ON THE ISOMORPHISM PROBLEM FOR SOME CLASSES OF COMPUTABLE ALGEBRAIC STRUCTURES

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Abstract. We establish that the isomorphism problem for the classes of computable nilpotent rings, distributive lattices, nilpotent groups, and nilpotent semigroups is Σ^1_1-complete, which is as complicated as possible. The method we use is based on uniform effective interpretations of computable binary relations into computable structures from the corresponding algebraic classes.

The notion of an isomorphism constitutes one of the most important equivalence relations between structures for the same language. In this paper we give new examples of natural classes of computable algebraic structures for which the isomorphism problem is Σ^1_1-complete. We follow the general method used by Hirschfeldt, Khoussainov, Shore, and Slinko [10]: For each of the classes $\mathcal{K}$, we describe an effective uniform method that transforms a computable binary relation $R \subseteq \omega^2$ into a computable model $\mathcal{M}_R \in \mathcal{K}$ so that $(\omega, R) \cong (\omega, S)$ is equivalent to $\mathcal{M}_R \cong \mathcal{M}_S$.

In general, we consider a class $\mathcal{K}$ of countable structures which is closed under isomorphism. We assume the universe of every structure is a subset of ω and identify each structure with its atomic diagram. Thus, a structure is *computable* if its atomic diagram is computable, and its *computable index* is a number e such that φ_e, the e^{th} partial computable function, is the characteristic function of its atomic diagram. We write $\mathcal{A}_e$ for the structure with a computable index e, provided that φ_e describes the atomic diagram of a structure in the given language. We let $I(K) = \{i \mid \mathcal{A}_i \in \mathcal{K}\}$ denote the index set of computable structures in $\mathcal{K}$, although we typically write $\mathcal{A}_i \in \mathcal{K}$ rather than $i \in I(\mathcal{K})$. The following definition was proposed by Goncharov and Knight in [7]. The *isomorphism problem for the computable structures in* $\mathcal{K}$ is

$$\{ (i, j) \mid \mathcal{A}_i, \mathcal{A}_j \in \mathcal{K} \mathbin{\&} \mathcal{A}_i \cong \mathcal{A}_j \}.$$

For a complexity class Γ, we say that a set X is Γ-*complete* if X is in Γ, and for all Y in Γ, Y is m-reducible to X.

Various authors have established that the isomorphism problem is Σ^1_1-complete for several well-known classes of computable structures. The proof of the Σ^1_1-completeness of the isomorphism problem for computable abelian p-groups, trees, Boolean algebras, linear orderings, and for arbitrary structures with at least one

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relation of arity at least 2 can be found in Goncharov and Knight [7]. The proof for a general case, as well as for structures with a single binary relation, can be found in Morozov [13, 12]. There are related results of Friedman and Stanley [6], where the authors established Borel completeness for the classes of countable trees, linear orderings, and fields of any fixed characteristic. Building upon Friedman's and Stanley's result for fields, Calvert [3] proved that the isomorphism problem for computable fields of any fixed characteristic, as well as for ordered real closed fields, is Σ^1_1-complete. Downey and Montalbán [5] proved that the isomorphism problem for computable torsion-free Abelian groups is Σ^1_1-complete. On the other hand, Calvert [3] showed that the isomorphism problem for computable vector spaces over a fixed infinite computable field, algebraically closed fields of a fixed characteristic, and Archimedean real closed fields is Π^0_3-complete. Calvert, Cenzer, Harizanov and Morozov [4] showed that the isomorphism problem for computable equivalence relations is Π^0_4-complete.

In this paper we prove the following result, and in addition, we use the same methods to give a new proof that the isomorphism problem for computable distributive lattices is Σ^1_1-complete.

Theorem 1. *For each of the following classes, the isomorphism problem for the computable structures is Σ^1_1-complete: nilpotent rings, 2-step nilpotent groups, and nilpotent semigroups.*

Although we use the same techniques as in [10], our results do not follow from [10], since there the authors actually give constructions transforming a binary relation R into a structure of the type $(\mathcal{M}_R, \bar{a})$, where $\mathcal{M}_R \in \mathcal{K}$ and $\bar{a} \in \mathcal{M}^{<\omega}$, and hence establish only the equivalence $(\omega, R) \cong (\omega, S) \Leftrightarrow (\mathcal{M}_R, \bar{a}) \cong (\mathcal{M}_S, \bar{b})$. However, we need to establish

$$(\omega, R) \cong (\omega, S) \Leftrightarrow \mathcal{M}_R \cong \mathcal{M}_S.$$

That is to say that our results establish definability without parameters, unlike [10]. In fact, a major effort in this paper is to refine and further develop transformations in [10] so that we are able to eliminate all extra constants that they might introduce.

