EFFECTIVENESS FOR INFINITE VARIABLE WORDS AND THE DUAL RAMSEY THEOREM

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Abstract. We examine the Dual Ramsey Theorem and two related combinatorial principles $\mathrm{VW}(k,l)$ and $\mathrm{OVW}(k,l)$ from the perspectives of reverse mathematics and effective mathematics. We give a statement of the Dual Ramsey Theorem for open colorings in second order arithmetic and formalize work of Carlson and Simpson [1] to show that this statement implies ACA_0 over RCA_0. We show that neither $\mathrm{VW}(2,2)$ nor $\mathrm{OVW}(2,2)$ is provable in WKL_0. These results give partial answers to questions posed by Friedman and Simpson [3].

1. Introduction

Our goal is to examine several combinatorial theorems from the point of view of effective mathematics and reverse mathematics. In effective mathematics and reverse mathematics one attempts to establish the computability theoretic and proof theoretic strengths, respectively, of the theorems of ordinary mathematics. In this section, we give a brief description of these programs as well as the statements and definitions for the combinatorics we will study.

In effective mathematics, one typically uses the notion of Turing computability to measure the strength of a theorem T. The most basic problem is to assume the objects in the hypotheses of T are computable and to ask how complicated, in terms of Turing degree or arithmetic complexity, the objects in the conclusion of the theorem are. For example, consider the statement that every abelian group G has a subgroup H consisting of the elements of finite order. To analyze this statement, we assume that G is a computable group, which means that its domain is a computable subset of $\mathbb{N}$ and its multiplication operator is a computable partial function defined on the domain of G. The question of how complicated H can be often breaks into two steps. First, we attempt to use a diagonalization argument to show that H is not always computable. Once we know that H is not always computable, we try to use a coding argument to produce a sharp upper bound on the complexity of H. It is not hard to show in this case that H can code the halting problem. That is, there is a computable group G for which the subgroup H consisting of the elements of finite order satisfies $\mathbf{0}' \leq_T H$. Since the elements of H clearly form a computably enumerable set, we know that $\mathbf{0}' \equiv_T H$ and therefore that H can be as complicated as its definition allows.

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In reverse mathematics, we measure the complexity of the set theoretic axioms required to prove a particular theorem T. Because ZF is too powerful to measure subtle differences in proof theoretic strength, we use second order arithmetic, Z_2, as our set theory. Once T is appropriately stated in Z_2, we look for the weakest subsystem of Z_2 which suffices to prove T. More formally, we work over a base theory RCA_0 which allows comprehension for Δ^0_1 formulas and we attempt to find a subsystem S extending RCA_0 such that S suffices to prove T and such that RCA_0 plus T suffices to prove all the axioms in S. The most common subsystems S which occur are (in increasing order of strength): WKL_0 (which states that every infinite tree of binary strings has an infinite path), ACA_0 (which allows comprehension for arithmetic formulas), ATR_0 (which allows sets to be formed by transfinite recursion using arithmetic formulas) and Π^1_1-CA_0 (which allows comprehension for Π^1_1 formulas). In each of these systems, the formulas used for comprehension are allowed to contain parameters.

Not surprisingly, these two approaches are closely connected. For example, if we work with a model $\mathcal{M}$ (called an ω-model) in which the first order part is the standard copy of the natural numbers, then $\mathcal{M}$ is model of RCA_0 if and only if the second order part of $\mathcal{M}$ is closed under Turing reducibility and Turing join. Therefore, the least ω-model of RCA_0 has the computable sets as its second order part. Similarly, $\mathcal{M}$ is a model of WKL_0 if and only if the second order part is closed under Turing reducibility, Turing join and the existence of an infinite path for each infinite subtree of $2^{<\omega}$, and $\mathcal{M}$ is a model of ACA_0 if and only if the second order part is closed under Turing reducibility, Turing join and the Turing jump. The least ω-model of ACA_0 has the arithmetic sets as its second order part. There is no least ω-model of WKL_0, but there is a model in which every set has low degree. (That is, every set X satisfies $X' = \mathbf{0}'$.)

These characterizations give a useful method for transferring results of effective mathematics into results of reverse mathematics. We illustrate this method with Ramsey's Theorem [5]. For any $A \subseteq \mathbb{N}$, let $[A]^k$ denote the set of all k element subsets of A. By a partition, we always mean a partition into nonempty disjoint pieces.

Ramsey's Theorem. *The following statement holds for all $k, l \geq 1$. For any partition $[\mathbb{N}]^k = C_0 \cup \cdots \cup C_{l-1}$, there is an infinite set A and an $i < l$ such that $[A]^k \subseteq C_i$.*

For a fixed k and l, we refer to the statement of Ramsey's Theorem as $\mathrm{RT}(k, l)$. Any infinite set all of whose k element subsets are contained in one of the partitions (such as A) is called a *homogeneous* set. Specker [10] gave the first results in effective mathematics concerning Ramsey's Theorem.

