i \ll_F f_j$ if $|mf_i| <_F |f_j|$ for every m. Notice that since each K_t is Archimedean, $f_i \ll f_j$ could be equivalently defined by $t_i <_T t_j$ where t_i is the T-least element of $\{t | f_i(t) \neq 0_{K_t}\}$ and t_j is the T-least element of $\{t | f_j(t) \neq 0_{K_t}\}$.

DEFINITION 4.2. Let T and K_t be as above. We say that an ordered group G is **isomorphic to a Hahn subgroup** of $\sum_T K_t$ if there is a sequence of functions f_g, indexed by G, such that

1. f_g, for $g \in G$, forms a Hahn subgroup of $\sum_T K_t$
2. $\forall t(f_{0_G}(t) = 0_{K_t})$
3. $\forall g, h \in G(f_{g+h} = f_g + f_h)$
4. $\forall g \in G(f_{-g} = -f_g)$
5. $\forall g, h \in G(g <_G h \rightarrow f_g <_F f_h)$

The following lemma lists several properties which are clear from the definitions.

LEMMA 4.3. *Let G be isomorphic to a Hahn subgroup of $\sum_T K_t$. For all $g, h \in G$ and all $n \in \mathbb{N}$, we have the following facts.*

1. $g <_G h \Leftrightarrow f_g <_F f_h$
2. $f_{ng}(t) = nf_g(t)$
3. $g \ll h \Leftrightarrow f_g \ll f_h$

We can now state our formal version of Hahn's Theorem.

HAHN'S THEOREM. *For every ordered abelian group G, there is a linear order T and a sequence of Archimedean ordered subgroups K_t of G such that G is order isomorphic to a Hahn subgroup of $\sum_T K_t$.*

THEOREM 4.4. *(RCA_0) The following are equivalent.*

1. *ACA_0.*
2. *Hahn's Theorem.*

This equivalence shows that Hahn's Theorem is not effective. One direction of Theorem 4.4 is supplied by Proposition 4.5, and the other direction is considered in Section 5.

PROPOSITION 4.5. *(RCA_0) Hahn's Theorem implies ACA_0.*

PROOF. Let f be a one-to-one function. By Theorem 1.3, it suffices to show that the range of f exists. We define a group G just as in the proof of Theorem 3.3, with generators x_n and relations $p_s x_{2n} = x_{2n+1}$ if $f(s) = n$. As before, it suffices to determine when $x_{2n} \approx x_{2n+1}$ to recover the range of f.

By assumption, G is isomorphic to some Hahn subgroup of $\sum_T K_t$, which we denote by f_g, for $g \in G$. Notice that for any pair x_{2n} and x_{2n+1}, we either have that $p_s x_{2n} = x_{2n+1}$ for some prime p_s, or $x_{2n} \ll x_{2n+1}$. If $x_{2n} \ll x_{2n+1}$, then since each K_t is Archimedean, there must be a $t \in T$ such that $f_{x_{2n}}(t) = 0_{K_t}$ and $f_{x_{2n+1}}(t) \neq 0_{K_t}$. On the other hand, if $p_s x_{2n} = x_{2n+1}$, then for every $t \in T$, $p_s f_{x_{2n}}(t) = f_{x_{2n+1}}(t)$. In particular, this means that for every t, $f_{x_{2n+1}}(t) \neq 0_{K_t}$ implies that $f_{x_{2n}}(t) \neq 0_{K_t}$. Therefore, we have the following Σ^0_1 condition for determining $x_{2n} \ll x_{2n+1}$.

$$x_{2n} \ll x_{2n+1} \Leftrightarrow \exists t(f_{x_{2n+1}}(t) \neq 0_{K_t} \wedge f_{x_{2n}}(t) = 0_{K_t})$$

Since the standard definition for $x_{2n} \ll x_{2n+1}$ is Π^0_1, we have that $x_{2n} \ll x_{2n+1}$ is Δ^0_1 definable. Therefore, the range of f is Δ^0_1 definable, as required. ⊣

§5. Proof of Hahn's Theorem. This section is devoted to the proof of Hahn's Theorem. The details of our proof are close to those presented in [6] and [1], but several modifications are necessary to avoid the use of Zorn's Lemma and Σ^1_1 induction. For the rest of this paper, we work in ACA_0.

