Ordered Groups: A Case Study In Reverse Mathematics *

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August 28, 2003

1 Introduction

The fundamental question in reverse mathematics is to determine which set existence axioms are required to prove particular theorems of mathematics. In addition to being interesting in their own right, answers to this question have consequences in both effective mathematics and the foundations of mathematics. Before discussing these consequences, we need to be more specific about the motivating question.

Reverse mathematics is useful for studying theorems of either countable or essentially countable mathematics. Essentially countable mathematics is a vague term that is best explained by an example. Complete separable metric spaces are essentially countable because, although the spaces may be uncountable, they can be understood in terms of a countable basis. Simpson (1985) gives the following list of areas which can be analyzed by reverse mathematics: number theory, geometry, calculus, differential equations, real and complex analysis, combinatorics, countable algebra, separable Banach spaces, computability theory, and the topology of complete separable metric spaces. Reverse mathematics is less suited to theorems of abstract functional analysis, abstract set theory, universal algebra, or general topology.

Section 2 introduces the major subsystems of second order arithmetic used in reverse mathematics: RCA_0, WKL_0, ACA_0, ATR_0 and $\Pi^1_1 - CA_0$. Sections 3 through 7 consider various theorems of ordered group theory from the perspective of reverse mathematics. Among the results considered are theorems on ordered quotient groups (including an equivalent of ACA_0), groups and semigroup conditions which imply orderability (WKL_0), the orderability of free groups (RCA_0), Hölder's Theorem (RCA_0), Mal'tsev's classification of the order types of countable ordered groups ($\Pi^1_1 - CA_0$) and the existence of strong divisible closures (ACA_0). Section 8 deals more directly with computability

*This material comes from the author's PhD thesis under the direction of Richard Shore, Cornell University issues and discusses the relationship between Π^0_1 classes and the space of orders on an orderable computable group. For more background and examples of equivalences between theorems and subsystems of second order arithmetic, see Friedman et al. (1983), Simpson (1985) or Simpson (1999). Our notation for reverse mathematics follows Friedman et al. (1983) and Simpson (1999) and for ordered groups follows Kokorin and Kopytov (1974) and Fuchs (1963).

2 Reverse Mathematics

The setting for reverse mathematics is second order arithmetic, Z_2. Z_2 uses a two sorted first order language, $\mathcal{L}_2$, which has both number variables and set variables. The number variables are denoted by lower case letters and are intended to range over ω. The set variables are denoted by capital letters and are intended to range over $\mathcal{P}(\omega)$. Because $\mathcal{L}_2$ has two types of variables, it also has two types of quantifiers: $\exists X, \forall X$ and $\exists x, \forall x$. The terms of $\mathcal{L}_2$ are built from the number variables and the constants $0, 1$ using the function symbols $+, \cdot$. Atomic formulas have the form $t_1 = t_2$, $t_1 < t_2$ or $t_1 \in X$ where t_1, t_2 are terms. General formulas are built from the atomic formulas using the standard logical connectives and the two types of quantifiers.

A model for $\mathcal{L}_2$ is a first order structure $\mathfrak{A} = \langle A, S_A, +_A, \cdot_A, 0_A, 1_A, <_A\rangle$. The number variables range over A, the set variables range over $S_A \subseteq \mathcal{P}(A)$, and the functions, constants, and relations are interpreted as indicated. The intended model for Z_2 is $\langle \omega, \mathcal{P}(\omega), +, \cdot, 0, 1, <\rangle$ and any model in which the number variables range over ω is called an ω-model.

The axioms for Z_2 consist of axioms specifying the ordered semiring properties of the natural numbers, an induction axiom,

$$\big(0 \in X \,\wedge\, \forall n(n \in X \rightarrow n+1 \in X)\big) \rightarrow \forall n(n \in X)$$

and a comprehension scheme for forming sets,

$$\exists X\, \forall n\, (n \in X \leftrightarrow \varphi(n))$$

where φ is any formula of $\mathcal{L}_2$ in which X does not occur freely. φ may contain other free variables as parameters. The following formula induction scheme is derivable from the comprehension scheme and the induction axiom:

$$\big(\varphi(0) \,\wedge\, \forall n(\varphi(n) \rightarrow \varphi(n+1))\big) \rightarrow \forall n \varphi(n).$$

The question of which set existence axioms are necessary to prove a particular theorem is answered by examining which instances of the comprehension scheme are required. Hence, we examine subsystems of Z_2 which arise from restricting the formulas over which the comprehension scheme applies. One consequence of limiting comprehension is that the formula induction scheme is also limited. We will not add full formula induction to the subsystems because this scheme is a disguised set existence principle. For a complete discussion of this issue, see Friedman et al. (1983) and Simpson (1999). A surprising observation is that the motivating question can be answered for a remarkable number of theorems by examining only five subsystems of Z_2. These subsystems are presented below in increasing order of strength.