In each of the first three sections, we present an effective method for a uniform transformation of a computable binary relation R into a computable structure $\mathcal{M}_R$ from a particular class and prove that all extra constants can be eliminated. In Section 1, we consider nilpotent rings, in Section 2, we consider distributive lattices, and in Section 3, we consider nilpotent semigroups. These codings are based on unpublished methods of Rabin and Scott [14].

Because the isomorphism problems for computable linear orders and for computable Boolean algebras are Σ^1_1-complete and because linear orders and Boolean algebras can be construed as distributive lattices, it follows that the corresponding problem for distributive lattices is Σ^1_1-complete. Therefore, the result on distributive lattices is not new. However, we have included our proof in Section 2 because the construction method is substantially different from that used in Goncharov and Knight [7]. The proof in [7] for linear orders uses the Kleene-Brouwer order on trees and makes use of the nontrivial fact that every Harrison order is isomorphic to $\omega_1^{CK}(1+\eta)+\alpha$ for some computable order α. Goncharov and Knight gave two proofs for Boolean algebras. The first proof uses the interval algebra of a linear order to obtain the result for Boolean algebras from the result for linear orders.

The second proof constructs Boolean algebras directly from trees after some preprocessing to put the trees in a suitable form.

In Section 4, we prove that the isomorphism problem for computable 2-step nilpotent groups is Σ^1_1-complete using the coding of a field F into a 2-step nilpotent group $H(F)$ by Mal'cev [11]. Mal'cev's coding used two parameters which we need to remove for our application. Subsequently, the first and fourth authors of this article developed a more detailed analysis of the connections between F and $H(F)$ with a group of coauthors in [1]. Therefore, there is some overlap between the material in Section 4 and in [1], so we will refer to the reader to [1] for the proofs of some algebraic facts used in our construction.

To conclude the introduction, note the following corollary of Theorem 1. We write $\mathcal{A} \cong_h \mathcal{B}$ if there exists a hyperarithmetical isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

Corollary 1. *In each of the following classes of structures: nilpotent rings, distributive lattices, 2-step nilpotent groups, and nilpotent semigroups, we have* (1) − (3)*:*

1. *There are computable structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A} \cong \mathcal{B}$ but $\mathcal{A} \not\cong_h \mathcal{B}$.*
2. *There exists a computable structure with Scott rank greater than or equal to ω_1^{CK}.*
3. *There exist a computable structure $\mathcal{M}$ with two tuples $\bar{a}$ and $\bar{b}$ of elements from its domain of equal length such that $(\mathcal{M}, \bar{a}) \cong (\mathcal{M}, \bar{b})$ but $(\mathcal{M}, \bar{a}) \not\cong_h (\mathcal{M}, \bar{b})$.*

Proof. (1) If $\mathcal{A} \cong \mathcal{B}$ would imply $\mathcal{A} \cong_h \mathcal{B}$ for all computable structures $\mathcal{A}$ and $\mathcal{B}$, then

$$\{ (i,j) \mid \mathcal{A}_i \cong \mathcal{A}_j \} = \{ (i,j) \mid \exists f \, [(f \text{ is hyperarithmetical}) \wedge f : \mathcal{A}_i \cong \mathcal{A}_j] \}.$$

This implies that the above set is Π^1_1, which is a contradiction.

(2) If all the structures in the class had Scott rank less than ω_1^{CK} then all pairs of isomorphic structures would be hyperarithmetically isomorphic (this could be easily proven using the back-and-forth method and the Scott formulas for tuples as in Barwise [2]), and we can use the same argument as in (1). This means that the class $\mathcal{K}$ contains a structure of Scott rank at least ω_1^{CK}.

(3) This follows directly from (2) and Goncharov et al. [8]. □

1. Nilpotent rings

Let R be an arbitrary binary irreflexive and symmetric relation on ω. We will transform such a computable relation R into a computable ring K_R in a uniform computable way. Moreover, we will have that $(K_R)^3$ is the trivial ring, hence K_R is nilpotent. Let $(R_i)_{i\in\omega}$ be a computable enumeration of such computable relations. Consider the following "disjointness condition" on a binary relation R.

