Computable Reductions and Reverse Mathematics

Reed Solomon ⋆

University of Connecticut, Storrs CT 06268

Abstract. Recent work in reverse mathematics on combinatorial principles below Ramsey's theorem for pairs has made use of a variety of computable reductions to give a finer analysis of the relationships between these principles. We use three concrete examples to illustrate this work, survey the known results and give new negative results concerning RT^1_k, SRT^2_ℓ and COH. Motivated by these examples, we introduce several variations of ADS and describe the relationships between these principles under Weihrauch and strong Weihrauch reductions.

1 Introduction

The general project of reverse mathematics is to formalize mathematical theorems in second order arithmetic and to determine which set-theoretic axioms are required to prove them. Typically, we take RCA_0, which includes the non-induction axioms of PA as well as the Δ^0_1-comprehension and Σ^0_1-induction schemes, to be the base subsystem of second order arithmetic over which we prove our equivalences. RCA_0 has the advantage that it allows a reasonable amount of coding and that theorems provable in RCA_0 roughly correspond to theorems for which existential objects can be found computably in the parameters. More specifically, an ω-model satisfies RCA_0 if and only if the second order part of the model is closed under Turing reducibility and the Turing join. The ω-models of other subsystems of second order arithmetic can be described in similarly natural computability-theoretic terms. These connections allow us to prove results in reverse mathematics using techniques and intuitions from computability theory.

While we will primarily be concerned with ω-models, it is worth noting the importance of non-ω-models. They play a crucial role in, for example, measuring levels of induction or conservation. More importantly for our purposes, they are used to separate Ramsey's theorem for pairs (RT^2_2) from its stable version (SRT^2_2). In the next section, we will define RT^2_2 and SRT^2_2 precisely, but for now we use them to make a motivational point. From a computability standpoint, the natural first step to separate these principles would be to prove that for every Δ^0_2 set D, either D or $\overline{D}$ contains an infinite low set. Unfortunately, this statement is false (see [6]), so this method fails at its initial step. However, Chong, Slaman

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and Yang [4] constructed a non-ω-model of RCA_0 in which this property holds and they used this model to prove that SRT^2_2 does not imply RT^2_2 over RCA_0.

One of the underlying motivations for the work presented here is the question of whether SRT^2_2 and RT^2_2 can be separated by an ω-model. Because the fundamental property of the Chong, Slaman and Yang model is false in any ω-model of RCA_0, it appears that new ideas are needed. There are a range of tools from proof theory and computability theory to compare the strengths of theorems and many of these tools give a finer analysis than the analysis given by provable equivalence over RCA_0. In this article, we will use several concrete examples to survey recent results in this direction using reductions which are more commonly used in computable analysis.

We begin by restricting our attention to ω-models and to combinatorial principles which have the Π^1_2 form

$$\forall X\,(\Phi(X) \to \exists Y \Psi(X,Y))$$

where $\Phi(X)$ and $\Psi(X,Y)$ are arithmetic. We refer to a statement P of this form as a *problem* and we refer to sets X which satisfy $\Phi(X)$ as *instances* of P. Given an instance X of P, we call a witness Y such that $\Psi(X,Y)$ a *solution* to X.

Given two problems P and Q of this form, consider how we might show that P holds in every ω-model of $\mathsf{RCA}_0 + Q$. Fix an ω-model of $\mathsf{RCA}_0 + Q$ and recall that the second order part of this model is a Turing ideal, so it is closed under Turing reducibility and the Turing join. We fix an instance X of P in this ideal and try to construct an instance $\widehat{X}$ of Q which is also in the ideal. The simplest way to ensure that $\widehat{X}$ is in the ideal is to make it computable from X. Because $\widehat{X}$ is in the ideal, it must have a solution $\widehat{Y}$ in the ideal and, because the ideal is closed under the join operation, $\widehat{Y} \oplus X$ is also in the ideal. If we are lucky, we can use $\widehat{Y} \oplus X$ to compute a solution Y to X which, because the ideal is closed under Turing reducibility, will also be in the ideal.