The weakest subsystem considered here is RCA_0, or Recursive Comprehension Axiom. It is obtained by restricting the comprehension scheme to Δ^0_1 formulas and allowing induction over Σ^0_1 formulas. Because RCA_0 contains Δ^0_1 comprehension and the comprehension scheme allows parameters, it follows that every model of RCA_0 is closed under Turing reduction. The minimum ω-model is $\langle \omega, REC, +, \cdot, 0, 1, < \rangle$ where $REC \subseteq \mathcal{P}(\omega)$ is the computable sets.

If a theorem is provable in RCA_0 then its effective or computable version is true. This process is illustrated in the following example. Given suitable definitions, Friedman et al. (1983) showed that RCA_0 suffices to prove that every field has an algebraic closure. This result implies that every computable field has a computable algebraic closure, a theorem first proved in Rabin (1960). Because of this connection to computable mathematics, we will not look for proofs in weaker subsystems. For examples of weaker subsystems, see Hatzikiriakou (1989).

RCA_0 is strong enough to establish the basic facts about the number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$, which makes RCA_0 a reasonable base theory over which to do coding. The set of natural numbers, $\mathbb{N}$, is defined by the formula $x = x$ and so $\forall n\, (n \in \mathbb{N})$. The integers and rationals are defined with the help of the standard pairing function, $\langle x, y \rangle$. Notice that we use ω to denote the standard natural numbers and $\mathbb{N}$ to denote the (possibly nonstandard) universe of a model of a subsystem of Z_2.

The real numbers do not form a set in Z_2, they must be represented by Cauchy sequences of rationals. Simpson (1999) gives the following definitions.

Definition 2.1. (RCA_0) A **sequence of rational numbers** is a function $f : \mathbb{N} \to \mathbb{Q}$ often denoted by $\langle q_k \,|\, k \in \mathbb{N} \rangle$. A **real number** is a sequence of rational numbers $\langle q_k \,|\, k \in \mathbb{N} \rangle$ such that $\forall k \forall i\, (\ |q_k - q_{k+i}| < 2^{-k}\ )$. If $x = \langle q_k \,|\, k \in \mathbb{N} \rangle$ and $y = \langle q'_k \,|\, k \in \mathbb{N} \rangle$ are real numbers, then we say $\mathbf{x} = \mathbf{y}$ if $\forall k\, (\,|q_k - q'_k| \leq 2^{-k+1}\,)$. The sum $\mathbf{x} + \mathbf{y}$ is the sequence $\langle q_{k+1} + q'_{k+1} \,|\, k \in \mathbb{N} \rangle$. The product $\mathbf{x} \cdot \mathbf{y}$ is the sequence $\langle q_{n+k} \cdot q'_{n+k} \,|\, k \in \mathbb{N} \rangle$ where n is the least natural number such that $2^n \geq |q_0| + |q_1| + 2$.

We write $x \in \mathbb{R}$ to mean that x is a sequence of rationals satisfying the required convergence rate, $0_\mathbb{R}$ to denote the sequence $\langle 0 \,|\, k \in \mathbb{N} \rangle$, and $1_\mathbb{R}$ to denote $\langle 1 \,|\, k \in \mathbb{N} \rangle$. With these definitions RCA_0 suffices to prove that the real number system obeys all the axioms of an Archimedean ordered field.

RCA_0 is also strong enough to define a set of unique codes and a length function for finite sequences. Fin_X denotes the set of finite sequences of elements of X. For any function $f : \mathbb{N} \to \mathbb{N}$, $f[n] = \langle f(0), \ldots, f(n-1) \rangle$.

Definition 2.2. (RCA_0) A **tree** is a set $T \subseteq \mathrm{Fin}_\mathbb{N}$ which is closed under initial segments. T is **binary branching** if every element of T has at most two successors. A **path** through T is a function $g : \mathbb{N} \to \mathbb{N}$ such that $g[n] \in T$ for all n.