Condition D: For all $l, t \in \omega$ with $l > 0$ and $0 \leqslant t \leqslant l^2$, there exist two disjoint l-element sets, $I_0 = \{q_0, \dots, q_{l-1}\}$ and $I_1 = \{q_l, \dots, q_{2l-1}\}$ such that for every $i, j \in I_0 \cup I_1$, if $R(i,j)$ then $(i \in I_0 \Leftrightarrow j \in I_1)$, and the cardinality of the set $R \cap (I_0 \times I_1)$ is t (and so the cardinality of the set $R \cap ((I_0 \times I_1) \cup (I_1 \times I_0))$ is $2t$).

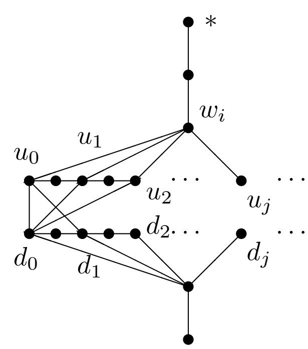
Lemma 1. *For every computable binary irreflexive and symmetric relation R, there exists a uniform effective transformation into a computable binary irreflexive and* *symmetric relation R^* such that R^* satisfies the Condition D and*

$$(\omega, R_i^*) \cong (\omega, R_j^*) \Leftrightarrow (\omega, R_i) \cong (\omega, R_j).$$

Proof. We describe a uniform procedure that given an algorithm recognizing an irreflexive symmetric relation $R \subseteq \omega^2$ constructs a computable binary relation $R^* \subseteq \omega^2$ that satisfies the condition of the lemma.

We give an idea of the proof while omitting some details. Specifically, we extend the relation R by attaching to each point $v \in \omega$ a separate copy of some fixed computable rigid structure $\mathcal{A}$ satisfying the Condition D so that every isomorphism between the two structures R_i^* and R_j^* obtained this way takes ω to ω, and takes each copy of the structure $\mathcal{A}$ we added in our construction to another such copy. We assume that all copies of $\mathcal{A}$ we add are disjoint.

Here is an idea of what such structure $\mathcal{A}$ might look like. We start with a structure that is depicted in Pic. 1. Here, the bold dots denote elements of this structure and the lines between them represent a binary symmetric relation.



Pic. 1. The structure $\mathcal{A}$. Initial stage.

To complete the construction of $\mathcal{A}$, we just add new lines joining some elements u_i and d_k so that we satisfy Condition D and still make our relation computable. Thus, the desired elements q_n in Condition D can be selected among these u_i and d_k, respectively.

Next, we extend R to R^* by attaching to each element $v \in \omega$ a new computable copy $\mathcal{A}_v$ of $\mathcal{A}$ by identifying this v and the element marked $*$ in the picture. Recall that we assume that all structures $\mathcal{A}_v$ are pairwise disjoint. Moreover, we have $\omega \cap \mathcal{A}_v = \{v\}$. Thus, we have extended the initial relation R by adding new elements and lines between them to obtain a new relation R^*. We can easily see that all isomorphisms between structures of the type R^* preserve structures of the type $\mathcal{A}_v$, and that the conditions of the lemma are satisfied. We leave the remaining details to the reader. $\square$

Hence, we will assume that the computable irreflexive symmetric relation R is chosen so that it satisfies Condition D.

Fix distinct elements $a, b, c_0, c_1, \ldots$ and let $A = \{a, b, c_0, c_1, \ldots\}$. For an arbitrary binary relation $R \subseteq \omega^2$, define a commutative ring K_R, the elements of which are formal linear expressions of the form $ma + nb + \sum_{i\in I} z_i c_i$, where $m, n, z_i \in \mathbb{Z}$, and the commutative multiplication operation satisfies the following conditions:

$$(\forall x)[ax = bx = 0] \quad \text{and} \quad c_i c_j = \begin{cases} b, & \text{if } R(i,j), \\ a, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

We can easily see that for all $x, y, z \in K_R$ we have: $(xy)z = x(yz) = 0$. This implies that K_R is an associative and nilpotent ring.

Our goal is to prove that for any two relations R and S satisfying Condition D, the isomorphism $K_R \cong K_S$ is equivalent to $(\omega, R) \cong (\omega, S)$. The first step is to establish that the set $\{b, -b\}$ is definable in K_R.