Many proofs of implications $Q \to P$ in RCA_0 proceed in this manner. However, it is worth noting that many variations are possible. For example, the proof could be more complicated in the sense that it requires us to solve several (possibly nested) instances of Q before computing a solution to X. On the other hand, the proof could be simpler in the sense that the solution $\widehat{Y}$ to $\widehat{X}$ might already be a solution to X. More generally, $\widehat{Y}$ might compute a solution to X without needing to reference the original instance X of P.

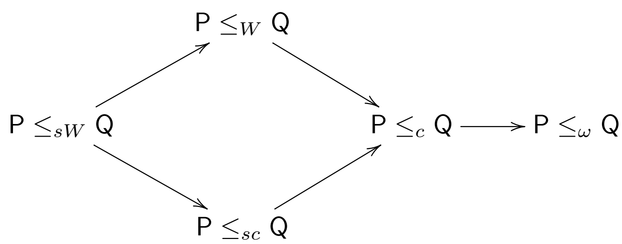
The following definition gives a framework in which we can begin to address these finer questions about exactly how the proof of P from Q (over ω-models of RCA_0) proceeds. In the next section, we will illustrate these reductions with a number of concrete examples. Hirschfeldt and Jockusch [9] give many additional examples as well as more general reduction procedures to extend this type of analysis.

Definition 1. *Let P and Q be problems.*

1. *P is ω-reducible to Q, denoted $P \leq_\omega Q$, if every ω-model of $\mathsf{RCA}_0 + Q$ is a model of P. (This reducibility has also been called computable entailment in a number of earlier articles.)*

2. P is computably reducible to Q, denoted $P \leq_c Q$, if every instance X of P computes an instance $\widehat{X}$ of Q such that if $\widehat{Y}$ is any solution to $\widehat{X}$, then there is a solution Y to X computable from $X \oplus \widehat{Y}$.
3. P is strongly computably reducible to Q, denoted $P \leq_{sc} Q$, if every instance X of P computes an instance $\widehat{X}$ of Q such that if $\widehat{Y}$ is any solution to $\widehat{X}$, then there is a solution Y to X computable from $\widehat{Y}$.
4. P is Weihrauch reducible to Q, denoted $P \leq_W Q$, if there are Turing functionals Φ and Δ such that if X is an instance of P, then Φ^X is an instance of Q and if $\widehat{Y}$ is any solution to Φ^X, then $\Delta^{X \oplus \widehat{Y}}$ is a solution to X.
5. P is strongly Weihrauch reducible to Q, denoted $P \leq_{sW} Q$, if there are Turing functionals Φ and Δ such that if X is an instance of P, then Φ^X is an instance of Q and if $\widehat{Y}$ is any solution to Φ^X, then $\Delta^{\widehat{Y}}$ is a solution to X.

It is straightforward to see that the following implications hold between these reductions. Hirschfeldt and Jockusch [9] prove that none of the given arrows reverse.



2 Three examples

In this section, we give three examples of relationships between principles below Ramsey's theorem for pairs to illustrate these reducibilities. To fix notation, let $[\omega]^2$ denote the set of pairs $\langle x, y\rangle$ with $x < y$. We frequently use k to denote the set $\{0, \dots, k-1\}$.

Definition 2. *A k-coloring of $[\omega]^2$ is a function $c : [\omega]^2 \to k$ and we write $c(x, y)$ in place of $c(\langle x, y\rangle)$. We say that c is stable if for every x, there is a color $i < k$ such that for every sufficiently large y, $c(x, y) = i$. That is, for every x, $\lim_y c(x, y)$ exists. For a stable coloring c, we say that x has limit color i if $\lim_y c(x, y) = i$.*

Given a coloring $c : [\omega]^2 \to k$, we say that $H \subseteq \omega$ is homogeneous if there is a color i such that $c(x, y) = i$ for all $x, y \in H$ with $x < y$. In this case, we say that H is homogeneous for color i. Similarly, if c is a stable coloring, we say that H is limit homogeneous if there is a color i such that $\lim_y c(x, y) = i$ for all $x \in H$. When we are dealing with more than one coloring, we will refer to sets which are homogenous for the coloring c as c-homogeneous sets.

