DOMINATING ORDERS, VERTEX PURSUIT GAMES AND COMPUTABILITY THEORY

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Abstract. In the vertex pursuit game of cops and robbers on finite graphs, the cop has a winning strategy if and only if the graph admits a dominating order. Such graphs are called constructible in the graph theory literature. This equivalence breaks down for infinite graphs and variants of the game have been proposed to reestablish partial connections between constructibility and being cop win. We answer an open question of Lehner about one of these variants by giving examples of weak cop win graphs which are not constructible. We show that the index set of computable constructible graphs is Π^1_1 hard and the index set of computable constructible locally finite graphs is Π^0_4 hard. Finally, we give an example of a computable constructible graph for which every dominating order computes $0''$.

1. Introduction

The game of cops and robbers is played on a fixed graph G by two players. To start, Player C (traditionally a cop, but perhaps a cat instead) chooses an initial vertex to occupy, then Player R (a robber, or perhaps a rat) chooses an initial vertex. In subsequent rounds, Player C moves to a vertex adjacent to her current position, followed by Player R moving to a vertex adjacent to his current position. If the players are ever on the same vertex, then the game ends and Player C wins. Otherwise, if the game continues through ω many rounds, Player R wins. This game is open, so for any fixed graph, one of the players has a winning strategy. We say G is *C-win* if Player C has a winning strategy, and G is *R-win* otherwise.

There are many variations on this game, but the standard version uses reflexive graphs (which allow each player to remain on their current vertex during their turn) and assumes the graph G is connected (otherwise Player R can win trivially by starting in a separate component from Player C). Throughout, we will assume our graphs are reflexive, connected and countable.

Nowakowski and Winkler [6], and independently Quilliot [7], characterized the C-win finite graphs using the following observation. Imagine the position of the players at the start of a round in which Player C wins. Player R cannot avoid capture, so every vertex adjacent to his current position must also be adjacent to Player C's position. That is, Player R's position is dominated by Player C's position. (A vertex y *dominates* x if $y \neq x$ and each vertex adjacent to x is also adjacent to y.)

They proved that a finite graph G is C-win if and only if G can be constructed in such a way that when each vertex is added to the graph, it is dominated by

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a previously added vertex. More formally, G is C-win if and only if there is an enumeration of the vertices $v_0, \ldots, v_k$ such that for each $i > 0$, v_i is dominated in the induced subgraph on $\{v_0, \ldots, v_i\}$. An order of the vertices with this property is called a *dominating order* (or sometimes a *construction order*) and a finite graph G is *constructible* if it admits such an order.

The notion of a dominating order extends naturally to infinite graphs using well orders. A *dominating order* of G is a well order of the vertices, often written as v_α, such that for each $\alpha > 0$, v_α is dominated in the induced subgraph on $\{v_\beta \mid \beta \leq \alpha\}$. Unfortunately, the equivalence between being C-win and being constructible breaks down for infinite graphs. The infinite chain graph

$$v_0 \text{ ——— } v_1 \text{ ——— } v_2 \text{ ——— } v_3 \cdots\cdots$$

is constructible and $v_0 \prec v_1 \prec v_2 \prec \cdots$ is a dominating order, but clearly Player R can win by "running down the chain" ahead of Player C. (We omit drawing the reflexive edges in this and future graphs.)

On many infinite graphs, Player R can win by simply "running away" from Player C (e.g. on any locally finite infinite graph, see Stahl [10]). To compensate for this deficiency, Lehner [5] (building on work of Chastand, Laviolette and Polat [3]) introduced the notion of a weak C-win graph. Player C gets a *weak win* either if she captures Player R in finitely many rounds or if the game continues for infinitely many rounds but Player R only visits each vertex finitely often. Equivalently, Player C fails to get a weak win only if Player R avoids capture forever and visits some vertex infinitely often. G is *weak C-win* if Player C has a strategy to weak win on G. By Borel determinacy, for any fixed graph, one of the two players has a winning strategy under the weak win criterion.

For finite G, being C-win and being weak C-win are equivalent, and so a finite graph is weak C-win if and only if it is constructible. Lehner [5] proved that if an infinite graph is constructible, then it is weak C-win. He asked whether the converse holds.