Since we are working in ACA_0, we can assume without loss of generality that G is divisible by embedding G into its divisible closure. Furthermore, we let $\mathrm{A}(G)$ be a set of positive Archimedean representatives, and we assume that $\mathrm{Rep} : G \to \mathrm{A}(G)$ is defined such that $\mathrm{Rep}(g) = h \Leftrightarrow (h \in \mathrm{A}(G) \wedge g \approx h)$.

Let $(T, \leq_T)$ be a linear order with $T = \mathrm{A}(G)$ and $x <_T y$ if and only if $y \ll_G x$. (Do not confuse $\leq_T$ with Turing reducibility, which plays no role here.) Notice that $T \subset G$, T is linearly independent, and $\mathrm{Rep}(t) = t$ for all $t \in T$. Also, we define the following convex subgroups.

$$H_t = \{h \in G | h \ll t \vee h \approx t\} \text{ and } H'_t \ \{h \in G | h \ll t\}$$

Lemma 5.1. *H_t and H'_t are both divisible convex subgroups of G, and H_t/H'_t is an Archimedean ordered group.*

Proof. H_t and H'_t are clearly convex and divisible since G is divisible. To show that H_t/H'_t is Archimedean, it suffices to show that $g \approx h$ for every $g, h \in H_t \setminus H'_t$. Suppose $g, h \in H_t \setminus H'_t$ and $g \ll h$. Since $h \in H_t$, we know $h \approx t$, so $g \ll t$. But then $g \in H'_t$, which is a contradiction. ⊣

Lemma 5.2. *For each $t \in T$, there is a divisible Archimedean subgroup $K_t \subset G$ such that $H_t = H'_t + K_t$. Here, $H'_t + K_t$ denotes the direct sum of these groups.*

Proof. This lemma is a direct consequence of Lemma 5.1 and Theorem 2.14. ⊣

Lemma 5.3. *For all $x, y, z \in G$, Rep$(x-y) \leq_T$ Rep$(x-z)$ implies that Rep$(x-y) \leq_T$ Rep$(y-z)$.*

Proof. Assume $\mathrm{Rep}(x-y) \leq_T \mathrm{Rep}(x-z)$, but $\mathrm{Rep}(y-z) <_T \mathrm{Rep}(x-y)$. By the definition of $<_T$, we have that $|n(x-y)| <_G |y-z|$ for all n. Applying the triangle inequality yields $|n(x-y)| <_G |x-y| + |x-z|$, which implies $(n-1)|x-y| <_G |x-z|$ for all n. Therefore, $(x-y) \ll (x-z)$, so $\mathrm{Rep}(x-z) <_T \mathrm{Rep}(x-y)$, which is a contradiction. ⊣

Lemma 5.4. *Let $x \in G$. There is a unique element $\rho_x \in K_{Rep(x)}$ such that either $\rho_x = x$ or $(x - \rho_x) \ll x$.*

Proof. Since $x \in H_{\mathrm{Rep}(x)}$ and $H_{\mathrm{Rep}(x)} = H'_{\mathrm{Rep}(x)} + K_{\mathrm{Rep}(x)}$, we can write x uniquely as $x = g + \rho_x$, where $g \in H'_{\mathrm{Rep}(x)}$ and $\rho_x \in K_{\mathrm{Rep}(x)}$. The result follows because $x - \rho_x = g \in H'_{\mathrm{Rep}(x)}$. ⊣

Using arithmetic comprehension, we define $T' = \{|\rho_t| \,|\, t \in T\}$. Since $|\rho_t| \approx t$, T' is also a set of Archimedean representatives. Furthermore, $|\rho_t| \in K_t$ for every $t \in T$. Therefore, having defined the groups K_t, we can assume without loss of generality (by replacing T by T' if necessary) that our choice T of Archimedean representatives has the property that $t \in K_t$ for every $t \in T$.