Definition 3. *We define the following versions of Ramsey's theorem.*

- *Ramsey's theorem for pairs* (RT^2_2). *Every coloring* $c : [\omega]^2 \to 2$ *has an infinite homogeneous set.*
- *Stable Ramsey's theorem for pairs* (SRT^2_2). *Every stable coloring* $c : [\omega]^2 \to 2$ *has an infinite homogeneous set.*
- *Limit Ramsey's theorem for pairs* (D^2_2). *Every stable coloring* $c : [\omega]^2 \to 2$ *has an infinite limit homogeneous set.*
- *Ramsey's theorem for singletons* (RT^1_k). *Every coloring* $c : \omega \to k$ *has an infinite homogeneous (i.e. monochromatic) set. (This principle is often called the pigeonhole principle.)*

Each of these statements can be written in the Π^1_2 form given above. In each case, the instances of the problem are the colorings of the appropriate type and the solutions to a given problem c are the infinite homogeneous sets of the appropriate type. We introduce one final principle which will be used in our third example.

Definition 4. *A set* Y *is cohesive for a sequence* $\langle X_n \mid n \in \omega \rangle$ *of subsets of* ω *if for every* n*, either* $Y \cap X_n$ *or* $Y \cap \overline{X_n}$ *is finite.*

- *Cohesive principle* (COH). *For every sequence* $\langle X_n \mid n \in \omega \rangle$ *of subsets of* ω*, there is an infinite cohesive set* Y*.*

Example 1. We consider the principles D^2_2 and SRT^2_2. D^2_2 arises naturally in computability theory by considering a set $D \leq_T 0'$ with a fixed Δ^0_2 approximation $f(x, s)$. By restricting the domain of $f(x, s)$ to values $x < s$, we can view f as a 2-coloring of $[\omega]^2$. The infinite limit homogeneous sets of this coloring are exactly the infinite subsets of D and of $\overline{D}$.

To determine the relationship between SRT^2_2 and D^2_2 under these reductions, notice that for a stable coloring $c : [\omega]^2 \to 2$, every homogeneous set is limit homogeneous (but not conversely). It follows that $\mathsf{D}^2_2 \leq_{sW} \mathsf{SRT}^2_2$ by letting the functionals Φ and Δ be the identity. That is, given an instance c of D^2_2, we map c to itself but view it as an instance of SRT^2_2. From any solution H to c as an instance of SRT^2_2 (i.e. an infinite homogeneous set H), we map H to itself but view it as a solution to the original instance c of D^2_2 (i.e. an infinite limit homogeneous set).

The non-trivial direction is to use D^2_2 to solve instances of SRT^2_2. A proof that $\mathsf{RCA}_0 + \mathsf{D}^2_2$ implies SRT^2_2 over ω-models goes as follows. Fix a stable coloring $c : [\omega]^2 \to 2$ as an instance of SRT^2_2. We regard c as an instance of D^2_2 and obtain an infinite limit homogeneous set L of color i. To produce a homogeneous set of color i, we thin L to $H = \{h_0, h_1, \ldots\}$. Let h_0 be the least element of L. Having defined h_n, let h_{n+1} be the least element of L such that $h_n < h_{n+1}$ and $c(h_j, h_{n+1}) = i$ for all $j \leq n$. (This proof works more generally in $\mathsf{RCA}_0 + \mathsf{D}^2_2$, but there is an induction issue. It uses $B\Sigma^0_2$ and hence relies on the fact that D^2_2 implies $B\Sigma^0_2$ as shown by Chong, Lempp and Yang [3].)