Weak König's Lemma. *Every infinite binary branching tree has a path.*

The second subsystem of Z_2 is called WKL_0 and contains the axioms of RCA_0 plus Weak König's Lemma. Because the effective version of Weak König's Lemma fails, WKL_0 is strictly stronger than RCA_0. The best intuition for WKL_0 is that Weak König's Lemma adds a compactness principle to RCA_0.

We can now give a more formal explanation of the goals and methods of reverse mathematics. Consider a theorem Thm of ordinary mathematics which is stated in $\mathcal{L}_2$. If $WKL_0 \vdash Thm$ and we cannot find a proof of Thm in RCA_0, then we attempt to show for each axiom φ of WKL_0 that $RCA_0 + Thm \vdash \varphi$. If we succeed then we have shown in RCA_0 that Thm is equivalent to the subsystem WKL_0 and that no subsystem strictly weaker than WKL_0 can prove Thm. In particular, we can stop looking for a proof of Thm in RCA_0. This process of proving axioms from theorems is the origin of the name reverse mathematics.

When trying to prove reversals, it is often helpful to use the fact that WKL_0 is equivalent to the existence of separating sets for pairs of functions with disjoint ranges. The following theorem gives other examples of equivalents of WKL_0.

Theorem 2.3. *(RCA_0) WKL_0 is equivalent to each of the following:*

1. *Every continuous function on $0 \leq x \leq 1$ is bounded (Simpson (1999)).*
2. *Gödel's Completeness Theorem for predicate logic (Simpson (1999)).*
3. *Every countable commutative ring has a prime ideal (Friedman et al. (1983)).*
4. *The Separable Hahn–Banach Theorem: if f is a bounded linear functional on a closed separable Banach space and if $\|f\| \leq r$, then f has an extension $\hat{f}$ to the whole space such that $\|\hat{f}\| \leq r$ (Brown and Simpson (1986)).*

Any theorem equivalent to WKL_0 fails to be effectively true. Although we have presented results in computable mathematics as consequences of results in reverse mathematics, frequently the results in computable mathematics come first. If the effective version of a theorem is known to hold, then its proof can often be translated into RCA_0. Similarly, if the computable version fails, then why it fails often gives a hint as to how to prove a reversal. Provability in WKL_0 also has consequences for the foundations of mathematics. Because WKL_0 is Π^0_2 conservative over primitive recursive arithmetic (see Parsons (1970)), it provides a modern rendering of Hilbert's Program. For a discussion of these issues see Simpson (1988), Drake (1989) and Feferman (1988).

The third subsystem of Z_2 is ACA_0, or Arithmetic Comprehension Axiom, and contains RCA_0 plus Σ^0_1 comprehension. Because Σ^0_1 comprehension suffices to define the Turing jump, models of ACA_0 are closed under the jump and the minimum ω-model consists of the arithmetic sets. To prove reversals, it is useful to know that ACA_0 is equivalent to the existence of the range of 1-1 functions.

Theorem 2.4. *(RCA_0) ACA_0 is equivalent to each of the following:*

1. *The Bolzano–Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence (Friedman (1976)).*
2. *Every countable vector space over $\mathbb{Q}$ has a basis (Friedman et al. (1983)).*
3. *Every countable commutative ring has a maximal ideal (Friedman et al. (1983)).*

Definition 2.5. (RCA_0) A **linear order** is a pair $(X, \leq_X)$ such that X is a set and $\leq_X$ is a binary relation on X which is reflexive, transitive and antisymmetric. A linear order is a **well order** if there is no function $f : \mathbb{N} \to X$ such that $f(n+1) <_X f(n)$ for all n.

The fourth system is ATR_0, or Arithmetic Transfinite Recursion, and includes ACA_0 plus axioms that allow arithmetic comprehension to be iterated along any well order. While ACA_0 is strong enough to prove that the n^{th} Turing jump $0^{(n)}$ exists, ATR_0 is required to construct the uniform upper bound $0^{(\omega)}$. For a formal description of the axioms of ATR_0 see Friedman et al. (1983). There are many equivalents of ATR_0 in the theory of ordinals (see Hirst (1994) and Friedman and Hirst (1990)), the simplest being the comparability of any two well orders (Friedman (1976)).