The strategy to prove Hahn's Theorem is to define a sequence of subgroups G_n, $n \in \mathbb{N}$, such that $G_n \subset G_{n+1}$ and $G = \cup G_n$. We show that G_0 is isomorphic to a Hahn subgroup of $\sum K_t$, and then extend this embedding to each G_n. Finally, we verify by induction that each G_n is isomorphic to a Hahn subgroup of $\sum K_t$.

Let G_0 be the divisible subgroup of G generated by $\cup_T K_t$. Since $t \in K_t$ and $K_t \cong \mathbb{Q}$ for all $t \in T$, G_0 is the smallest divisible subgroup of G containing T. The elements of G_0 are the sums $\sum_I q_t t$, where $I \subset T$ is finite and $q_t \in \mathbb{Q} \setminus \{0\}$. Since the elements of T are linearly independent, these finite sums are unique. To show that G_0 is isomorphic to a Hahn subgroup of $\sum K_t$, let

$$f_t(t') = \begin{cases} t & \text{if } t' = t \\ 0_{K_{t'}} & \text{otherwise} \end{cases}$$

for all $t \in T$. For $g \in G_0$ with $g \notin T$, write $g = \sum_{t \in I} q_t t$ and define $f_g(t') = \sum_{t \in I} q_t f_t(t')$. Since $f_g(t') \neq 0_{K_{t'}}$ for only finitely many t', the functions f_g, for $g \in G$, define a Hahn subgroup of $\sum_T K_t$.

Our next goal is to define G_n for $n \geq 1$, by presenting a sequence of elements g_n to generate these subgroups. Let g_1 be the $\mathbb{N}$-least element of G which is not in G_0, and let g_{n+1} be the $\mathbb{N}$-least element of G which is not a solution to an equation of the form $mx = h + c_1 g_1 + \cdots + c_n g_n$ with $h \in G_0$, $m, c_1, \ldots, c_n \in \mathbb{Z}$, and $m \neq 0$. Less formally, g_{n+1} is the $\mathbb{N}$-least element not in the divisible subgroup of G generated by G_0 together with $g_1, \ldots, g_n$. If no such element exists, then G is generated by G_0 and $g_1, \ldots, g_n$, so do not define g_{n+1} and proceed with the construction using the finite set $g_1, \ldots g_n$.

We define G_n to be the divisible subgroup generated by G_0 and $g_1, \ldots, g_n$. It is clear that $G = \cup G_n$ and that every $g \in G_{n+1} \setminus G_n$ can be uniquely written as a sum $g = h + q g_{n+1}$ where $h \in G_n$ and $q \in \mathbb{Q}$.

We also define the following sets uniformly for all $i \geq 1$.

$$S_i = \{\mathrm{Rep}(g_i - y) | y \in G_{i-1}\} \subset T$$

LEMMA 5.5. *For each $i \geq 1$, S_i has no T-greatest element.*

PROOF. Suppose S_i does have a T-greatest element. Let $z \in G_{i-1}$ be such that $\mathrm{Rep}(g_i - y) \leq_T \mathrm{Rep}(g_i - z)$ for all $y \in G_{i-1}$. By Lemma 5.4, there is a $\rho \in K_{\mathrm{Rep}(g_i - z)}$ such that either $g_i - z = \rho$ or $g_i - z - \rho \ll g_i - z$.