This proof shows that $\mathsf{SRT}^2_2 \leq_c \mathsf{D}^2_2$. From an instance c of SRT^2_2, we compute an instance of D^2_2 (via the identity function) and then use an arbitrary D^2_2 solution L together with the original SRT^2_2 coloring c to compute an SRT^2_2 solution H to

c. However, notice that the computation of H from L is non-uniform because it depends on knowing the limit color for L and that the thinning process uses the original SRT^2_2 instance c. Thus this proof leaves open the questions of whether $\mathsf{SRT}^2_2 \leq_W \mathsf{D}^2_2$ (can the proof be made uniform?) and whether $\mathsf{SRT}^2_2 \leq_{sc} \mathsf{D}^2_2$ (is the reference to the original coloring c necessary to compute H from L?). Both of these reductions fail.

Theorem 5 (Dzhafarov [7]). $\mathsf{SRT}^2_2 \not\leq_W \mathsf{D}^2_2$ *and* $\mathsf{SRT}^2_2 \not\leq_{sc} \mathsf{D}^2_2$.

The proof of Theorem 5 uses a generalization of Seetapun's method of cone avoidance for RT^2_2 and in fact shows something stronger. Let $\mathsf{D}^2_{<\infty}$ be the problem for which the instances are given by pairs $\langle c, k\rangle$ where $c : [\omega]^2 \to k$ is a stable coloring and the solutions to $\langle c, k\rangle$ are the infinite limit homogeneous sets. The stronger result from Dzhafarov [7] is that $\mathsf{SRT}^2_2 \not\leq_W \mathsf{D}^2_{<\infty}$ and $\mathsf{SRT}^2_2 \not\leq_{sc} \mathsf{D}^2_{<\infty}$.

Example 2. We consider the principles RT^1_k for varying numbers of colors. These principles differ in a significant way from those in the first example. Given $c : \omega \to k$, there is an infinite homogeneous set H computable from c by non-uniformly fixing a color i for which there are infinitely many x such that $c(x) = i$. Therefore, when comparing RT^1_k and RT^1_ℓ, we immediately have $\mathsf{RT}^1_k \leq_c \mathsf{RT}^1_\ell$ because $\leq_c$ allows non-uniformity and access to the original RT^1_k coloring. Furthermore, if $k \leq \ell$, then every k-coloring is an ℓ-coloring and we immediately have $\mathsf{RT}^1_k \leq_{sW} \mathsf{RT}^1_\ell$. Therefore, we only consider potential reductions from RT^1_k to RT^1_ℓ when $\ell < k$ and the reduction is stronger than $\leq_c$.

It is instructive to consider a combinatorial proof of RT^1_3 from RT^1_2. Fix a coloring $c : \omega \to 3$ and use it to define a coloring $d : \omega \to 2$ by $d(x) = 0$ if $c(x) = 0$ or $c(x) = 1$ and $d(x) = 1$ if $c(x) = 2$. Let H be an infinite d-homogeneous set. If H is d-homogeneous for color 1, then it is a c-homogeneous set for color 2 and we are done. On the other hand, if H is d-homogeneous for color 0, then we need another application of RT^1_2 to get our c-homogeneous set. Let d_0 be the restriction of d to H. Applying RT^1_2 to d_0 gives an infinite d_0-homogeneous set $H_0 \subseteq H$ which is clearly c-homogeneous as well. This proof does not use the original coloring to thin out a homogeneous set, but it does have a split between cases (so is non-uniform) and it potentially uses two applications of RT^1_2 to solve a single instance of RT^1_3. The following theorem collects a number of recent results related to the connection between these principles.

Theorem 6. *The following negative results hold for all* $0 < \ell < k$.

- $\mathsf{RT}^1_k \not\leq_{sW} \mathsf{RT}^1_\ell$ *(Dorais, Dzhafarov, Hirst, Mileti and Shafer [5])*
- $\mathsf{RT}^1_k \not\leq_W \mathsf{RT}^1_\ell$ *(independently by Hirschfeldt and Jockusch [9], by Patey [11] and by Rakotoniaina [12])*
- $\mathsf{RT}^1_k \not\leq_{sc} \mathsf{RT}^1_\ell$ *(Dzhafarov [7])*

The most interesting result concerning these principles comes from refining the forcing techniques to remove any computable dependence between the instances of RT^1_k and RT^1_ℓ.