The last and most powerful subsystem is $\Pi^1_1 - CA_0$ which is ACA_0 plus Π^1_1 comprehension. This system is strong enough to define Kleene's $\mathcal{O}$ and hence models of $\Pi^1_1 - CA_0$ are closed under the hyperjump. $\Pi^1_1 - CA_0$ is strictly stronger than ATR_0 and is useful for proving that certain recursions terminate. An example of this phenomenon is presented in Section 6. Equivalents of $\Pi^1_1 - CA_0$ include versions of the Cantor–Bendixson Theorem for $\mathbb{N}^{\mathbb{N}}$ and $2^{\mathbb{N}}$ (Friedman (1976), Kreisel (1959)).

3 Ordered Quotient Groups

Definition 3.1. (RCA_0) A **partially ordered (p.o.) group** is a group G together with a partial order $\leq_G$ on the elements of G such that for any $a, b, c \in G$, if $a \leq_G b$ then $a \cdot_G c \leq_G b \cdot_G c$ and $c \cdot_G a \leq_G c \cdot_G b$. If the order is linear, then $(G, \leq_G)$ is called a **fully ordered (f.o.) group**. A group which admits a full order is called an **O-group**

We frequently drop the subscripts on $\cdot_G$ and $\leq_G$. The following examples, especially the second one, are useful to keep in mind.

Example 3.2. The additive groups $(\mathbb{R}, +)$, $(\mathbb{Q}, +)$, and $(\mathbb{Z}, +)$ with the standard orders are all f.o. groups.

Example 3.3. Let G be the free abelian group with ω generators a_i for $i \in \omega$. Elements of G have the form $\sum_{i \in I} r_i a_i$ where $I \subseteq \omega$ is a finite set, $r_i \in \mathbb{Z}$ and $r_i \neq 0$. To compare that element with $\sum_{j \in J} q_j a_j$, let $K = I \cup J$ and define $r_k = 0$ for $k \in K \setminus I$ and $q_k = 0$ for $k \in K \setminus J$. The full order is given by $\sum_{i \in I} r_i a_i < \sum_{j \in J} q_j a_j$ if and only if $r_k < q_k$ where k is the largest element of K for which $r_k \neq q_k$.

Definition 3.4. (RCA_0) The **positive cone** of a p.o. group $(G, \leq_G)$ is the set $P(G, \leq_G) = \{g \in G \mid 1_G \leq_G g\}$.

The equation $ab^{-1} \in P(G) \leftrightarrow b \leq a$ shows that RCA_0 suffices to define the positive cone from the order and vice versa and also that the Turing degree of the order and the positive cone are the same. There is a classical set of algebraic conditions that determines if an arbitrary subset of a group is the positive cone of some full or partial order. This algebraic characterization is useful for building orders by guessing at approximations on a binary branching tree.

Definition 3.5. (RCA_0) If G is a group and $X \subseteq G$, then $X^{-1} = \{g^{-1} \mid g \in X\}$. X is a **full subset of G** if $X \cup X^{-1} = G$ and X is a **pure subset of G** if $X \cap X^{-1} \subseteq \{1_G\}$.

Theorem 3.6. *(RCA_0) A subset P of a group G is the positive cone of some partial order on G if and only if P is a normal pure semigroup with identity. Furthermore, P is the positive cone of a full order if and only if in addition P is full.*

If H is a normal subgroup of G, then we define G/H in Z_2 by picking the $\leq_{\mathbb{N}}$-least element of each equivalence class. Since $aH = bH \leftrightarrow a^{-1}b \in H$, RCA_0 suffices to form the quotient group. If H is a convex normal subgroup, then the order on G induces an order on G/H.

Definition 3.7. (RCA_0) H is a **convex** subgroup of G if for any $a, b \in H$ and $g \in G$, we have that $a \leq g \leq b$ implies $g \in H$. If H is a convex normal subgroup, then the **induced order** on G/H is defined for $a, b \in G/H$ by $a \leq_{G/H} b \leftrightarrow \exists h \in H(a \leq_G bh)$.

This definition is Σ^0_1, so it is not clear whether RCA_0 is strong enough to prove the existence of the induced order. It turns out that RCA_0 suffices if G is an f.o. group but not if G is a p.o. group.

Theorem 3.8. *(RCA_0) For every f.o. group $(G, \leq)$ and every convex normal subgroup H, the induced order on G/H exists.*

Theorem 3.9. *(RCA_0) The following are equivalent:*

1. ACA_0
2. *For every p.o. group $(G, \leq_G)$ and every convex normal subgroup H, the induced order on G/H exists.*

In terms of computable mathematics, Theorem 3.8 shows that the induced order on the quotient of a computably fully ordered computable group by a computable convex subgroup is computable. However, by Theorem 3.9, if the computable group is only computably partially ordered, then the induced order could be as complicated as $0'$.