Lemma 2. *Let $l > 0$ and t be such that $0 \leqslant t \leqslant l^2$. Every element of the form $2la \pm 2tb$ is a square in K_R.*

Proof. We will use Condition D for l and t. Let I_0 and I_1 be as in the definition of Condition D, and let $I = I_0 \cup I_1$. Hence the cardinality of the set $R \cap I^2$ is $2t$. Assign the coefficients $z_i = -1$ to all members of I_0, and the coefficients $z_i = 1$ to all members of I_1. Now,

$$\left(\sum_{i\in I} z_i c_i\right)^2 = 2la + \sum_{i,j\in I \wedge R(i,j)} (-1)b = 2la - 2tb.$$

If we set all coefficients to $z_i = 1$, we obtain

$$\left(\sum_{i\in I} z_i c_i\right)^2 = 2la + \sum_{i,j\in I \wedge R(i,j)} b = 2la + 2tb.$$

This completes the proof. □

Let $d \in K_R$, $d \neq 0$. We call the family

$$\{y \mid (\exists r, s \in \mathbb{Z})(r^2 + s^2 \neq 0 \;\&\; rd = sy)\} - \{0\}$$

the *direction of* d and denote it by $\langle d \rangle$.

Note the following:

(1) For every $k \neq 0$, we have $\langle d \rangle = \langle kd \rangle$.
(2) For every $d \in K_R - \{0\}$, the direction $\langle d \rangle$ is definable by the infinitary formula $\varphi(y, d)$:

$$(y \neq 0) \wedge \bigvee_{r,s\in\mathbb{Z} \& r^2+s^2\neq 0} (rd = sy).$$

Lemma 3. *If $m \neq 0$, then the direction $\langle ma + nb \rangle$ contains a nontrivial square.*

Proof. Without loss of generality, assume that $n \geqslant 0$. Take an integer $p > 0$ so that $pn \leqslant (pm)^2$. By Lemma 2, the element $2pma + 2pnb \in \langle ma + nb \rangle$ is a square. □

Lemma 4. *The direction $\langle b \rangle$ contains no nontrivial squares.*

Proof. Take an arbitrary element $x = ma + nb + \sum_{i\in I} z_i c_i$ and consider its square

$$x^2 = \left(ma + nb + \sum_{i\in I} z_i c_i\right)^2 = \left(\sum_{i\in I} z_i c_i\right)^2 = \sum_{i\in I} z_i^2 a + \sum_{i,j\in I \wedge R(i,j)} z_i z_j b. \quad (1)$$

Hence, it follows that if $x^2 \neq 0$, then x^2 has a nontrivial coefficient for a. Hence x^2 does not belong to $\langle b \rangle$. ☐

Lemma 5. *Let K_R^2 be a subring in K_R generated by squares. Then b is defined in K_R, up to a nonzero coefficient from $\mathbb{Z}$, by the following condition $U(x)$:*

$x \neq 0$, $x \in K_R^2$, and $\langle x \rangle$ is the unique direction that contains no nontrivial squares.

Proof. Of course, K_R^2 is definable by a computable infinitary formula. Since all squares are linear combinations of a and b, every element they generate can also be presented in this form. It follows from Lemma 2 that $2b \in K_R^2$. Also, $\langle 2b \rangle = \langle b \rangle$. By Lemma 3 and Lemma 4, $\langle b \rangle$ is the only direction of an element in K_R^2 which contains no nontrivial square. ☐

The direction $\langle b \rangle$ is definable, and the property $(\forall y \in \langle b \rangle)[\bigvee_{m \in \mathbb{Z}}(y = mx)]$ defines the set $\{b, -b\}$. Hence the set $\{b, -b\}$ is also definable in K_R by some computable infinitary formula (which can easily be written).

The next task is to distinguish the element a. It follows from (1) in the proof of Lemma 4 that a is the only square element x such that for all $m \in \mathbb{Z}$, $x + mb$ is not divisible by any natural number greater than 1. Since $\pm b$ is definable in K_R, a is definable in K_R by a computable infinitary formula expressing the following property:

$$\exists y(x = y^2) \ \& \ \bigwedge_{m \in \mathbb{Z} \& n > 1} \forall u(nu \neq x + mb).$$

Thus, we have established the following

Lemma 6. *The element b is definable, up to the coefficient ± 1, in K_R by an infinitary computable formula. The element a is definable in K_R by an infinitary computable formula.* ☐

Unfortunately, it is impossible to define the elements c_i, although it is possible to distinguish them up to some element of the form $ma + nb$. This follows from the following two lemmas.