First, suppose $g_i - z = \rho$. Then, $g_i = z + \rho$, which implies that $g_i \in G_{i-1}$ since $z \in G_{i-1}$ and $\rho \in G_0$. This contradicts our choice of g_i. Second, suppose $g_i - (z + \rho) \ll g_i - z$. By the definition of $<_T$, $\mathrm{Rep}(g_i - z) <_T \mathrm{Rep}(g_i - (z + \rho))$. Since $z + \rho \in G_{i-1}$, this inequality contradicts our choice of z. ⊣

We next choose a sequence of elements of T which are cofinal in S_i. Let $u_{i,0}$ be the $\mathbb{N}$-least element of S_i, and let $u_{i,j+1}$ be the $\mathbb{N}$-least element of S_i for which $u_{i,j} <_T u_{i,j+1}$. By Lemma 5.5, such elements always exist. Define U_i to be the set of $u_{i,j}$ for $j \in \mathbb{N}$, and let $z_{i,j}$ be the $\mathbb{N}$-least element of G_{i-1} such that $u_{i,j} = \mathrm{Rep}(g_i - z_{i,j})$. Notice that each $z_{i,j}$ can be uniquely written as

$$z_{i,j} = h_{i,j} + \sum_{k=1}^{i-1} q_{i,j,k} g_k$$

for some $h_{i,j} \in G_0$ and $q_{i,j,k} \in \mathbb{Q}$.

We are now in a position to define a function $F(i,t) : (\mathbb{N} \setminus \{0\}) \times T \to \cup_{t \in T} K_t$, and then set $f_{g_i}(t) = F(i,t)$.

$$F(i,t) = \begin{cases} f_{h_{i,0}}(t) + \sum_{k=1}^{i-1} q_{i,0,k} F(k,t) & \text{if } t < u_{i,0} \\ f_{h_{i,j}}(t) + \sum_{k=1}^{i-1} q_{i,j,k} F(k,t) & \text{if } u_{i,j-1} \leq t < u_{i,j} \\ 0_{K_t} & \text{otherwise} \end{cases} \tag{1}$$

Claim. For every $i \geq 1$, $\{t \in T | F(i,t) \neq 0_{K_t}\}$ is a well ordered subset of T.

Since the statement of the claim is Σ^1_1, we cannot prove it by induction. Instead, we fix an arbitrary i_0, and prove the statement for this index. For a contradiction, assume that $\{t | F(i_0, t) \neq 0_{K_t}\}$ is not well ordered, and fix an infinite sequence c_n, $n \in \mathbb{N}$, in T such that $c_{n+1} <_T c_n$ and $F(i_0, c_n) \neq 0_{K_{c_n}}$ for every n.

Since $F(i_0, c_0) \neq 0_{K_{c_0}}$, we can fix j_0 such that $c_0 < u_{i_0,j_0}$, and hence $c_n < u_{i_0,j_0}$ for every n. There must be some interval of the form $(-\infty, u_{i_0,0})$ or $(u_{i_0,j'-1}, u_{i_0,j'})$, for some $j' \leq j_0$, in which the tail of the c_n sequence lies. That is, one of the following two cases must hold.

$$\exists n \forall m \geq n (c_m < u_{i,0}) \text{ or}$$
$$\exists j' \leq j (1 < j \wedge \exists n \forall m \geq n (c_m < u_{i_0,j'}) \wedge \forall m (u_{i,j'-1} < c_m))$$

Therefore, there is a fixed $j \leq j_0$ and $n \in \mathbb{N}$ such that for all $m \geq n$,

$$F(i_0, c_m) = f_{h_{i_0},j}(c_m) + \sum_{k=1}^{i_0-1} q_{i_0,j,k} F(k, c_m).$$

By arithmetic comprehension, we can fix $i_1 <_{\mathbb{N}} i_0$ such that $F(i_1, c_m) \neq 0_{K_{c_m}}$ for infinitely many m. Since this argument could now be repeated for i_1, it is clear that by downward arithmetic induction, we have that for all x with $2 \leq x \leq i_0$, there exists i such that $0 < i < x$ and $F(i, c_n) \neq 0_{K_{c_m}}$ for infinitely many c_n. In particular, after thinning out the c_n sequence, we can assume that there is an infinitely descending sequence c_n such that $F(1, c_n) \neq 0_{K_{c_n}}$ for all n. However,

$$F(1,t) = \begin{cases} f_{h_{1,0}}(t) & \text{if } t < u_{i,0} \\ f_{h_{1,j}}(t) & \text{if } u_{i,j-1} \leq t < u_{i,j} \\ 0_{K_t} & \text{otherwise.} \end{cases}$$

As above, we can fix j and n such that for all $m \geq n$, $F(1, c_m) = f_{h_{1,j}}(c_m)$. This equation gives the desired contradiction since $h_{1,j} \in G_0$, and hence has finite support.