Theorem 1.1. *There is a computable partition $[\mathbb{N}]^2 = C_0 \cup C_1$ which has no computable (or even Σ^0_1) homogeneous set.*

Because the computable sets form the second order part of an ω-model of RCA_0, the theorem implies that RCA_0 does not suffice to prove $\mathrm{RT}(2, 2)$. Jockusch [4] extended Theorem 1.1 by proving that a computable partition of $[\mathbb{N}]^2$ need not have a Σ^0_2 homogeneous set. He proved even more for partitions of $[\mathbb{N}]^3$.

Theorem 1.2. *There is a computable partition $[\mathbb{N}]^3 = C_0 \cup C_1$ such that any homogeneous set A satisfies $\mathbf{0}' \leq_T A$.*

Since there is an ω-model of WKL_0 in which all the sets are low, Theorem 1.2 shows that WKL_0 does not suffice to prove $\mathrm{RT}(3,2)$. In fact, the theorem says quite a bit more. Because the formulas used in the comprehension scheme for ACA_0 are allowed to contain parameters, the subsystem ACA_0 is equivalent to one in which the axioms only state comprehension for Σ^0_1 formulas. The proof that there are partitions for $\mathrm{RT}(3,2)$ which encode $\mathbf{0}'$ also shows that any model of $\mathrm{RT}(3,2)$ and RCA_0 must be closed under Σ^0_1 comprehension. Therefore, $\mathrm{RT}(3,2)$ suffices to prove ACA_0 over RCA_0. Since ACA_0 can prove $\mathrm{RT}(3,2)$, Theorem 1.2 completely classifies $\mathrm{RT}(3,2)$ in terms of reverse mathematics. (For more details, see Section III.7 in [7].)

In this article, we perform a similar (though less comprehensive) analysis for several variations of Ramsey's Theorem which were introduced by Carlson and Simpson [1] and later used by Simpson [6] to study initial segment constructions in degree theory. We begin with a classical statement of the Dual Ramsey Theorem as proved in [1]. Let $(\mathbb{N})^{\mathbb{N}}$ be the set of all partitions of $\mathbb{N}$ into infinitely many pieces and let $(\mathbb{N})^k$ be the set of all partitions of $\mathbb{N}$ into k pieces. For partitions X and Y of $\mathbb{N}$, we say Y is coarser than X if every partition block in X is contained in a partition block in Y. If $X \in (\mathbb{N})^{\mathbb{N}}$, then $(X)^k$ is the set of all $Y \in (\mathbb{N})^k$ which are coarser than X.

Dual Ramsey Theorem. *The following statement holds for all $k, l \geq 1$. For any partition $(\mathbb{N})^k = C_0 \cup \cdots \cup C_{l-1}$ in which each set C_i is Borel, there exists an $X \in (\mathbb{N})^{\mathbb{N}}$ and an $i < l$ such that $(X)^k \subseteq C_i$. (We discuss the topology in Section 2.)*

Friedman and Simpson [3] asked for an analysis of the Dual Ramsey Theorem in reverse mathematics. The only result to date is from Slaman [8] and states that the Dual Ramsey Theorem is provable in Π^1_1-CA_0. In Section 2, we carefully formalize (in Z_2) a version of the Dual Ramsey Theorem in which each C_i is required to be open.

Open Dual Ramsey Theorem. *For any partition $(\mathbb{N})^k = C_0 \cup \cdots \cup C_{l-1}$ in which each set C_i is open, there exists an $X \in (\mathbb{N})^{\mathbb{N}}$ and an $i < l$ such that $(X)^k \subseteq C_i$.*

We let $\mathrm{ODRT}(k,l)$ stand for this theorem with fixed values of k and l. Formalizing a proof given by Carlson and Simpson [1], we show that $\mathrm{ODRT}(k+1,l)$ implies $\mathrm{RT}(k,l)$ over RCA_0. From the discussion above, this immediately shows that for $k \geq 3$ and $l \geq 2$, $\mathrm{ODRT}(k,l)$ is not provable in WKL_0. It also shows that for $k \geq 4$ and $l \geq 2$, $\mathrm{ODRT}(k,l)$ implies ACA_0.

Friedman and Simpson [3] also asked for an analysis of Thereom 6.3 from Carlson and Simpson [1], a strengthening of one of the crucial lemmas in the proof of the Dual Ramsey Theorem. We state this principal in two different forms and with slightly different terminology than [3]. Let A denote a nonempty finite alphabet and let $\mathrm{Var} = \{x_n \mid n \in \mathbb{N}\}$ be an infinite set of variables which is disjoint from A.

Definition 1.3. An *infinite variable word* is an $\mathbb{N}$-sequence of elements of $A \cup \mathrm{Var}$ in which each x_i occurs at least once and at most finitely often. A *finite variable word* is a finite initial segment of an infinite variable word.

Definition 1.4. A finite or infinite variable word is *ordered* if all occurrences of x_i come before the first occurrence of x_{i+1}.

$W(x_i)$ denotes a (finite or infinite) variable word in which the variable x_i occurs at least once (although other variables can occur) and place($W(x_i)$) denotes the location of the first occurrence of x_i in W. For $c \in A$, $W(c/x_i)$ denotes the word obtained by substituting c into W for all occurrences of the variable x_i.