4 Group and Semigroup Conditions

For the rest of this article, we will be concerned only with fully ordered groups and will use the term ordered group to mean fully ordered group. There are a number of group conditions which imply full orderability. The simplest is given by the following classical theorem.

Theorem 4.1. *Torsion free abelian groups are O-groups.*

The effective content of Theorem 4.1 was first explored in Downey and Kurtz (1986). They constructed a computable group classically isomorphic to $\prod_\omega \mathbb{Z}$ which has no computable order. Hatzikiriakou and Simpson (1990) used a similar construction in the context of reverse mathematics to show that Theorem 4.1 is equivalent to WKL_0. It follows by the Low Basis Theorem that every computable torsion free abelian group has an order of low Turing degree.

Torsion free nilpotent groups (defined below) are also O-groups. Once the method for presenting nilpotent groups in Z_2 is specified, the Hatzikiriakou and Simpson (1990) result can be extended to show that this group condition is also equivalent to WKL_0. Presenting nilpotent groups raises the question of how hard it is to define the center of a group, or more generally, to define the terms in the upper central series. The following theorem shows that RCA_0 is not strong enough.

Theorem 4.2. *(RCA_0) The following are equivalent:*

1. *ACA_0*
2. *For each group G, the center of G, denoted $C(G)$, exists.*

In fact, in the proof of Theorem 4.2 we can use the simplest possible non-abelian group: a classically nilpotent group G in which the quotient of G by its center is abelian. Therefore, as a corollary, there is a computable (infinitely generated) nilpotent group whose center is as complicated as $0'$. This result shows a contrast with the situation for finitely generated nilpotent groups.

Theorem 4.3 (Baumslag et al. (1991)). *The center of a finitely generated nilpotent group is computable.*

Because of Theorem 4.2, if we want to work with nilpotent groups in RCA_0 we need a code for them which explicitly gives the terms of the upper central series.

Definition 4.4. (RCA_0) The pair $N \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ is a **code for a nilpotent group** G if the first $n+1$ columns of N satisfy

1. $N_0 = \langle 1_G \rangle$
2. $N_1 = C(G)$
3. $N_n = G$

4. For $0 \leq i < n$, $N_{i+1} = \pi^{-1}(C(G/N_i))$ where $\pi : G \to G/N_i$ is the standard projection.

A group G is **nilpotent** is there is such a code (N, n) for G.

Theorem 4.5. *(RCA_0) The following are equivalent.*

1. WKL_0

2. Every torsion free nilpotent group is an O-group.

Notice that because every abelian group is nilpotent, the Hatzikiriakou and Simpson (1990) result already shows that (2) ⇒ (1) in Theorem 4.5.

Another classical theorem states that any direct product of O-groups is an O-group. In Z_2, both finite and restricted countable direct products can be represented by finite sequences. Using a lexicographic order, RCA_0 suffices to show that the finite direct product of O-groups is an O-group. However, defining an order on a countable direct product is more difficult because of the difference between

$$\forall i \ \exists P \ (P \text{ is a full order of } A_i)$$

and

$$\exists P \ \forall i \ (\text{the } i^{th} \text{ column of } P \text{ is a full order of } A_i).$$

Theorem 4.6. *(RCA_0) The following are equivalent:*

1. WKL_0

2. If $\forall i$ *(A_i is an O-group) then* $G = \prod_{i \in \mathbb{N}} A_i$ *is an O-group.*

As above, this theorem implies that there is a computable direct product of computably orderable computable groups, in fact of torsion free abelian groups, which is not computably orderable. This result is not implied by the Downey and Kurtz (1986) example mentioned above because although their group was classically isomorphic to $\prod_\omega \mathbb{Z}$, it did not decompose computably.

In addition to studying group conditions that imply orderability, algebraists have also studied semigroup conditions. Theorems 4.7, 4.8 and 4.9 give semigroup conditions for the existence of a full order or the extension of a partial order. $S(a_1, \ldots, a_n)$ denotes the normal semigroup generated by $a_1, \ldots a_n$. To handle these semigroups in RCA_0, we use a function $s(A, n, m, x)$ such that x is in the normal semigroup generated by the elements of $A = \{a_1, \ldots, a_n\}$ if and only if there exist n and m such that $s(A, n, m, x) = 1$. Thus, while RCA_0 cannot in general prove the existence of the normal semigroup, it can work with it as an enumerable object. When the theorems are restated in terms of this function, they are all equivalent to WKL_0.