Theorem 7 (independently by Hirschfeldt and Jockusch [9] and by Patey [11]). *For any $\ell < k$, there is a coloring $c : \omega \to k$ such that every coloring $d : \omega \to \ell$ (computable from c or not) has an infinite homogeneous set H which does not compute any infinite c-homogeneous set.*

The proof of Theorem 7 given in Patey [11] shows that the coloring $c : \omega \to k$ can even be made low.

Example 3. We consider the relationship between COH and the principles above. The interest in these relationships is partly motivated by the fact that RCA_0 proves the equivalence of RT^2_2 and $\mathsf{SRT}^2_2 + \mathsf{COH}$. Cholak, Jockusch and Slaman [2] introduced this equivalence as an important tool for constructing solutions to instances of RT^2_2 in two steps by using COH to pass from a general 2-coloring to a stable 2-coloring and then using SRT^2_2 to solve the stable coloring. Therefore, we would like to know whether COH is reducible (by any of these reductions) to SRT^2_k for some $k \geq 2$ or to $\mathsf{SRT}^2_{<\infty}$. Partial answers to these questions are given by the following theorem.

Theorem 8 (Dzhafarov [7]). $\mathsf{COH} \not\leq_W \mathsf{SRT}^2_{<\infty}$ *and* $\mathsf{COH} \not\leq_{sc} \mathsf{SRT}^2_2$.

Dzhafarov's proof that $\mathsf{COH} \not\leq_{sc} \mathsf{SRT}^2_2$ uses a new tree labeling method for constructing solutions to SRT^2_2. The version of this method given in [7] is closely tied to coloring pairs and it was left as an open question whether this result could be extended to larger exponents. We will return to this question below.

A second motivation for studying the relationships between COH and other Ramsey principles is that COH can be recast as a sequential version of Ramsey's Theorem for Singletons in which the solution is allowed to make finitely many errors. We say that Y is *almost homogeneous* for a coloring $c : \omega \to k$ if there is a finite set F such that $Y - F$ is homogeneous for c. Under a suitable coding, COH is sW-equivalent to the statement that for every sequence $\langle c_k \mid k \geq 1\rangle$ of colorings $c_k : \omega \to k$, there is an infinite set Y such that Y is almost homogeneous for every coloring c_k. (For the details of this coding, see [8].) From this perspective, we would like to understand the precise relationship between RT^1_k and SRT^2_ℓ as a stepping stone to determine whether COH can be reduced (by $\leq_{sc}$, $\leq_c$ or $\leq_\omega$) to SRT^2_ℓ or to $\mathsf{SRT}^2_{<\infty}$. As above, $\mathsf{RT}^1_k \leq_c \mathsf{SRT}^2_\ell$ for any k and ℓ because $\leq_c$ allows access to the RT^1_k coloring, and it is not hard to see that $\mathsf{RT}^1_k \leq_{sW} \mathsf{SRT}^2_\ell$ whenever $k \leq \ell$. Therefore, our interest is in comparing RT^1_k and SRT^2_ℓ when $\ell < k$. The following theorem answers this question for W-reducibility.

Theorem 9 (Hirschfeldt and Jockusch [9]). *For all $\ell < k$,* $\mathsf{RT}^1_k \not\leq_W \mathsf{SRT}^2_\ell$.

In recent work, we were able to answer a number of the questions left open by the results above. Our main result is to show that $\mathsf{RT}^1_k \not\leq_{sc} \mathsf{SRT}^2_\ell$ when $\ell < k$ in a strong form analogous to Theorem 7.