For any variable word W, a *substitution instance* of W is a word V of the same length as W in which all occurrences of each variable in W have been substituted by some letter from A. We allow different variables to be substituted by different letters from A. Frequently, we want to substitute for all the variables in W except for one distinguished variable. We indicate such a situation by saying that $V(x_i)$ is a substitution instance of all the variables in W except x_i.

For $n \in \mathbb{N}$, we let $W|n$ (called the *restriction* of W to n) denote the initial segment of W of length n. If x_i occurs in W and $n = \text{place}(W(x_i))$, then $W|n$ is the initial segment of W that ends just before the first occurrence of the variable x_i.

For an infinite variable word W, we let $W(A)$ denote the set of all $\alpha \in A^{<\mathbb{N}}$ such that $|\alpha| = \text{place}(W(x_i))$ for some $i \in \mathbb{N}$ and α is a substitution instance of W restricted to $|\alpha|$. That is, $W(A)$ contains all substitution instances of initial segments of W formed by cutting off W just before the first occurrence of x_i for each $i \in \mathbb{N}$.

Given these definitions, we let VW(k,l) be the statement that if $|A| = k$ and $c : A^{<\mathbb{N}} \to l$ is an l-coloring of $A^{<\mathbb{N}}$, then there is an infinite variable word W such that every element of $W(A)$ has the same color. (In this situation, we say that W is a *homogeneous* word for c.) We let OVW(k,l) be the same statement except that we require W to be an infinite ordered variable word. There is a proof of OVW(k,l) for all k and l in Carlson and Simpson [1, Theorem 6.3]. Friedman and Simpson [3] asked for an analysis of OVW(k,l) and in particular, they asked if this statement is effectively true. In Section 3, we show that neither OVW$(2,2)$ nor VW$(2,2)$ is provable in WKL_0 and therefore neither is effectively true.

In the last section, we list several open questions.

Our notation is standard. It follows [9] for computability theory, [7] for reverse mathematics and [1] for the Dual Ramsey Theorem. The reader who is interested in more background is referred to [2] for effective mathematics and [7] for reverse mathematics.

2. Open Dual Ramsey Theorem

We begin with the definitions necessary to state the Open Dual Ramsey Theorem in RCA_0. Throughout, we let $k \in \mathbb{N}$ stand for both the number k and the set $\{0, \dots, k-1\}$.

Definition 2.1. (RCA_0) For any $k \geq 1$, a *partition* of $\mathbb{N}$ into k blocks is a function f from $\mathbb{N}$ onto k such that $f(0) = 0$ and for all $0 < i < k$ and all $n \in \mathbb{N}$, if $f(n) = i$, then there is an $m < n$ such that $f(m) = i - 1$.

Such a function f partitions $\mathbb{N}$ into the blocks $K_i = \{n \in \mathbb{N} \mid f(n) = i\}$, for $0 \leq i < k$. Notice that this definition guarantees that if f and g are distinct functions which partition $\mathbb{N}$ into k blocks, then these partitions are different in the sense that neither can be obtained from the other by a permutation of the index labels on the blocks.

We can represent the class of all such functions as an open set in $k^{\mathbb{N}}$. Define P_k to be the set of all strings $\sigma \in k^{<\mathbb{N}}$ such that

(1) $\sigma(0) = 0$,
(2) $\forall i < k\, \exists n < |\sigma|\, (\sigma(n) = i)$,
(3) $\forall i < (k-1)\, \forall n < |\sigma|\, (\sigma(n) = i + 1 \rightarrow \exists m < n\, (\sigma(m) = i))$.

Because this definition uses only bounded quantification, RCA_0 suffices to prove the existence of P_k.

For any $\sigma \in k^{<\mathbb{N}}$, we let $[\sigma]$ denote the set of all functions $f : \mathbb{N} \rightarrow k$ such that $\sigma \subset f$. (We do not put any requirements here that f be onto k.) For any set $P \subset k^{<\mathbb{N}}$, we let $\mathcal{O}(P)$ denote the union of $[\sigma]$ for $\sigma \in P$. Recall that in the standard product topology on $k^{\mathbb{N}}$ (where k is given the discrete topology), the sets $[\sigma]$ for $\sigma \in k^{<\mathbb{N}}$ form a basis of clopen sets. Therefore, for any set $P \subset k^{<\mathbb{N}}$, $\mathcal{O}(P)$ is a union of basic open sets and hence $\mathcal{O}(P)$ is open. Our notation $\mathcal{O}(P)$ is intended to indicate that $\mathcal{O}(P)$ is the open set generated by the basic open sets represented in P. Later, we will also deal with closed sets generated as the set of paths through a tree. A tree is an nonempty set $T \subseteq k^{<\mathbb{N}}$ (or $T \subseteq \mathbb{N}^{<\mathbb{N}}$) such that if $\sigma \in T$ and $\tau \subset \sigma$, then $\tau \in T$. Let $[T]$ denote the set of infinite paths through T and recall that $[T]$ is a closed subset of $k^{\mathbb{N}}$.

Classically, it is clear that the class of all partitions of $\mathbb{N}$ into exactly k pieces is equal to $\mathcal{O}(P_k)$. By the comments above, $\mathcal{O}(P_k)$ is an open set in $k^{\mathbb{N}}$. Furthermore, RCA_0 is strong enough to prove that $\mathcal{O}(P_k)$ is not empty. For example, fix any $\sigma \in P_k$ and let f be the function such that $\sigma \subset f$ and for all $x \geq |\sigma|$, $f(x) = 0$. Therefore, we use $\mathcal{O}(P_k)$ to represent the class of all partitions of $\mathbb{N}$ into k blocks in RCA_0.