We answer Lehner's question in Section 2 by giving an effective method to transform trees $T \subseteq \omega^{<\omega}$ into graphs G_T such that G_T admits a dominating order if and only if T is well founded. We prove that G_T is always weak C-win, and therefore, for any non-well founded tree T, G_T provides a negative answer to Lehner's question.

This transformation also gives information about the possible ranks of constructible graphs. The *rank* (or *construction time*) of a constructible graph G is the least order type of a dominating order for G. Using the transformation T_G, we show in Section 2 that the ranks of constructible graphs are unbounded in the countable ordinals. By way of contrast, Lehner [5] showed that the rank of a constructible locally finite graph is at most ω.

Ivan, Leader and Walters [4] recently extended these two results. They give a clever construction of a graph G that is C-win (not just weak C-win) but not constructible. They build G using a finite graph closely related to one of our coding graphs in Section 3. By carefully gluing together copies of this graph, they create an infinite graph in which Player C can "get ahead of" Player R on an infinite path, but which does not allow Player R to use the same shortcut to escape as he is chased towards a root node and eventually captured in finite time. They also show how to combine copies of this finite graph in an inductive manner to realize each countable ordinal as the rank of a graph.

Our transformation from trees to graphs gives a lower bound for the complexity of the index set of constructible computable graphs (which is Σ^1_2 by its definition). Because the transformation is effective and being well founded is Π^1_1-complete, it follows that this index set is Π^1_1-hard. In Section 2, we use the fact that the rank of a locally finite graph is bounded by ω to show the index set for constructible locally finite computable graphs is Σ^1_1. While we do not know the precise complexity of these two index sets in the arithmetic or analytic hierarchy, these results show their complexities are not the same.

Our original motivation for this project was to understand the complexity of dominating orders on constructible computable graphs. We turn to this question in Section 3, where we describe a framework for building such graphs that is suitable for coding a single jump. Using this framework, we build a locally finite constructible computable graph G such that every dominating order on G computes $0'$ and we investigate some subtleties in giving an effective version of Lehner's result that the rank of a locally finite graph is bounded by ω.

In Section 4, we give a second framework, this time designed to code two jumps. We use this method to build a locally finite constructible computable graph G for which every dominating order computes $0''$, and to prove that the index set of locally finite constructible computable graphs is Π^0_4-hard. Although these results seem to exhaust our particular construction method, we do not see any reason to suppose they are optimal.

We end this section with a summary of notation, conventions and formal definitions. A *graph* consists of a vertex set V and a symmetric reflexive edge relation $E \subseteq V \times V$. We assume our graphs are connected, countable and $V \subseteq \mathbb{N}$. We say $x, y \in V$ are *neighbors* if $E(x, y)$ and we define $N_G[x] = \{v \in G \mid E(x, v)\}$. The vertex y *dominates* x if $x \neq y$ and $N_G[x] \subseteq N_G[y]$. Because our graphs are reflexive, if y dominates x, then x and y are connected by an edge. A graph is *locally finite* if $N_G[x]$ is finite for all x.

For an order $\prec$ of V and $x \in V$, let $V_{\preceq x} = \{v \in G \mid v \preceq x\}$, let $G_{\preceq x}$ be the induced subgraph on $V_{\preceq x}$, and let $N_{\preceq x}[v]$ be the neighbor set of v in $G_{\preceq x}$. A *dominating order of* G is a well order $\prec$ of V such that for all $x \in V$, if x is not the $\prec$-least element of V, then x is dominated in $G_{\preceq x}$. G is called *constructible* if it admits a dominating order. The *rank* of a constructible graph G is the least ordinal α such that G has a dominating order of type α.

A *tree* $T \subseteq \omega^{<\omega}$ is a set of finite strings which is closed under initial segments. For finite strings $\sigma, \tau \in \omega^{<\omega}$, we write $\sigma \sqsubseteq \tau$ to denote that σ is an initial segment of τ, we let $|\sigma|$ denote the length of σ and we use λ for the empty string. We say τ is an *immediate successor* of σ if $\sigma \sqsubseteq \tau$ and $|\tau| = |\sigma| + 1$. A node $\sigma \in T$ is a *leaf* if it has no immediate successors. A tree T is *well founded* if it does not contain an infinite path.