Theorem 10 (Dzhafarov, Patey, Solomon and Westrick [8]). *For all $\ell < k$, there is a coloring $c : \omega \to k$ such that every stable coloring $d : [\omega]^2 \to \ell$ has an infinite homogeneous set which does not compute an infinite c-homogeneous set.*

The technique for proving this theorem involves an extension and a simplification of the tree labeling method used to build solutions to SRT^2_2 in Dzhafarov's proof of $\mathsf{COH} \not\leq_{sc} \mathsf{SRT}^2_2$ in Theorem 8. It remains an open questions whether the coloring c can be low as in Patey's proof of Theorem 7. There are two corollaries below to Theorem 10. The first corollary follows immediately and the second corollary follows by viewing COH as a sequential form of RT^1_k allowing finitely many errors.

Corollary 11 (Dzhafarov, Patey, Solomon and Westrick [8]). *For all $\ell < k$, $\mathsf{RT}^1_k \not\leq_{sc} \mathsf{SRT}^2_\ell$.*

Corollary 12 (Dzhafarov, Patey, Solomon and Westrick [8]). $\mathsf{COH} \not\leq_{sc} \mathsf{SRT}^2_{<\infty}$.

3 Variations on ADS

In this section, we consider variations of the ascending/descending sequence principle ADS (defined below) which are motivated by the combinatorial relationships between RT^2_2, SRT^2_2 and D^2_2. Throughout this section, when we say $(L, \leq_L)$ is a linear order, we assume that $L \subseteq \omega$. That is, we assume our algebraic structures are coded in the natural numbers. We use $\leq$ to denote the usual order on ω.

Definition 13. *Let $(L, \leq_L)$ be a linear order and let $S \subseteq L$. We say that S is an*

- *ascending sequence if for all $x, y \in S$, $x \leq y$ implies $x \leq_L y$;*
- *descending sequence if for all $x, y \in S$, $x \leq y$ implies $x \geq_L y$;*
- *ascending chain if for all $x \in S$, there are only finitely many $y \in S$ with $y \leq_L x$;*
- *descending chain if for all $x \in S$, there are only finitely many $y \in S$ with $y \geq_L x$.*

Note that if S is an ascending (descending) chain, then S is isomorphic to ω (or ω^*). However, it is given as a chain in L rather than as an ascending (descending) sequence in the sense that the elements of S do not necessarily appear in ascending (descending) order when enumerated in increasing $\leq$-order.

Definition 14. *We define the following principles.*

- *Ascending/descending sequence principle (ADS): Every infinite linear order has an infinite ascending or descending sequence.*
- *Ascending/descending chain principle (ADC): Every infinite linear order has an infinite ascending or descending chain.*

There principles have the required Π^1_2 form given in the introduction and the combinatorial relationship between ADC and ADS is similar to the combinatorial relationship between D^2_2 and SRT^2_2. For example, ADC and ADS have the same instances and an ADS-solution to $(L, \leq_L)$ is also an ADC-solution to $(L, \leq_L)$. Therefore, $\mathsf{ADC} \leq_{sW} \mathsf{ADS}$ via identity functionals just as $\mathsf{D}^2_2 \leq_{sW} \mathsf{SRT}^2_2$.

More importantly for our analogy, every ADC-solution S to $(L, \leq_L)$ can be thinned out to an ADS-solution H to $(L, \leq_L)$. To see why, suppose $S = \{s_0, s_1, \ldots\}$ is an infinite ascending chain. We define an infinite ascending sequence $H = \{h_0, h_1, \ldots\}$ as follows. Let h_0 be the $\leq$-least element of S. Having defined h_n, let h_{n+1} be the $\leq$-least element of S such that $h_n < h_{n+1}$ and $h_n <_L h_{n+1}$. A similar argument works in the case when S is a descending chain. As with SRT^2_2 and D^2_2, the computation of H from S is non-uniform (because it requires knowing whether S is ascending or descending) and uses the ADS-instance $(L, \leq_L)$. Therefore, the thinning process shows $\mathsf{ADS} \leq_c \mathsf{ADC}$ exactly as with $\mathsf{SRT}^2_2 \leq_c \mathsf{D}^2_2$. Before considering whether this reduction can be made uniform, we introduce two notions of stability in this context.