We will also need to consider the class of all partitions of $\mathbb{N}$ into an arbitrary, possibly infinite, number of blocks as well as the class of all partitions of $\mathbb{N}$ into infinitely many blocks. For this purpose, we define $\mathcal{T}$ to be the set of all strings $\sigma \in \mathbb{N}^{<\mathbb{N}}$ such that

(1) $\sigma(0) = 0$,
(2) $\forall i\, \forall n < |\sigma|\, (\sigma(n) = i + 1 \rightarrow \exists m < n\, (\sigma(m) = i))$.

Notice that $\mathcal{T}$ is a tree with no dead ends and that for any $\sigma \in \mathcal{T}$ and any $n < |\sigma|$, $\sigma(n) \leq n$. Therefore, $\mathcal{T}$ is a recursively branching tree. Classically, $[\mathcal{T}]$ contains all the partitions of $\mathbb{N}$ into either finitely or infinitely many blocks. Formally, a block of a partition $f \in [\mathcal{T}]$ is a set of the form $\{x \in \mathbb{N} \mid f(x) = i\}$, where i is an element of the range of f. RCA_0 suffices to prove the existence of $\mathcal{T}$ and to prove that $[\mathcal{T}]$ is not empty. For example, the identity function $f(n) = n$ is an element of $[\mathcal{T}]$.

If $f \in [\mathcal{T}]$ and $\mathrm{range}(f) = \mathbb{N}$, then f represents a partition of $\mathbb{N}$ into infinitely many blocks. A *refinement* of f is another partition which possibly collapses some of the partitioned blocks of f into larger blocks, but does not break up any of the blocks of f. Formally, $g \in [\mathcal{T}]$ is a refinement of f if $\forall i, j (f(i) = f(j) \rightarrow g(i) = g(j))$. For $f \in [\mathcal{T}]$, let $(f)^{\leq k}$ denote the class of all refinements of f into at most k blocks.

In order to represent $(f)^{\leq k}$, we define the tree $\mathcal{T}(f, k)$ to be the set of all strings $\sigma \in \mathcal{T}$ such that $\mathrm{range}(\sigma) \subset k$ and $\forall i, j < |\sigma| (f(i) = f(j) \rightarrow \sigma(i) = \sigma(j))$. RCA_0 suffices to prove the existence of $\mathcal{T}(f, k)$ from the parameters f and k. Furthermore, for any given k, RCA_0 can prove that if $\mathrm{range}(f) = \mathbb{N}$, then $[\mathcal{T}(f, k)]$ is nonempty since RCA_0 can form the path g_k such that $g_k(i) = f(i)$ if $f(i) < k$ and $g_k(i) = 0$ otherwise. g_k collapses all blocks after the k^{th} block into the first block, and leaves the blocks corresponding to $f(n) = 1, \ldots, f(n) = k-1$ unchanged. Since classically, $(f)^{\leq k}$ is exactly $[\mathcal{T}(f,k)]$, we use $[\mathcal{T}(f,k)]$ to represent the class of all refinements of f into at most k blocks in RCA_0.

An *open l-coloring* of $\mathcal{O}(P_k)$ is given by sets $S_0, \ldots, S_{l-1} \subset P_k$ such that

1. $\mathcal{O}(P_k) = \mathcal{O}(S_0) \cup \cdots \cup \mathcal{O}(S_{l-1})$,
2. for any $i \neq j$, $S_i \cap S_j = \emptyset$,
3. if $\sigma \in S_i$, then $\tau \in S_i$ for all $\sigma \subset \tau$ where $\tau \in P_k$.

Notice that the sets $\mathcal{O}(S_i)$ are pairwise disjoint by Condition (3). Therefore, the sets $\mathcal{O}(S_i)$ form a disjoint open cover of $\mathcal{O}(P_k)$, which we view as a coloring of $\mathcal{O}(P_k)$.

We can now state the Open Dual Ramsey Theorem for k partitions and l colors in RCA_0. For fixed k and l, we denote this statement by $\mathrm{ODRT}(k,l)$.

Theorem 2.2. *Let $S_0, \ldots, S_{l-1}$ be an open coloring of $\mathcal{O}(P_k)$. Then there is a partition $f \in [\mathcal{T}]$ and an $i < l$ such that range$(f) = \mathbb{N}$ and for all partitions $\alpha \in \mathcal{O}(P_k)$, if $\alpha \in [\mathcal{T}(f,k)]$, then $\alpha \in \mathcal{O}(S_i)$.*

Before proceeding, we should note that, classically, this theorem is a direct consequence of the Dual Ramsey Theorem. As mentioned above, our coloring consists of open (and hence Borel) sets. Thus, the classical Dual Ramsey Theorem implies that there is a partition f of $\mathbb{N}$ into infinitely many pieces such that all refinements of f into exactly k blocks have the same color. Any $\alpha \in [\mathcal{T}(f,k)]$ is a partition of $\mathbb{N}$ into at most k blocks. But, if this α is in $\mathcal{O}(P_k)$, then α must actually be a partition of $\mathbb{N}$ into exactly k pieces. Therefore, this statement of the Open Dual Ramsey Theorem is a direct consequence of the Dual Ramsey Theorem.