Lemma 7. *Assume that $(m_i)_{i \in \omega}$ and $(n_i)_{i \in \omega}$ are sequences of integers. Then the linear extension of the mapping given by*

$$a \mapsto a, \quad b \mapsto b, \quad c_i \mapsto c_i + m_i a + n_i b$$

is an automorphism of K_R. Thus, the family $\{c_i \mid i \in \omega\}$ cannot be defined within K_R.

Proof. This is a routine check. ☐

Lemma 8. *We have that $x^2 = a$ if and only if $x = \pm c_i + ma + nb$ for some $i \in \omega$ and $m, n \in \mathbb{Z}$.*

Proof. It follows from (1) in the proof of Lemma 4. ☐

Lemma 9. *For any relations R_{i_0} and R_{i_1} satisfying Condition D, we have that*

$$K_{R_{i_0}} \cong K_{R_{i_1}} \Leftrightarrow (\omega, R_{i_0}) \cong (\omega, R_{i_1}).$$

Proof. The conclusion follows from how we recognize the elements $\pm c_i$, up to some $ma + nb$, in K_R for $R \in \{R_{i_0}, R_{i_1}\}$. The property that elements of the form $\pm c_i + ma + nb$ contain the same element c_i is equivalent to the statement that their product equals $\pm a$. The binary relations are definable from the ring structure by the fact that

$$R(i,j) \ \Leftrightarrow\ (\pm c_i + ma + nb)(\pm c_j + m'a + n'b) \in \{b, -b\}. \qquad \square$$

It remains to note that we can construct, uniformly in i, a computable structure isomorphic to K_{R_i} in order to establish the following

Theorem 2. *The isomorphism problem for computable nilpotent rings is Σ^1_1-complete.*

Remark. The same result can be obtained for rings with units. To do so, consider the rings K^1_R, the elements of which are the formal expressions

$$k + ma + nb + \sum_{i\in I} z_i c_i,$$

where $k, m, n \in \mathbb{Z}$. We impose the same rules on a, b, and the c_i as before. The subring K_R of K^1_R can be defined within K^1_R by the formula

$$x^3 = 0.$$

Then we can use the above results.

2. Distributive lattices

We will now focus on the computable isomorphism problem for distributive lattices by transforming a binary irreflexive and symmetric relation into a distributive lattice. The idea of our proof is partly contained in an unpublished manuscript by Rabin and Scott [14]. Consider a set $S = A \cup B$, where $A = \{a_i \mid i \in \omega\}$ and $B = \{b_i \mid i \in \omega\}$ are disjoint sets of distinct elements a_i and b_i, respectively. Partition B into infinite sets $B_{x,y}$ for a pair of different $x, y \in \omega$ so that the relation $\{(x,y,n) \mid b_n \in B_{x,y}\}$ is computable, and the following conditions are satisfied:

$$B = \textstyle\bigcup_{x,y\in\omega\&\ x\neq y} B_{x,y},$$

$$\{x,y\} \neq \{z,t\} \Rightarrow B_{x,y} \cap B_{z,t} = \varnothing,$$

$$B_{x,y} = B_{z,t} \Leftrightarrow \{x,y\} = \{z,t\}.$$

Let R be a given infinite binary irreflexive and symmetric relation on ω. Without loss of generality, we may assume that the domain of R contains at least three elements. Define a lattice L_R as the lattice of subsets of $A \cup B$ generated by the sets:

$$A,$$

$$\{x\} \ \text{ for all } x \in A \cup B,$$

$$u_{x,y} =_{def} \{a_x, a_y\} \cup B_{x,y} \text{ for all } x, y \text{ such that } R(x,y).$$

Since the operations of this lattice are the usual set-theoretic intersection and union, the lattice is distributive.

We will now show how to construct for a given computable relation R, a computable isomorphic copy of L_R. Notice first that every element of L_R is a finite union of intersections of generators. By examining all possible intersections, we conclude that they can only be of the following types: A, finite subsets of $A \cup B$ and $u_{x,y}$. Thus every element of L_R can be represented as the union of a finite family of one-element sets, some sets $u_{x,y}$, and possibly the set A. By omitting one-element sets that are contained in other sets in a union like this one, we obtain the unique "canonical representation" of elements. Thus, it is possible to effectively reduce every representation to a canonical one. Therefore, given a computable relation R, we can construct, in a uniform effective manner, a computable lattice isomorphic to L_R.