For an excellent introduction to the game of cops of robbers, see Bonato and Nowakowski [2]. Our notation and terminology in computability theory follows Ash and Knight [1], Sacks [8] and Soare [9].

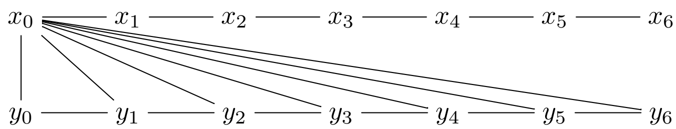
2. Trees and graphs

The main result of this section is a transformation from tree to graphs. Whenever we describe the edge relation for a graph, we implicitly assume we take the reflexive and symmetric closure of the described relation.

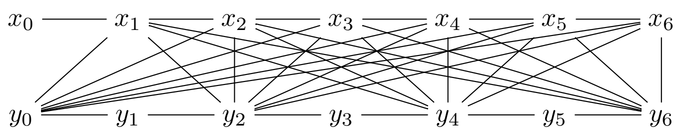
A reader who is primarily interested in a graph theoretic counter-example to Lehner's question can ignore the computability theory. The fact that the transformation in Theorem 2.1 is uniformly computable is irrelevant to the counter-example.

Theorem 2.1. *There is a computable functional that uniformly transforms trees $T \subseteq \omega^{<\omega}$ into graphs G_T such that G_T is constructible if and only if T is well founded.*

Proof. We start by describing a finite graph H that will be our main building block. H has vertices $V_H = \{x_i \mid i \leq 6\} \cup \{y_i \mid i \leq 6\}$ and edge relation defined by $E_H(x_0, y_i)$ for $i \leq 6$, $E_H(x_i, y_j)$ for $1 \leq i \leq 6$ and j even, and both $E(x_i, x_{i+1})$ and $E(y_i, y_{i+1})$ for $i \leq 5$. Note that x_0 is connected to each y_j vertex



while for $1 \leq i \leq 6$, x_i is only connected to the even index y_j vertices.



Fix a tree $T \subseteq \omega^{<\omega}$. To orient with the pictures of H above, we view T growing downward, so the successors of a node σ are positioned below σ. To form G_T, we replace each node $\sigma \in T$ with vertices v_i^σ for $i \leq 6$ and edges $E(v_i^\sigma, v_{i+1}^\sigma)$. When τ is an immediate successor of σ in T, we connect the nodes v_i^σ and v_j^τ as the nodes x_i and y_j are connected in H.

More formally, the vertices of G_T are $V_T = \{v_i^\sigma \mid i \leq 6 \text{ and } \sigma \in T\}$. For each $\sigma \in T$, we add edges $E_T(v_i^\sigma, v_{i+1}^\sigma)$ for $\sigma \in T$ and $i \leq 5$. When τ is an immediate successor of σ, we add edges $E_T(v_0^\sigma, v_i^\tau)$ for $i \leq 6$, and $E_T(v_i^\sigma, v_j^\tau)$ for $1 \leq i \leq 6$ and j even. This completes the description of G_T.

It is clear that G_T is uniformly computable from T. We prove that G_T is constructible if and only if T is well founded in the following series of lemmas. To simplify the notation, we fix T and drop the subscript T from G_T.

Lemma 2.2. *For any dominating order $\prec$ on G, any immediate successor pair $\sigma \sqsubseteq \tau$ in T, and any $k \leq 3$, there can be at most three nodes of the form v_i^σ such that $v_i^\sigma \prec v_{2k}^\tau$.*

Proof. Suppose four nodes $v_{j_0}^\sigma$, $v_{j_1}^\sigma$, $v_{j_2}^\sigma$ and $v_{j_3}^\sigma$ satisfy $v_{j_\ell}^\sigma \prec v_{2k}^\tau$. Consider which node dominates v_{2k}^τ in $G_{\preceq v_{2k}^\tau}$. We have $\{v_{j_0}^\sigma, v_{j_1}^\sigma, v_{j_2}^\sigma, v_{j_3}^\sigma, v_{2k}^\tau\} \subseteq N_{\preceq v_{2k}^\tau}[v_{2k}^\tau]$. By the construction of G, the only nodes which connect to four v_i^σ nodes are of the form:

(1) v_{2u}^τ, or
(2) v_w^ρ for the unique ρ such that $\rho \sqsubseteq \sigma$ is an immediate successor pair in T, or
(3) v_{2u}^μ for any $\mu \neq \tau$ such that $\sigma \sqsubseteq \mu$ is an immediate successor pair in T.