Theorem 4.7 (Fuchs (1958)). *A partial order P on a group G can be extended to a full order if and only if for any finite set of nonidentity elements, $a_1, \ldots a_n \in G$, there is a sequence $\epsilon_1, \ldots \epsilon_n$ with $\epsilon_i = \pm 1$ such that*

$$P \cap S(a_1^{\epsilon_1}, \ldots a_n^{\epsilon_n}) = \emptyset.$$

Theorem 4.8 (Łos (1954) and Ohnishi (1952)). *A group G is an O-group if and only if for any finite sequence of nonidentity elements $a_1, \dots a_n \in G$, there exists a sequence $\epsilon_1, \dots \epsilon_n$ with $\epsilon_i = \pm 1$ such that*

$$1_G \notin S(a_1^{\epsilon_1}, \dots a_n^{\epsilon_n}).$$

Theorem 4.9 (Lorenzen (1949)). *A group G is an O-group if and only if for any finite sequence of nonidentity elements $a_1, \dots a_n \in G$*

$$\bigcap S(a_1^{\epsilon_1}, \dots a_n^{\epsilon_n}) = \emptyset$$

where the intersection extends over all sequences $\epsilon_1, \dots \epsilon_n$ with $\epsilon_i = \pm 1$.

5 Free Groups

It is well known that every group is isomorphic to the quotient of a free group. RCA_0 suffices to develop the machinery for free groups and to prove this classical theorem. A similar result holds for ordered groups: every ordered group is the epimorphic image of a ordered free group under an order preserving epimorphism (called an o-epimorphism). Downey and Kurtz (1986) asked about the effective content of this theorem.

The standard proof does not go through in RCA_0. However, because of an algebraic trick in Revesz (1986), the effective version of this theorem will hold if the free group on a countable number of generators is computably orderable. Again, the standard proof that every free group is orderable does not work in RCA_0. However, a modification of the proof that the free product of two O-groups is an O-group does work in RCA_0. Taking the two O-groups to be copies of $\mathbb{Z}$ we get the following theorem.

Theorem 5.1. *(RCA_0) The free group on two generators is an O-group.*

Since the free group on a countable number of generators embeds into the free group on two generators in a nice way, we get the following corollary and the desired result.

Corollary 5.2. *(RCA_0) The free group on a countable number of generators is an O-group.*

Theorem 5.3. *(RCA_0) Any ordered group is the o-epimorphic image of an ordered free group.*

Because the proof goes through in RCA_0, the theorem is effectively true. This result gives a surprising answer to the question from Downey and Kurtz (1986).

6 Hölder's Theorem and Mal'tsev's Theorem

Once a group is known to be orderable, it is natural to ask how many orders it admits and to try to classify them. These questions are quite difficult and the main tool for attacking them in the abelian case is Hölder's Theorem, which states that every Archimedean ordered group is order isomorphic to a subgroup of the additive group of the reals. Mal'tsev proved a related result which classifies the possible order types for a countable ordered group.

Recall that $\mathbb{R}$ is not a set in Z_2 and that the reals are defined by convergent sequences of rationals. Hence it is not immediately clear what is meant by a subgroup of $\mathbb{R}$ in Z_2.

Definition 6.1. (RCA_0) If a is an element of an ordered group, then $|\,a\,|$ denotes whichever of a or a^{-1} is positive. If G is an ordered group, then $a \in G$ is **Archimedean less that** $b \in G$ if $|\,a^n\,| < |\,b\,|$ for all $n \in \mathbb{N}$. If there are $n, m \in \mathbb{N}$ such that $|\,a^n\,| \geq |\,b\,|$ and $|\,b^m\,| \geq |\,a\,|$, then a and b are **Archimedean equivalent**. G is an **Archimedean ordered group** if G is ordered and all pairs of nonidentity elements are Archimedean equivalent.

Definition 6.2. (RCA_0) A **subgroup of** $(\mathbb{R}, +_{\mathbb{R}})$ is a sequence of reals $A = \langle\, r_n \mid n \in \mathbb{N}\,\rangle$ together with a function $+_A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and a distinguished number $i \in \mathbb{N}$ such that

1. $r_i = 0_{\mathbb{R}}$
2. $n +_A m = p$ if and only if $r_n +_{\mathbb{R}} r_m = r_p$
3. $(\mathbb{N}, +_A)$ satisfies the group axioms with i as the identity element.