For any string $\sigma \in \mathcal{T}$ or partition $f \in [\mathcal{T}]$, we define

$$B_\sigma = \{i \mid 0 < i < |\sigma| \wedge \forall j < i\,(\sigma(j) < \sigma(i))\}$$
$$B_f = \{i \mid 0 < i \wedge \forall j < i\,(f(j) < f(i))\}.$$

In words, B_f represents the $\leq_{\mathbb{N}}$-least elements of the blocks of f, after the first block. We can now give the main theorem of this section. Both the statement and the proof of this theorem are formalizations in RCA_0 of Theorem 3.1 from [1].

Theorem 2.3. *(RCA_0) $ODRT(k+1,l)$ implies $RT(k,l)$.*

Proof. Recall that $\mathrm{RT}(k,l)$ says that for any l-coloring of the k-element subsets of $\mathbb{N}$, there is an infinite set X such that all k-element subsets of X have the same color. Therefore, we fix a coloring of the k-elements subsets of $\mathbb{N}$ written as a partition of the k-element subsets of $\mathbb{N}$ into l pairwise disjoint nonempty pieces $C_0, \ldots, C_{l-1}$. Define the sets U_i for $i < l$ by

$$U_i = \{\sigma \in P_{k+1} \mid B_\sigma \in C_i\}.$$

By the definition of B_σ, it is clear that if $\sigma \in U_i$ and $\sigma \subset \tau$ for any $\tau \in P_{k+1}$, then $\tau \in U_i$. It is also clear that the $U_0, \ldots, U_{l-1}$ are disjoint and $\mathcal{O}(U_0) \cup \cdots \cup \mathcal{O}(U_{l-1}) = \mathcal{O}(P_{k+1})$. Therefore, we can apply the Open Dual Ramsey Theorem to the coloring of $\mathcal{O}(P_{k+1})$ given by $U_0, \ldots, U_{l-1}$.

Fix $f \in [\mathcal{T}]$ and $i < l$ as in the statement of the Open Dual Ramsey Theorem. Consider B_f. Since range$(f) = \mathbb{N}$, we have that B_f is infinite. We claim that the set of k-element subsets of B_f is contained in C_i, and hence B_f is the required infinite homogeneous set.

Fix $\{y_1, \ldots, y_k\} \subset B_f$ and without loss of generality assume $y_1 < \cdots < y_k$. We will show that $\{y_1, \ldots, y_k\} \in C_i$. First, we construct $\alpha \in \mathcal{O}(P_{k+1})$ such that $\alpha \in [\mathcal{T}(f, k+1)]$ and $B_\alpha = \{y_1, \ldots, y_k\}$. Fix n_i such that $f(y_i) = n_i$. We define $\alpha(x) = 0$ if $f(x) \notin \{n_1, \ldots, n_k\}$ and $\alpha(x) = i$ if $f(x) = n_i$. Thus, we have collapsed all blocks in f other than the blocks containing one of the y_i elements into a single block in α, and we have left each block in f containing one of the y_i elements unchanged in α. Therefore, α has exactly $k+1$ blocks and $B_\alpha = \{y_1, \ldots, y_k\}$.

By our choice of f, we know $\alpha \in \mathcal{O}(U_i)$. Therefore, if we let $\sigma \subset \alpha$ be an initial segment such that σ maps onto $k+1$, then we must have $B_\sigma = \{y_0, \ldots, y_{k-1}\}$ and also $B_\sigma \in C_i$. Therefore, $\{y_0, \ldots, y_{k-1}\} \in C_i$ as required. □

Corollary 2.4. *For any $k \geq 3$ and $l \geq 2$, ODRT(k, l) is not provable in WKL$_0$.*

Proof. This follows from Theorem 2.3 and the fact that for $k \geq 2$ and $l \geq 2$, RT(k, l) is not provable in WKL$_0$. □

Corollary 2.5. *(RCA$_0$) For any $k \geq 4$ and any $l \geq 2$, ODRT(k, l) implies ACA$_0$.*

Proof. This follows from Theorem 2.3 and the fact that for $k \geq 3$ and $l \geq 2$, RT(k, l) implies ACA$_0$. □

3. Infinite variable words

In this section, we work with the finite alphabet $A = \{a, b\}$. We begin with the statement of our main theorem.

Theorem 3.1. *Let $A = \{a, b\}$. There is a computable two-coloring $c\colon A^{<\mathbb{N}} \to 2$ such that $W(A)$ is not homogeneous for any Δ^0_2 infinite ordered variable word W.*

Corollary 3.2. *WKL_0 does not suffice to prove $OVW(2, 2)$.*

Proof. By Theorem 3.1, any model of OVW$(2, 2)$ which contains the computable sets must contain non-low sets. However, there is an ω-model of WKL$_0$ containing all the computable sets in which every set is low. (See Corollary VIII.2.18 in [7].) □

To connect this statement to VW$(2, 2)$, we use the following simple lemma, which holds for any nonempty alphabet A.