For (1), v_{2u}^τ cannot dominate v_{2k}^τ in $G_{\preceq v_{2k}^\tau}$ because v_{2u}^τ is connected to v_{2k}^τ if and only if $u = k$. For (2), v_w^ρ is not connected to v_{2k}^τ because $\rho \sqsubset \sigma \sqsubset \tau$ and hence τ is not an immediate successor of ρ in T. For (3), v_{2u}^μ is not connected to v_{2k}^τ because $|\mu| = |\tau|$ and $\mu \neq \tau$. Therefore, none of these nodes can dominate v_{2k}^τ in $G_{\preceq v_{2k}^\tau}$, contradicting the fact that $\prec$ is a dominating order. □

For a dominating order $\prec$ on G and a node $\sigma \in T$, let $m^\sigma_\prec$ denote the $\prec$-greatest vertex in $\{v^\sigma_{2\ell} \mid \ell \leq 3\}$. That is, $m^\sigma_\prec$ is the greatest even index vertex in G with superscript σ.

Lemma 2.3. *For any dominating order $\prec$ on G and any immediate successor pair $\sigma \sqsubseteq \tau$, $v^\tau_{2k} \prec m^\sigma_\prec$ for all $k \leq 3$. In particular, $m^\tau_\prec \prec m^\sigma_\prec$.*

Proof. Fix a dominating order $\prec$ on G and a successor pair $\sigma \sqsubseteq \tau$. Suppose there is a node v^τ_{2k} such that $m^\sigma_\prec \prec v^\tau_{2k}$. Since $v^\sigma_{2\ell} \preceq m^\sigma_\prec \prec v^\tau_{2k}$ for all $\ell \leq 3$, there are four nodes of the form v^σ_i such that $v^\sigma_i \prec v^\tau_{2k}$, contradicting Lemma 2.2. □

Lemma 2.4. *If T has an infinite path, then G does not have a dominating order.*

Proof. Let f be an infinite path in T. Assume for a contradiction that G has a dominating order $\prec$. Let $\sigma_n = f \upharpoonright n$. For each n, $\sigma_n \sqsubseteq \sigma_{n+1}$ is an immediate successor pair, so by Lemma 2.3, $m^{\sigma_{n+1}}_\prec \prec m^{\sigma_n}_\prec$. Therefore, $\prec$ contains an infinite descending chain, contradicting the fact that $\prec$ is a well order. □

It remains to show that if T is well-founded, then G has a dominating order. We construct the dominating order using two decompositions of T. The first decomposition is by levels, where the level of $\sigma \in T$ is the finite ordinal $|\sigma|$. For $n \in \omega$, let $L_n = \{\sigma \in T \mid |\sigma| = n\}$. For $\sigma, \tau \in L_n$, $\sigma <_{L_n} \tau$ (σ is *left of* τ) if $\sigma \neq \tau$ and $\sigma(i) < \tau(i)$ for the least $i < n$ such that $\sigma(i) \neq \tau(i)$. For each n, $(L_n, <_{L_n})$ is a well order. We combine these orders on L_n to get a well order $<^*_L$ on T defined by

$$\sigma <^*_L \tau \text{ if and only if } |\sigma| < |\tau| \text{ or } \big(|\sigma| = |\tau| \text{ and } \sigma <_{L_{|\sigma|}} \tau\big).$$

The second decomposition of T uses the standard notion of ordinal rank on a well-founded tree. For a leaf $\sigma \in T$, $\text{rank}_T(\sigma) = 0$. For a non-leaf node $\sigma \in T$,

$$\text{rank}_T(\sigma) = \sup\{\text{rank}_T(\tau) + 1 \mid \tau \text{ is an immediate successor of } \sigma\}.$$