We use $F(i,t)$ to define $f_g(t)$ for each $g \in G$ as follows. Write $g = h + \sum_{i \in I} q_i g_i$, where $h \in G_0$, I is finite, and $q_i \in \mathbb{Q} \setminus \{0\}$. Define $f_g(t) = f_h(t) + \sum_I q_i f_{g_i}(t)$. Since the support of each f_{g_i} is well ordered, the support of f_g is also well ordered. Also, notice that $f_{g+h} = f_g + f_h$ and $f_{-g} = -f_g$.

It remains to show by induction on i that G_i is isomorphic to the Hahn subgroup formed by the functions f_h for $h \in G_i$. In addition, we show that the following definition for $F(i,t)$ is equivalent to the one given by Equation (1).

$$F(i,t) = \begin{cases} f_{z_{i,j}}(t) & \text{if } t < u_{i,j} \\ 0_{K_t} & \text{otherwise} \end{cases} \tag{2}$$

We have already established the base case for G_0, so assume the induction hypotheses hold for G_{i-1}.

By definition,

$$f_{z_{i,j}}(t) = f_{h_{i,j}}(t) + \sum_{k=1}^{i-1} q_{i,j,k} F(k,t).$$

Therefore, to show that Equations (1) and (2) are equivalent, it suffices to consider $t <_T u_{i,j} <_T u_{i,l}$ and show that $f_{z_{i,j}}(t) = f_{z_{i,l}}(t)$. Applying Lemma 5.3 to $\mathrm{Rep}(g_i - z_{i,j}) <_T \mathrm{Rep}(g_i - z_{i,l})$ yields $\mathrm{Rep}(g_i - z_{i,j}) \leq_T \mathrm{Rep}(z_{i,j} - z_{i,l})$, and hence $z_{i,j} - z_{i,l} \lessapprox u_{i,j}$. Since $z_{i,j}$, $z_{i,l}$ and $u_{i,j}$ are all in G_{i-1}, we have by the induction hypothesis that $f_{z_{i,j}} - f_{z_{i,l}} \lessapprox f_{u_{i,j}}$.

Claim. $(f_{z_{i,j}} - f_{z_{i,l}}) = 0_{K_t}$ for all $t <_T u_{i,j}$

Suppose not. Let t_0 be the T-least element for which this fails. Then $f_{z_{i,j}} - f_{z_{i,l}}$ and $f_{u_{i,j}}$ are compared in the order on $\sum_T K_t$ at the argument t_0. But, $f_{u_{i,j}}(t) = 0_{K_t}$ for all $t <_T u_{i,j}$, so $f_{u_{i,j}}(t_0) = 0_{K_t}$. Therefore, since $|f_{z_{i,j}}(t_0) - f_{z_{i,l}}(t_0)| > 0_{K_t}$, we have that $f_{u_{i,j}} \ll f_{z_{i,j}} - f_{z_{i,l}}$, which gives the contradiction and proves the claim.

Claim. The embedding of G_i has the cut property.

Consider any $x \in G_i$, and let C be the cut determined by t_0.

$$Cf_x(t) = \begin{cases} f_x(t) & \text{if } t <_T t_0 \\ 0_{K_t} & \text{otherwise} \end{cases}$$

We need to show that there is a $y \in G_i$ such that $Cf_x = f_y$. Because $x \in G_i$, it follows that $x = h + qg_i$ for some $h \in G_{i-1}$ and $q \in \mathbb{Q}$. Therefore, $Cf_x = Cf_h + qCf_{g_i}$. By the Hahn embedding of G_{i-1}, there is an $h' \in G_{i-1}$ such that $f_{h'} = Cf_h$. We consider two cases for Cf_{g_i}. If $t_0 \leq u_{i,j}$ for some j, then $Cf_{g_i} = Cf_{z_{i,j}}$, and by induction, there is a $z \in G_{i-1}$ such that $f_z = Cf_{z_{i,j}}$. If $u_{i,j} <_T t_0$ for all j, then $Cf_{g_i} = f_{g_i}$. In either case, the claim is proved.