Lemma 3.3. *Let A be a finite nonempty alphabet. If W is an infinite variable word such that $W(A)$ is homogeneous for the coloring c, then there exists an infinite ordered variable word V such that $V(A)$ is homogeneous for c and $V \leq_T W'$. (W' denotes the Turing jump of W.)*

Proof. We form V by choosing a subsequence $\{y_i\}_{i \in \mathbb{N}}$ of the variables $\{x_m\}_{m \in \mathbb{N}}$. Let $y_0 = x_0$. Assume that y_i has been determined to be x_{k_i}. Using W' we can find the last occurrence of x_{k_i} in W. Let $x_{k_{i+1}}$ be the first variable which has its first occurrence in

W after the last occurrence of x_{k_i}. Set $y_{i+1} = x_{k_{i+1}}$. This defines the sequence $\{y_i\}$ of variables. Let V be the word obtained from W by substituting an arbitrary letter $a \in A$ for each variable $x_j \notin \{y_i\}$ and by substituting x_i for y_i. It is clear that V is an infinite ordered variable word and that $V(A)$ is a subset of $W(A)$. Therefore, $V(A)$ is homogeneous for c. □

Corollary 3.4. *WKL_0 does not suffice to prove $VW(2,2)$.*

Proof. By Theorem 3.1 and Lemma 3.3, any ω-model of VW$(2,2)$ which contains the computable sets must contain a non-low set. That is, suppose c is a computable coloring as in Theorem 3.1 and W is an infinite variable word with low Turing degree for which $W(A)$ is homogeneous. By Lemma 3.3, there is an infinite ordered variable word V which is computable in the Turing jump of W (and hence is Δ^0_2) for which $V(A)$ is homogeneous. This contradicts the choice of c. □

We begin the proof of Theorem 3.1 with a series of definitions and technical lemmas needed to show that our eventual construction succeeds.

Definition 3.5. A finite set $W_1(x_{i_1}), \ldots, W_n(x_{i_n})$ of finite variable words with distinguished variables is *admissible* if for all $j \neq k \leq n$, place$(W_j(x_{i_j})) \neq$ place$(W_k(x_{i_k}))$.

Definition 3.6. Let $s \in \mathbb{N}$ and let $W_1(x_{i_1}), \ldots, W_n(x_{i_n})$ be an admissible set of words such that for all $m \leq n$, $|W_m| < s$. Let k_m be such that $|W_m| + k_m = s$. The (undirected and labeled) *graph G induced by* $W_1(x_{i_1}), \ldots, W_n(x_{i_n})$ on A^s is defined as follows. The nodes of G are the elements of A^s, the set of words of length s over A. If $\alpha, \beta \in A^s$, then we put an edge labeled by W_m between α and β if and only if there is a substitution instance $\widehat{W}_m(x_{i_m})$ of all the variables in W_m except x_{i_m} and a string $\delta \in A^{k_m}$ such that either

$$\big(\alpha = \widehat{W}_m(a/x_{i_m}) * \delta \text{ and } \beta = \widehat{W}_m(b/x_{i_m}) * \delta\big) \text{ or}$$
$$\big(\alpha = \widehat{W}_m(b/x_{i_m}) * \delta \text{ and } \beta = \widehat{W}_m(a/x_{i_m}) * \delta\big).$$

Whenever we use the term "induced graph," we assume that it has been induced by an admissible set of words as described above. Notice that since the variable x_{i_j} is assumed to occur in the word W_j, there cannot be a node α with an edge labeled by W_j from α to itself. Furthermore, by the following lemma, dropping the edge labels of an induced graph G results in an ordinary undirected graph without multiple edges between nodes. This property plays no role in our construction, but it is included to give the reader some intuition about these graph structures.

Lemma 3.7. *Let G be an induced graph on A^s. For any pair of distinct elements $\alpha, \beta \in A^s$, there is at most one labeled edge between α and β.*

Proof. Suppose not and fix α, β and $j \neq k$ such that there are edges labeled by W_j and W_k between α and β. Since place$(W_j(x_{i_j})) \neq$ place$(W_k(x_{i_k}))$, we can assume without loss of generality that place$(W_j(x_{i_j})) <$ place$(W_k(x_{i_k}))$. The fact that α and β differ at the location place$(W_j(x_{i_j}))$ contradicts the existence of an edge labeled by W_k between α and β. □

Lemma 3.8. *Let G be an induced graph on A^s. G has only even length cycles.*

Proof. Fix any cycle in G and a node α in this cycle. Assume that the labels that occur in the cycle are $W_1, \dots, W_n$ and that $\text{place}(W_1(x_{i_1})) < \cdots < \text{place}(W_n(x_{i_n}))$. It suffices to show by induction that each of these labels occurs on an even number of edges in the cycle.

Consider the label W_1. Imagine starting at α, traveling around the cycle exactly once and keeping track of the letter which occurs in position $\text{place}(W_1(x_{i_1}))$. Since $\text{place}(W_1(x_{i_1}))$ is the least among the places of the distinguished variables of the labels in this cycle, the letter in this position can change only when we cross an edge labeled by W_1. Since the letter in $\text{place}(W_1(x_{i_1}))$ must have its original value when we return to α after going around the cycle, there must be an even number of edges labeled by W_1.