Because T is well-founded, every node in T is assigned an ordinal rank by transfinite recursion and the largest rank is assigned to the root node λ. Let $R_\alpha = \{v \in T \mid \text{rank}_T(v) = \alpha\}$ be the set of nodes in T of rank α. Each set R_α is countable, so we can fix well orders $<_\alpha$ such that $(R_\alpha, <_\alpha)$ has order type $\leq \omega$ for each $\alpha \leq \text{rank}_T(\lambda)$. We combine these orders on R_α to get a well order $<^*_r$ on T defined by

$$v^\sigma_i <^*_r v^\tau_j \text{ if and only if } \text{rank}_T(\sigma) < \text{rank}_T(\tau) \text{ or } \big(\text{rank}_T(\sigma) = \text{rank}_T(\tau) = \alpha \text{ and } \sigma <_\alpha \tau\big).$$

Lemma 2.5. *If T does not have an infinite path, then G has a dominating order.*

Proof. Fix T with no infinite path. We define a dominating order $\prec$ on G. The nodes v^σ_0, $\sigma \in T$, form an initial segment of the dominating order with $v^\sigma_0 \prec v^\tau_0$ if and only if $\sigma <^*_L \tau$. We verify that $\prec$ has the dominating property on this initial segment. The least element under $\prec$ is v^λ_0 since λ is the only node with length 0. For $\tau \neq \lambda$, let σ be such that $\sigma \sqsubseteq \tau$ is an immediate successor pair. Since $|\sigma| < |\tau|$, we have $\sigma <^*_L \tau$ and hence $v^\sigma_0 \prec v^\tau_0$. Moreover, v^σ_0 dominates v^τ_0 in $G_{\preceq v^\tau_0}$ because the only vertices connected to v^τ_0 in $G_{\preceq v^\tau_0}$ are v^τ_0 and v^σ_0.

We order the remaining elements of G as follows. To make $\{v^\sigma_0 \mid \sigma \in T\}$ an initial segment, set $v^\tau_0 \prec v^\sigma_i$ for all $\tau, \sigma \in T$ and $i \geq 1$. For $\tau, \sigma \in T$ and $i, j \geq 1$, set

$$v^\sigma_i \prec v^\tau_j \text{ if and only if } \big(\sigma = \tau \text{ and } i < j\big) \text{ or } \big(\sigma \neq \tau \text{ and } \sigma <^*_r \tau\big).$$

If the order type of $(T, <^*_r)$ is β, then the type of $(\{v^\sigma_i \mid \sigma \in T \text{ and } 1 \leq i \leq 6\}, \prec)$ is $6 \cdot \beta$ because each node σ splits into a discrete interval $v^\sigma_1 \prec \ldots \prec v^\sigma_6$. In particular,

$\prec$ well orders $\{v_i^\sigma \mid \sigma \in T \text{ and } 1 \leq i \leq 6\}$. Since $\prec$ also well orders the initial segment $\{v_0^\sigma \mid \sigma \in T\}$, it follows that $\prec$ well orders G.

To finish the proof, note that each vertex v_i^σ with $i \geq 1$ is dominated in $G_{\preceq v_i^\sigma}$ by v_{i-1}^σ because the neighbors of v_i^σ in $G_{\preceq v_i^\sigma}$ are (1) v_{i-1}^σ, (2) v_{2k}^τ for each immediate successor τ of σ and each $k \leq 3$, and (3) v_0^μ when $\sigma \neq \lambda$ and σ is an immediate successor of μ. Each of these nodes is connected to v_{i-1}^σ. □

This completes the proof of Theorem 2.1. □

The key property of the graphs G_T is that they are all weak C-win.

Theorem 2.6. *For every tree $T \subseteq \omega^{<\omega}$, G_T is weak C-win.*

Proof. Player C starts at the vertex v_0^λ and has two basic strategies. Her initial strategy is to use nodes of the form v_0^σ to try to chase Player R down an infinite path in T, and so weak win because Player R never visits a node infinitely often.

More specifically, suppose it is Player C's turn, Player C is at v_0^σ and Player R is at v_j^τ with $\sigma \sqsubseteq \tau$ and $|\sigma| < |\tau|$. If τ is an immediate successor of σ, then there is an edge from v_0^σ to v_j^τ and so Player C can win immediately. Otherwise, if $|\sigma| + 1 < |\tau|$, then let $m = \tau(|\sigma|)$. Player C moves from v_0^σ to $v_0^{\sigma^\frown m}$.