Claim. If $x \neq y \in G_i$, then $f_x \neq f_y$.

Suppose $x \neq y$, $x = h_1 + q_1 g_i$, $y = h_2 + q_2 g_i$, and $f_x = f_y$. Then, $f_{h_1} + q_1 f_{g_i} = f_{h_2} + q_2 f_{g_i}$, which implies $f_w = f_{g_i}$, where $w = (1/(q_2 - q_1))(h_1 - h_2) \in G_{i-1}$. Therefore, for any j and all $t <_T u_{i,j}$, $f_w(t) = f_{g_i}(t) = F(i,t) = f_{z_{i,j}}(t)$. If t_0 is T-least such that $f_w(t_0) - f_{z_{i,j}}(t_0) \neq 0_{K_t}$, then $u_{i,j} <_T t_0$ for all j, and therefore, $f_w - f_{z_{i,j}} \ll f_{u_{i,j}}$ for all j. Since $w, z_{i,j}, u_{i,j} \in G_{i-1}$, the induction hypothesis implies that $w - z_{i,j} \ll u_{i,j}$ for all j. Also, since $u_{i,j} = \mathrm{Rep}(g_i - z_{i,j})$, and $u_{i,j} <_T \mathrm{Rep}(w - z_{i,j})$, it follows from Lemma 5.3 that $\mathrm{Rep}(g_i - z_{i,j}) \leq_T \mathrm{Rep}(g_i - w)$ for all j. But, U_i is cofinal in S_i, so there is a $z_{i,j}$ with $\mathrm{Rep}(g_i - w) <_T \mathrm{Rep}(g_i - z_{i,j})$, which gives the desired contradiction.

We extend this argument by considering any $w \in G_i$, and letting

$$W = \{t \in T | f_w(t) \neq 0_{K_t}\}$$
$$\text{and } Z = \{t \in T | f_{g_i}(t) \neq 0_{K_t}\}.$$

If there exists a $t \in W$ such that $t' <_T t$ for all $t' \in Z$, then let t_0 be the least such. There must be a $t <_T t_0$ for which $f_{g_i}(t) \neq f_w(t)$. To see why, assume there is no such t, and let C be the cut determined by t_0. Since $w \in G_{i-1}$, there is a $w' \in G_{i-1}$ such that $Cf_w = f_{w'}$. However, $f_{g_i}(t) = f_w(t)$ for all $t <_T t_0$ and $f_{g_i}(t) = 0_{K_t}$ for all $t \geq_T t_0$, so $f_{w'} = f_{g_i}$, which contradicts the last claim.

Claim. The embedding of G_i preserves order.

By induction, we know that for any $g, h \in G_{i-1}$, $g \leq_G h$ if and only if $f_g \leq f_h$ in $\sum_T K_t$. To prove this claim for all $g, h \in G_i$, we split into cases.

First, consider $y \in G_{i-1}$ with $y <_G g_i$, and assume $f_{g_i} \leq f_y$. By the last claim, $f_y \neq f_{g_i}$, so $f_{g_i} < f_y$. If t_0 is the T-least argument for which $f_{g_i}(t_0) \neq f_y(t_0)$, then $f_{g_i}(t_0) <_{K_{t_0}} f_y(t_0)$. By the extension of the last claim, we know that t_0 does not exceed all the points in $\{t \in T | f_{g_i}(t) \neq 0_{K_t}\}$. Therefore, there is a $z_{i,j}$ for which $t_0 <_T \text{Rep}(g_i - z_{i,j}) = u_{i,j}$. It follows that $f_{g_i}(t_0) = f_{z_{i,j}}(t_0) < f_y(t_0)$ and $f_{z_{i,j}}(t) = f_y(t)$ for all $t < t_0$, and hence, $f_{z_{i,j}} < f_y$. Since $z_{i,j}, y \in G_{i-1}$, the induction hypothesis implies that $z_{i,j} <_G y$.