Definition 15. *Let $(L, \leq_L)$ be a linear order. An element $x \in L$ is called small if there are only finitely many $y \in L$ such that $y <_L x$. Similarly, $x \in L$ is called large if there are only finitely many $y \in L$ such that $x <_L y$. We say that L is stable if every $x \in L$ is either small or large.*

If L is stable, then L is isomorphic to either $\omega + k$, $k + \omega^*$ or $\omega + \omega^*$. The established definition for the stable version SADS of ADS restricts the instances of the problem to linear orders isomorphic to $\omega + \omega^*$, that is, to stable linear orders with infinitely many small elements and infinitely many large elements. In the context of ω-models of RCA_0 or of $\leq_c$-reductions, neglecting to consider linear orders L of type $\omega + k$ or $k + \omega^*$ does not matter because L can (non-uniformly) compute an infinite ascending sequence (in the case of $\omega + k$) or descending sequence (in the case of $k + \omega^*$). In our analogy with SRT^2_2, the orders $\omega + k$ and $k + \omega^*$ correspond to stable colorings $c : [\omega]^2 \to 2$ in which there is a fixed color i such that almost every x has limit color i. However, in the context of uniform reductions, it is important not to discount these trivial cases. To account for these cases, we consider two stable versions of ADS and of ADC.

Definition 16. *We define the following notions of stability for* ADS *and* ADC*.*

- SADS*: Every stable linear order L with infinitely many small and large elements has an infinite ascending or descending sequence.*
- SADC*: Every stable linear order L with infinitely many small and large elements has an infinite ascending or descending chain.*
- General-ADS*: Every stable linear order L has an infinite ascending or descending sequence.*
- General-ADC*: Every stable linear order L has an infinite ascending or descending chain.*

It is straightforward to check that these four principles are equivalent under $\leq_c$ and that the following relationships hold under $\leq_{sW}$.

- $\mathsf{SADS} \leq_{sW} \mathsf{General\text{-}ADS} \leq_{sW} \mathsf{ADS}$

– SADC $\leq_{sW}$ General-ADC $\leq_{sW}$ ADC
– SADC $\leq_{sW}$ SADS
– General-ADC $\leq_{sW}$ General-ADS

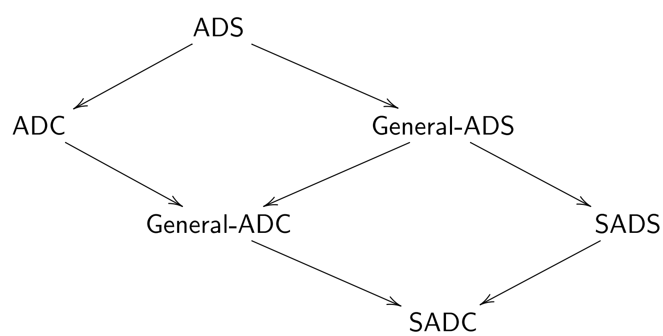
Taken together with these reductions, the following theorem gives us a complete picture of the relationships between the six principles introduced in this section under $\leq_W$ and $\leq_{sW}$.

Theorem 17 (Astor, Dzhafarov, Solomon and Suggs [1]). *We have the following negative results concerning* $\leq_W$.

(1) SADS $\not\leq_W$ ADC.
(2) General-ADC $\not\leq_W$ SADS.
(3) ADC $\not\leq_W$ General-ADS.

The first statement in Theorem 17 is proved by a Seetapun-style forcing construction while the second statement is proved with a simpler forcing notion. The last statement follows from the fact that computable instances of SADC have low solutions and that there are computable instances of ADC which have no low solutions (as shown in [10]).

The relationships between these six principles under $\leq_W$ and $\leq_{sW}$ are summarized by the following diagram. The arrows indicate an $\leq_{sW}$ reduction while the missing arrows indicate the failure of a $\leq_W$ reduction.



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