Player R has two options to react to this strategy. He can move down the tree, occupying vertices of the form v_j^τ for strings τ with (occasionly) increasing length, so that he keeps at least two tree levels between his position and Player C's position. However, if he does this forever, then he will occupy each vertex only finitely often, allowing Player C to weak win.

On the other hand, he could allow Player C's level in the tree to "catch up" to his until on his turn, he is at a vertex v_j^τ and Player C is at a vertex v_0^σ with τ an immediate successor of σ. Player R can now move to a vertex v_k^σ, occupying a vertex associated to the same node of T as Player C's position. Of course, he will pick $k \geq 1$ to avoid losing immediately.

At this point, Player C switches to her back-up strategy which is guaranteed to catch Player R in finitely many more rounds. Specifically, Player C moves to v_1^σ. Player R has four options for his move.

(1) He moves to v_{k-1}^σ or v_{k+1}^σ, or stays put on v_k^σ. In this case, Player C moves to v_2^σ. If Player R continues to take this option, Player C chases him down the finite chain of elements associated to σ and wins in a finite round.
(2) He moves to $v_{2\ell}^\rho$ for some immediate successor ρ of σ on T. However, v_1^σ is also connected to $v_{2\ell}^\rho$, so in this case, he loses immediately. (In fact, every vertex v_i^σ is connected to $v_{2\ell}^\rho$, so no matter where Player C is in the row corresponding to σ, she can win immediately.)
(3) He moves to v_0^μ where μ is the immediate predecessor of σ in T. As in (2), v_1^σ (and in fact, every vertex v_i^σ) is connected to v_0^μ, so he loses immediately.
(4) If k is even, then he can move to $v_{k'}^\mu$ for $1 \leq k' \leq 6$, where μ is the immediate predecessor of σ. The vertex v_1^σ (or v_i^σ for any odd i) is not connected to $v_{k'}^\mu$, so Player C cannot necessarily win immediately. However, she can move to v_0^μ, so the players are again occupying vertices associated to the same node of T with Player C at the 0 indexed node. The key point is that the length of the corresponding node on T has decreased.

Repeating this analysis, Player R has no choice but to move slowly up the tree towards the root, perhaps taking a few turns at each level to move along the row of vertices corresponding to a particular node in the tree. When Player R reaches a vertex of the form v_k^λ corresponding to the root, option (4) is no longer available and he loses in finitely many more rounds. $\square$

Corollary 2.7. *For any tree $T \subseteq \omega^{<\omega}$ that is not well-founded, the graph G_T is weak C-win but not constructible.*

Corollary 2.8. *There is a locally finite graph G that is weak C-win but not constructible.*

Proof. Let T be an infinite finitely branching tree. For example, take a single infinite path $T = \{0^n \mid n \in \omega\}$. G_T is weak C-win by Theorem 2.6, is locally finite because T is finitely branching, and is not constructible by Theorem 2.1. $\square$

In addition to giving examples of weak C-win graphs that are not constructible, Theorem 2.1 gives information about the index set of computable constructible graphs. This index set is (at worst) Σ^1_2 since G is constructible if and only if there is a binary relation $\prec$ on G such that $\prec$ is a well order that satisfies the domination condition. The domination condition is arithmetical, but to say $\prec$ is a well order is Π^1_1, and hence the definition is Σ^1_2. Since the index set of well founded computable trees is Π^1_1-complete and the functional in Theorem 2.1 is computable, we get the following corollary.

Corollary 2.9. *The index set of computable constructible graphs is Π^1_1-hard.*

Using the following proposition, we can contrast this situation with the index set of computable locally finite graphs that are constructible.

Proposition 2.10 (Lehner [5])**.** *A countable locally finite graph is constructible if and only if admits a dominating order of type $\leq \omega$.*

Proposition 2.11. *The index set of computable locally finite constructible graphs is Σ^1_1.*

Proof. By Proposition 2.10, a locally finite graph is constructible if and only there is a linear order on G in which every vertex has finitely many predecessors and which satisfies the domination condition. Saying that a linear order has the finite predecessor property is arithmetical, so the entire statement is Σ^1_1. $\square$

We return to the question of proving a lower bound on the complexity of this index set in Section 4. The next theorem shows there is no analog of Proposition 2.10 for general countable constructible graphs. The *rank* of a constructible graph G is the least ordinal α such that G has a dominating order of type α.