We now have that $z_{i,j} <_G y <_G g_i$, so $0_G <_G y - z_{i,j} <_G g_i - z_{i,j}$. These inequalities imply that $y - z_{i,j} \lessapprox g_i - z_{i,j}$, and so $u_{i,j} = \text{Rep}(g_i - z_{i,j}) \leq_T \text{Rep}(y - z_{i,j})$. Then, $\text{Rep}(y - z_{i,j}) \ll u_{i,j}$, which implies that $f_{y - z_{i,j}} \ll f_{u_{i,j}}$, and hence $f_y(t) = f_{z_{i,j}}(t)$ for all $t \leq_T u_{i,j}$. In particular, $f_y(t_0) = f_{z_{i,j}}(t_0)$, since $t_0 <_T u_{i,j}$, which contradicts our choice of t_0, finishing the proof for the case when $y <_G g_i$.

The case when $g_i <_G y$ with $y \in G_{i-1}$ is similar. To finish the claim, it suffices to check that for $q \in \mathbb{Q}$, $0 <_G qg_i + y$ implies that $qf_{g_i} + f_y$ is positive in $\sum_T K_t$. This case splits into subcases for $q > 0$ and $q < 0$. For the case when $q > 0$, notice that $0 <_G qg_i + y$ implies that $-(1/q)y <_G g_i$. Since $-(1/q)y \in G_{i-1}$, we can apply the previous case to conclude that $-(1/q)f_y < f_{g_i}$. Rewriting this inequality yields that $qf_{g_i} + f_y$ is positive. The case for $q < 0$ is similar.

This completes the proof that each G_i is isomorphic to the Hahn subgroup f_g, $g \in G_i$. Since $G = \cup G_i$, it is easy to verify that the sequence of functions f_g, $g \in G$ is a Hahn subgroup which is isomorphic to G.

REFERENCES

[1] A.H. Clifford, *Note on hahn's theorem on ordered abelian groups*, ***Proceedings of the American Mathematical Society***, vol. 5 (1954), pp. 860–863.

[2] Rodney G. Downey and S. A. Kurtz, *Recursion theory and ordered groups*, ***Annals of Pure and Applied Logic***, vol. 32 (1986), pp. 137–151.

[3] H.M. Friedman, S.G. Simpson, and R.L. Smith, *Countable algebra and set existence axioms*, ***Annals of Pure and Applied Logic***, vol. 25 (1983), pp. 141–181.

[4] L. Fuchs, ***Partially ordered algebraic systems***, Pergamon Press, 1963.

[5] K. Hatzikiriakou and S.G. Simpson, *wkl_0 and orderings of countable abelian groups*, ***Contemporary Mathematics***, vol. 106 (1989), pp. 170–177.

[6] M. Hausner and J.G. Wendel, *Ordered vector spaces*, ***Proceedings of the American Mathematical Society***, vol. 3 (1952), pp. 977–981.

[7] A.I. Kokorin and V.M. Kopytov, ***Fully ordered groups***, Halsted Press, 1974.

[8] S.G. Simpson, ***Subsystems of second order arithmetic***, Springer-Verlag, 1999.

[9] Reed Solomon, *Reverse mathematics and fully ordered groups*, ***Notre Dame Journal of Formal Logic***, vol. 39 (1998), pp. 157–189.

[10] ———, *Ordered groups: a case study in reverse mathematics*, ***The Bulletin of Symbolic Logic***, vol. 5 (1999), pp. 45–58.

[11] ———, $\pi^1_1 - ca_0$ *and the order types of countable ordered groups*, ***The Journal of Symbolic Logic***, (To appear).

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