Assume by induction that each label W_i for $i < m$ occurs an even number of times in the cycle and consider the label W_m. As before, imagine starting at α, traveling once around the cycle and keeping track of the letter which occurs in position $\text{place}(W_m(x_{i_m}))$. The letter in this position will change every time we cross an edge labeled W_m, the letter cannot change when we cross an edge labeled W_k with $k > m$, and the letter may change when we cross an edge labeled W_l for $l < m$. However, if the letter changes when we cross an edge labeled W_l, then it must change every time we cross such an edge. Therefore, by induction, there are an even number of changes at position $\text{place}(W_m(x_{i_m}))$ caused by edges labeled W_l with $l < m$ and none caused by edges labeled W_k for $k > m$. Since the letter in $\text{place}(W_m(x_{i_m}))$ must have its original value when we return to α after going around the cycle, there must be an even number of edges labeled by W_m. □

By Lemma 3.8, any induced graph G can be two-colored. That is, there is a map from the the unlabeled version of G to $\{0, 1\}$ such that no two nodes connected by an edge have the same value. We now state the main combinatorial lemma used to prove Theorem 3.1.

Lemma 3.9. *Let $W_1(x_{i_1}), \dots, W_n(x_{i_n})$ be an admissible set of finite variable words such that for all $m \le n$, $|W_m| < s$. Then, there is a two-coloring of A^s such that for each $W_m(x_{i_m})$ the following property holds. Let k_m be such that $|W_m(x_{i_m})| + k_m = s$. For each substitution instance $\widehat{W}_m(x_{i_m})$ of all the variables in W_m except x_{i_m} and for all $\alpha \in A^{k_m}$*

$$c(\widehat{W}_m(a/x_{i_m}) * \alpha) \neq c(\widehat{W}_m(b/x_{i_m}) * \alpha).$$

Proof. Let G be the graph induced on A^s by the admissible set of finite variable words. If $\widehat{W}_m(x_{i_m})$ and α are as in the statement of the lemma, then there is an edge labeled by W_m from $\widehat{W}_m(a/x_{i_m}) * \alpha$ to $\widehat{W}_m(b/x_{i_m}) * \alpha$. Therefore, any two-coloring c of G has the desired property. □

Lemma 3.10. *Let $c : A^{<\mathbb{N}} \to \{0, 1\}$ be a two-coloring and let W be a finite ordered variable word in which $x_0, \dots, x_e$ occur. Suppose there is a $k_0 \in \mathbb{N}$ such that the following property holds. For each $k > k_0$, there exists $i_k \le e$ such that for all substitution* *instances* $\widehat{W}(x_{i_k})$ *of all variables in* W *except* x_{i_k} *and all* $\alpha \in A^k$*, we have*

$$c(\widehat{W}(a/x_{i_k}) * \alpha) \neq c(\widehat{W}(b/x_{i_k}) * \alpha).$$

Then, W *is not an initial segment of any homogeneous infinite ordered variable word* U *in which all occurrences of* $x_0, \dots, x_e$ *in* U *occur in* W*.*

Proof. Let U be any infinite ordered variable word such that $W \subset U$ and all occurrences of $x_0, \dots, x_e$ in U occur in W. We need to show that U is not homogeneous for the coloring c. Fix a variable x_m such that $\text{place}(U(x_m)) > |W| + k_0$. Set $k = \text{place}(U(x_m)) - |W|$ and note that $k > k_0$. Let $i_k \leq e$ be the value chosen for k in the statement of the lemma.

Choose any substitution instance $\widehat{W}(x_{i_k})$ of all the variables in W except x_{i_k}. Choose a string $\alpha \in A^k$ such that $\widehat{W}(a/x_{i_k}) * \alpha$ and $\widehat{W}(b/x_{i_k}) * \alpha$ are both in $U(A)$ (if one of them is contained in $U(A)$, then the other must also be). By hypothesis,

$$c(\widehat{W}(a/x_{i_k}) * \alpha) \neq c(\widehat{W}(b/x_{i_k}) * \alpha).$$

Therefore, U is not homogeneous. □

We can now prove Theorem 3.1.

Proof. We construct a computable coloring c of $A^{<\mathbb{N}}$ such that no Δ^0_2 infinite ordered variable word is homogeneous for c. Our construction proceeds in stages and at stage s we color all strings in A^s. Because the construction is effective, our coloring is computable. We meet the requirements $\mathcal{R}_e$: if $U_e(n) = \lim_s \varphi_e(n, s)$ is a total function which represents an infinite ordered variable word (under a fixed effective coding of $A \cup \text{Var}$), then $U_e(A)$ is not homogeneous for c. Here, φ_e is the standard effective list of all computable partial functions. By the Limit Lemma, if we meet all the $\mathcal{R}_e$ requirements, then we will have successfully diagonalized against all Δ^0_2 infinite ordered variable words.