Hölder's Theorem. *(RCA_0) Every Archimedean ordered group is order isomorphic to a subgroup of the naturally ordered group* $(\mathbb{R}, +)$.

The standard proofs of Hölder's Theorem use Dedekin cuts to define the embedding, but in Z_2 it is easier to work with Cauchy sequences. RCA_0 suffices to use the order on an Archimedean ordered group G to construct approximations to reals which give an isomorphic copy of G. Thus, RCA_0 proves Hölder's Theorem.

By contrast, Mal'tsev's Theorem requires strong set existence axioms. The proof uses a transfinite recursion, which suggests that it probably lies in the scope of ATR_0. However, proving that the recursion terminates actually requires $\Pi^1_1 - CA_0$. This result is interesting because there are relatively few known equivalences of $\Pi^1_1 - CA_0$.

Theorem 6.3. *(RCA_0) The following are equivalent:*

1. $\Pi^1_1 - CA_0$
2. *Let G be a countable ordered group. There is a well order α and $\epsilon = 0$ or 1 such that $\mathbb{Z}^\alpha \mathbb{Q}^\epsilon$ is the order type of G (Mal'tsev (1949)).*
3. *Let G be a countable abelian ordered group. There is a well order α and $\epsilon = 0$ or 1 such that $\mathbb{Z}^\alpha \mathbb{Q}^\epsilon$ is the order type of G (Mal'tsev (1949)).*

7 Strong Divisible Closure

There are several naturally occurring notions of closure in algebra including the algebraic closure of a field, the real closure of an ordered field and the divisible closure of an abelian group. From the perspective of reverse mathematics, there are three questions to ask about each notion of closure. We state them in terms of the divisible closure. How hard is it to prove that divisible closures exist? How hard is it to prove that the divisible closure is unique? How hard is it to prove that each abelian group G has a divisible closure D for which G is isomorphic to a subgroup of D (D is called a strong divisible closure)? Friedman et al. (1983) considered the analogous questions for algebraic and real closures, and given the appropriate definitions they proved the following results.

Theorem 7.1 (Friedman et al. (1983)). *(RCA_0)*

1. *$RCA_0 \vdash$ every countable field has an algebraic closure.*
2. *$WKL_0 \Leftrightarrow$ every countable field has a unique algebraic closure.*
3. *$ACA_0 \Leftrightarrow$ every countable field has a strong algebraic closure.*

Theorem 7.2 (Friedman et al. (1983)). *(RCA_0)*

1. *$RCA_0 \vdash$ every countable ordered field has a real closure.*
2. *$RCA_0 \vdash$ every countable ordered field has a unique real closure.*
3. *$ACA_0 \Leftrightarrow$ every countable ordered field has a strong real closure.*

We consider the divisible closure in more detail.

Definition 7.3. (RCA_0) Let D be an abelian group. D is **divisible** if for all $d \in D$ and all $n \geq 1$, there exists a $c \in D$ such that $nc = d$. Here, we are using the additive notation of abelian groups, so nc refers to c added to itself n times.

Definition 7.4. (RCA_0) Let A be an abelian group. A **divisible closure** of A is a divisible abelian group D together with a monomorphism $h : A \to D$ such that for all $d \in D, d \neq 1_D$, there exists $n \in \mathbb{N}$ such that $nd = h(a)$ for some $a \in A, a \neq 1_A$.

Theorem 7.5 (Friedman et al. (1983)). *(RCA_0)*

1. *$RCA_0 \vdash$ Every abelian group has a divisible closure.*
2. *$ACA_0 \Leftrightarrow$ every abelian group has a unique divisible closure.*

Friedman et al. (1983) did not consider the strong divisible closure. Downey and Kurtz (1986) showed that the uniqueness of divisible closures becomes easier to prove if the group is ordered.

Theorem 7.6 (Downey and Kurtz (1986)). *(RCA_0) Every ordered abelian group G has an ordered divisible closure $h : G \to D$ such that h is order preserving. This divisible closure is unique up to order preserving isomorphism.*

Because of the drop from ACA_0 to RCA_0 in the proof of uniqueness with the addition of an order, it is reasonable to look at the strong divisible closure for both abelian groups and ordered abelian groups. The existence of the strong divisible closure, however, turns out to be equivalent to ACA_0 even when the group is ordered.