We now show that the relation R can be recovered from L_R. As usual, we call the minimal nonzero elements *atoms.* In this case, atoms are exactly one-element sets. Atoms can be distinguished by the following formula

$$\mathtt{at}\,(x) = \exists! y\,[y \neq x \;\wedge\; y \leqslant x].$$

We denote the least element by 0.

Lemma 10. *There exists a first-order formula, which in every L_R defines elements that are nonzero unions of possibly some sets $u_{m,n}$ and possibly A.*

Proof. Consider the formula $Q(x)$ which states:

> $x \neq 0$, and for each atom $y \leqslant x$, there is no least upper bound for the family $\{z \leqslant x \mid z \cap y = 0\}$.

The conclusion follows from the description of the elements of L_R. □

Lemma 11. *There exists a formula which defines A in every lattice L_R.*

Proof. Let the formula $Q(x)$ be as in the proof of Lemma 10. Let $Q^*(x)$ be the formula which defines the minimal elements satisfying the formula $Q(x)$.

Since the domain of R contains at least three elements, the element A can be defined by the formula that asserts

> $Q^*(x)$ and for each $y \neq x$ such that $Q^*(y)$, the element $x \cap y$ is greater than exactly two atoms. □

Given Lemma 11, we can use A in further formulas.

Lemma 12. *For all infinite $R_i, R_j \subseteq \omega^2$, the following equivalence holds:*

$$L_{R_i} \cong L_{R_j} \Leftrightarrow (\omega, R_i) \cong (\omega, R_j).$$

Proof. To prove the statement, it suffices to establish that R is always definable from L_R. Indeed, R is isomorphic to the relation defined by the formula $\psi_R(x, y)$:

$$x, y \leqslant A \;\wedge\; x \neq y \;\wedge\; \exists u\,[Q^*(u) \;\wedge\; u \neq A \;\wedge\; x, y \leqslant u]. \qquad \square$$

Since the computable isomorphism problem for infinite binary irreflexive and symmetric relations on ω is Σ^1_1-complete, we have proved the following theorem.

Theorem 3 (Goncharov and Knight)**.** *The computable isomorphism problem for computable distributive lattices is Σ^1_1-complete.*

3. Nilpotent semigroups

We follow the same pattern as in Section 2. The reasoning is essentially the same, but the construction is much easier. We will outline the transformation and the proof.

Assume that an irreflexive and symmetric relation $R \subseteq \omega^2$ is given. The set of elements of the semigroup S_R is

$$\{0, b\} \cup \{c_i \mid i \in \omega\}.$$

These elements will form a commutative semigroup by satisfying the following multiplication rules:

$$c_i c_j = \begin{cases} b, & \text{if } R(i,j), \\ 0, & \text{if } \neg R(i,j), \end{cases}$$
$$bx = 0,$$
$$0x = 0\ .$$

We note that every $x \in S_R$ satisfies $x^3 = 0$. Thus, 0 is definable, and b is definable as the only element $x \neq 0$ that is a product. Moreover, the semigroup S_R is nilpotent. We now sketch an idea how to recover R from S_R. The domain of the relation will be $\{c_i \mid i \in \omega\} = S_R - \{0, b\}$. The relation $\{(x,y) \mid xy = b\}$ on this domain is isomorphic to R. Thus, $S_{R_{i_0}} \cong S_{R_{i_1}}$ is equivalent to $(\omega, R_{i_0}) \cong (\omega, R_{i_1})$. Hence we have the following result.

Theorem 4. *The isomorphism problem for computable nilpotent semigroups is Σ^1_1-complete.*

4. 2-step nilpotent groups

Our approach in this section is somewhat different. We modify Mal'cev's coding of fields into 2-step nilpotent groups to obtain a computable transformation without parameters that preserves isomorphisms. Since the isomorphism problem for computable fields of any fixed characteristic is Σ^1_1-complete, it follows that the isomorphism problem for computable 2-step nilpotent groups is Σ^1_1-complete.

For a field F, let $H(F)$ denote the multiplicative group of matrices

$$h(a,b,c) = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix},$$

where $a, b, c \in F$. These groups are called Heisenberg groups and were studied by Mal'cev [11], who established some of their properties.