At stage s, we let $U_{e,s}$ be our approximation to U_e. We define $U_{e,s}$ to be the word formed by the longest initial segment of $\varphi_{e,s}(0, s), \dots, \varphi_{e,s}(s-1, s)$ which converges. Notice that $|U_{e,s}| \leq s$ and that if U_e is an infinite ordered variable word, then we will eventually see longer and longer initial segments of $U_{e,s}$ which are correct. At stage 0 we color the empty sequence arbitrarily and do nothing else.

Construction at Stage s.

For each $e \leq s$, check if $U_{e,s}$ is a finite variable word in which $x_0, \dots, x_{e+1}$ all occur. If not, then we ignore $U_{e,s}$ at this stage. If so, then define $V_{e,s}$ to be the initial segment of $U_{e,s}$ which ends just before the first occurrence of x_{e+1}. If $V_{e,s}$ is a finite ordered variable word, then we say $V_{e,s}$ *requires attention*. Otherwise, we ignore $V_{e,s}$ at this stage.

Let $V_{j_0,s}, \dots, V_{j_n,s}$ be the words requiring attention and assume that $j_0 < j_1 < \cdots < j_n$. For each $m \leq n$, pick a *pivot variable* x_{i_m} for $V_{j_m,s}$ such that $i_m \leq j_m$ and for any $m \neq p$,

$$\text{place}(V_{j_m,s}(x_{i_m})) \neq \text{place}(V_{j_p,s}(x_{i_p})).$$

These choices can be made by starting with $V_{j_0,s}$ and proceeding to $V_{j_n,s}$ since each $V_{j_m,s}$ has $j_m + 1$ variables occurring in it.

Next, color the strings in A^s such that the following property holds for each $V_{j_m,s}$. Let k_m be such that $|V_{j_m,s}| + k_m = s$. For each substitution instance $\widehat{V}_{j_m,s}(x_{i_m})$ of all the variables in $V_{j_m,s}$ except x_{i_m} and for all $\alpha \in A^{k_m}$,

$$c(\widehat{V}_{j_m,s}(a/x_{i_m}) * \alpha) \neq c(\widehat{V}_{j_m,s}(b/x_{i_m}) * \alpha).$$

The coloring can be defined consistent with these requirements by the Lemma 3.9. (If no words require attention at stage s, then color A^s arbitrarily.) Proceed to stage $s+1$.
End of Construction

It remains to check that we have met the requirements $\mathcal{R}_e$. Assume that U_e is total and represents an infinite ordered variable word. Set $n = \text{place}(U(x_{e+1}))$ and fix a stage $t > n$ such that $U_{e,t}(i) = U_e(i)$ for all $i \leq n$. For every stage $s \geq t$, $V_{e,s}$ is defined and is equal to $U_e|n$. Therefore, $V_{e,s}$ has reached a limit which we call V_e. Furthermore, V_e requires attention at every stage $s \geq t$. Let $k_0 = t - |V_e|$. Consider any $k > k_0$, let $s = t + (k - k_0)$ and notice that $|V_e| + k = s$. At stage s of our construction, we pick a pivot variable for V_e from among $x_0, \ldots, x_e$. Denote this variable by y_s. Our coloring at stage s guarantees that for all strings $\alpha \in A^k$ and all substitution instances $\widehat{V}_e(y_s)$ of all variables in V_e except y_s, we have $c(\widehat{V}_e(a/y_s) * \alpha) \neq c(\widehat{V}_e(b/y_s) * \alpha)$. By Lemma 3.10, we know that V_e is not the initial segment of an infinite ordered variable word which is homogeneous for c. However, by assumption, V_e is an initial segment of U_e and therefore, we have met requirement $\mathcal{R}_e$. □

4. Open Problems

Problem 1. In the previous section, we used a diagonalization argument to show that OVW$(2,2)$ and VW$(2,2)$ are not provable in WKL$_0$. Can this be replaced with a coding argument which shows that, for at least some values of k and l, OVW(k,l) and VW(k,l) imply ACA$_0$?

Problem 2. Determine the set theoretic strength required to prove OVW(k,l) and VW(k,l). The classical proof of OVW(k,l) given in Carlson and Simpson [1] goes beyond arithmetic comprehension. Can these principles be proved in ACA$_0$?

Problem 3. Close the gap between Π^1_1-CA$_0 \vdash$ ODRT(k,l) and RCA$_0$ + ODRT$(k,l) \vdash$ ACA$_0$.

There is another closely related infinite variable word principle which we will denote VWI(k,l). Given a finite alphabet A, we say that an $\mathbb{N}$-sequence W of elements from $A \cup \text{Var}$ is a *variable word with infinitely occurring variables* if each variable x_n occurs at least once in W. Note that variables are allowed to occur infinitely often, but are not required to do so. The principle VWI(k,l) is the same as VW(k,l) except the homogeneous word is allowed to be a variable word with infinitely occurring variables. VWI(k,l) is clearly implied by both OVW(k,l) and VW(k,l), but our diagonalization techniques in Section 3 do not immediately work for VWI$(2,2)$. VW(k,l) is of interest since it is the actual combinatorial principal required by Carlson and Simpson in their proof of the Dual Ramsey Theorem.

Problem 4. Is VWI(k, l) proof theoretically simpler than VW(k, l)? In particular, is it provable in WKL_0 or even effectively true?

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