Definition 7.7. (RCA_0) Let A be an abelian group. A **strong divisible closure** of A is a divisible closure $h : A \to D$ such that h is an isomorphism of A onto a subgroup of D. If A is an ordered group then we also require that D is ordered and h is order preserving.

Theorem 7.8. *(RCA_0) The following are equivalent:*

1. ACA_0
2. *Every abelian group has a strong divisible closure.*
3. *Every ordered Archimedean group has a strong divisible closure.*

8 Π^0_1 Classes

For an orderable computable group G, we define the spaces of orders

$$\mathbb{X}(G) = \{\, P \subset G \,|\, P \text{ is a positive cone on } G \,\}.$$

So far, we know that $\mathbb{X}(G)$ need not have a computable member but that it must have a member of low Turing degree.

Definition 8.1. A **Π^0_1 class** is the set of paths through a computable binary branching tree. A Π^0_1 class is called a **Π^0_1 class of separating sets** if its members are exactly the characteristic functions for the separating sets of some pair of disjoint computably enumerable sets.

Theorem 8.2 (Jockusch Jr. and Soare (1972)). *There exist disjoint computably enumerable sets A and B such that $A \cup B$ is coinfinite and for any separating sets C and D, either $C \equiv_T D$ or C and D are Turing incomparable.*

Metakides and Nerode (1979) examined the spaces of orders on orderable computable fields. They proved both that for any computable field F, there is a Π^0_1 class $\mathcal{C}$ and a degree preserving bijection from $\mathbb{X}(F)$ to $\mathcal{C}$ and that for any Π^0_1 class $\mathcal{C}$, there is a computable field F and a degree preserving bijection from $\mathbb{X}(F)$ to $\mathcal{C}$. The first of these results holds for orderable groups with essentially the same proof. Downey and Kurtz (1986) asked if something similar to the second result holds for computable torsion free abelian groups.

For groups, if $\leq$ is an order, then so is $\leq'$ defined by $a \leq' b$ if and only if $b \leq a$. This property does not hold for fields since -1 can never be positive in an ordered field. Because of this property, if G has an order of some degree, then G has at least two orders of that degree. Hence, the best result you could hope for is that for any Π^0_1 class $\mathcal{C}$, there is a computable torsion free abelian group G and a 2-1 degree preserving map from $\mathbb{X}(G)$ to $\mathcal{C}$. A much weaker result would be that for every Π^0_1 class of separating sets, there is a computable torsion free abelian group such that

$$\{\deg(P) \mid P \in \mathbb{X}(G)\} \;=\; \{\deg(f) \mid f \in \mathcal{C}\}.$$

Unfortunately, even this weaker result is false. Lemma 8.4 follows from examining the classification of orders on torsion free abelian groups of finite rank with an eye towards computability issues (see Teh (1960)).

Definition 8.3. Let G be a torsion free abelian group. The elements $g_0, \ldots, g_n$ are **linearly independent** if for any constants $c_0, \ldots, c_n$ from $\mathbb{Z}$ we have that $c_0 g_0 + \cdots + c_n g_n = 0_G$ implies that each $c_i = 0$. The cardinality of a maximal linear independent set is called the **rank** of G.

Lemma 8.4. *Let G be a computable torsion free abelian group.*

1. *If the rank of G is 1, then G has exactly two orders, both of which are computable.*
2. *If the rank of G is finite and greater than 1, then G has $2^{\aleph_0}$ orders and in particular has orders of every Turing degree.*
3. *If the rank of G is infinite, then G has $2^{\aleph_0}$ orders and in particular has orders of every degree greater than $0'$.*

Let $\mathcal{C}$ be the Π^0_1 class of separating sets from Theorem 8.2. It is clear from Lemma 8.4 that for any computable torsion free abelian group G

$$\{\deg(P) \mid P \in \mathbb{X}(G)\} \;\neq\; \{\deg(f) \mid f \in \mathcal{C}\}.$$

Hence, if we want to use the space of orders on a group to represent an arbitrary Π^0_1 classes, we need to look beyond abelian groups. Unfortunately, a similar result holds for orderable nilpotent groups, so we need to look even beyond them. The question of whether using all orderable computable groups suffices to represent arbitrary Π^0_1 classes is still open.

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