It is straightforward to check that $H(F)$ is a 2-step nilpotent group with center $Z(H(F))$ consisting of elements of the form $h(0,0,x)$. The field addition is connected to the group multiplication by $h(0,0,\alpha) \cdot h(0,0,\beta) = h(0,0,\alpha+\beta)$. Mal'cev showed how to recover the field multiplication using parameters for the noncommuting elements $h(1,0,0)$ and $h(0,1,0)$. To remove the use of these parameters, we show they can be replaced by any pair of noncommuting elements.

A direct calculation shows that

$$[h(a_0,b_0,c_0), h(a_1,b_1,c_1)] = h\left(0, 0, \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}\right).$$

Therefore, $[h(a_0,b_0,c_0), h(a_1,b_1,c_1)] \neq 1$ if and only if $\begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} \neq 0$ and for any $(a_0,b_0) \neq (0,0)$,

$$[h(a_0,b_0,c_0), h(a_1,b_1,c_1)] = 1 \Leftrightarrow \exists \alpha \left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \alpha \cdot \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \right).$$

The following lemma is the main technical tool to show that we can recover the field multiplication in $H(F)$ using any pair of noncommuting elements. We refer the reader to [1, Lemma 2.2] for a proof of this result.

Lemma 13. *Let* $w_0 = h(a_0, b_0, c_0)$ *and* $w_1 = h(a_1, b_1, c_1)$ *be such that* $[w_0, w_1] \neq 1$. *Let* $D = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix}$, $x = h(0, 0, \alpha \cdot D)$, $y = h(0, 0, \beta \cdot D)$ *and* $z = h(0, 0, \gamma \cdot D)$. *Then*

$$\alpha \cdot \beta = \gamma \Leftrightarrow \exists x' \exists y' \left([x', w_0] = [y', w_1] = 1 \wedge [w_0, y'] = y \wedge [x', w_1] = x \wedge z = [x', y'] \right). \tag{2}$$

Theorem 5 (Mal'cev, Morozov)**.** *Assume* F_0 *and* F_1 *are fields such that* $H(F_0) \cong H(F_1)$. *Then* $F_0 \cong F_1$.

Proof. Fix $H(F_0) \cong H(F_1)$. Let $w_0, w_1 \in H(F_0)$ be any pair of noncommuting elements. (Note that such elements always exist, for instance Mal'cev's parameters $w_0 = h(0, 1, 0)$ and $w_1 = h(1, 0, 0)$ are noncommuting.) Define a model

$$F(H(F_0), w_0, w_1) = (Z(H(F_0)), \oplus_{w_0,w_1}, \odot_{w_0,w_1})$$

on the center $Z(H(F_0))$ with the operations $\oplus_{w_0,w_1}$ and $\odot_{w_0,w_1}$ defined by

$$x \oplus_{w_0,w_1} y = x \cdot y$$

$$x \odot_{w_0,w_1} y = z \Leftrightarrow \exists x' \exists y' \left([x', w_0] = [y', w_1] = 1 \wedge \wedge [w_0, y'] = y \wedge [x', w_1] = x \wedge z = [x', y'] \right).$$

Let D be as in Lemma 13. The mapping $\varphi_0(\alpha) = h(0, 0, \alpha \cdot D)$ defines an isomorphism from F_0 onto $F(H(F_0), w_0, w_1)$.

Let $\psi : H(F_0) \to H(F_1)$ be a group isomorphism and let $v_i = \psi(w_i)$ for $i < 2$. The elements v_0 and v_1 are noncommuting, so as above, we have an isomorphism $\varphi_1 : F_1 \to F(H(F_1), v_0, v_1)$. Clearly, $\psi \restriction Z(H(F_0))$ is a field isomorphism from $F(H(F_0), w_0, w_1)$ onto $F(H(F_1), v_0, v_1)$. Therefore, $\varphi_1^{-1} \circ \psi \circ \varphi_0$ is an isomorphism from F_0 onto F_1. □

Now we are ready to prove Σ^1_1-completeness of the isomorphism problem for computable groups isomorphic to $H(F)$. For every computable field, we can uniformly effectively construct a computable group G_F isomorphic to $H(F)$. By Theorem 5, the condition $G_{F_i} \cong G_{F_j}$ is equivalent to $F_i \cong F_j$, and, since the isomorphism problem for computable fields of a fixed characteristic is Σ^1_1-complete, we have our final theorem.

Theorem 6. *The isomorphism problem for computable 2-step nilpotent groups is* Σ^1_1*-complete.*

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