Theorem 2.12. *Let $T \subseteq \omega^{<\omega}$ be a well-founded tree. The rank of G_T as a constructible graph is greater than or equal to $rank_T(\lambda)$.*

Proof. Fix T. Let $\alpha = \mathrm{rank}_T(\lambda)$ and let $\prec$ be a dominating order on G. We show the order type of $(G_T, \prec)$ is at least α. Below, we drop the subscript T.

Recall that for $\sigma \in T$, m^σ is the $\prec$-greatest element of $\{v^\sigma_{2k} \mid k \leq 3\}$. Let β_σ be the order type of $(G_{\preceq m^\sigma}, \prec)$. We claim that for every $\sigma \in T$, $\mathrm{rank}_T(\sigma) \leq \beta_\sigma$. The theorem follows from this claim because $\alpha = \mathrm{rank}_T(\lambda) \leq \beta_\lambda \leq$ order-type$(G, \prec)$.

We prove the claim by induction on $\mathrm{rank}_T(\sigma)$. When $\mathrm{rank}_T(\sigma) = 0$, we have $0 \leq \beta_\sigma$ trivially. Suppose $\mathrm{rank}_T(\sigma) > 0$. If τ is an immediate successor of σ, then

$\mathrm{rank}_T(\tau) < \mathrm{rank}_T(\sigma)$, and so by the induction hypothesis, $\mathrm{rank}_T(\tau) \leq \beta_\tau$. By Lemma 2.3, $m^\tau \prec m^\sigma$ and hence $G_{\preceq m^\tau} \subsetneq G_{\preceq m^\sigma}$ and so $\beta_\tau < \beta_\sigma$. Therefore,

$$\begin{aligned} \mathrm{rank}_T(\sigma) &= \sup\{\, \mathrm{rank}_T(\tau)+1 \mid \tau \text{ is immediate successor of } \sigma \} \\ &\leq \sup\{\, \beta_\tau + 1 \mid \tau \text{ is immediate successor of } \sigma \} \\ &\leq \beta_\sigma. \end{aligned}$$

□

Corollary 2.13. *The ranks of countable constructible graphs are cofinal in ω_1.*

3. Comb graphs and coding one jump

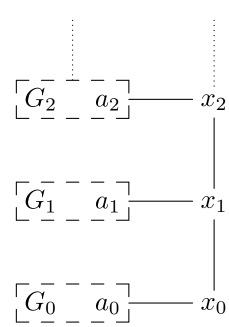
In this section, we develop a general framework to code information into dominating orders. The following useful fact is straightforward to verify.

Lemma 3.1. *Let $\prec$ be a dominating order on G. For every $v \in G$, the induced subgraph $G_{\preceq v}$ is connected.*

Definition 3.2. Let $G_i = (V_i, E_i)$, $i \in \omega$, be a sequence of disjoint connected graphs, each with a designated node a_i. The *comb graph $G = (V, E)$ with spine x_i, connectors a_i and teeth graphs G_i* is the graph defined by

$$V = \bigcup_{i \in \omega} (V_i \cup \{x_i\}) \quad \text{and} \quad E = \bigcup_{i \in \omega} (E_i \cup \{\langle x_i, x_{i+1}\rangle, \langle x_i, a_i\rangle\}).$$

A comb graph looks like



where the notation $[G_i \quad a_i]$ —— x_i indicates that G_i is attached to G by connecting x_i to a_i, but making no other connections between x_i and nodes in G_i. Note that if the sequence of graphs G_i with distinguished elements a_i is uniformly computable, then the corresponding comb graph has a computable presentation.

Lemma 3.3. *Let G be a comb graph with teeth graphs G_i, let $\prec$ be a dominating order on G, and let $\prec_i$ be the restriction of $\prec$ to G_i. For all i, $\prec_i$ is a dominating order on G_i, and for all i except possibly one, $\prec_i$ has least element a_i.*

Proof. Let $\prec$ be a dominating order of G with $\prec$-least element v_0. Let i_0 be such that either $v_0 = x_{i_0}$ or $v_0 \in G_{i_0}$.

Three observations follow from Lemma 3.1. First, x_{i_0} is the $\prec$-least element of the form x_j. Second, if $x_{i_0} \prec a_{i_0}$, then x_{i_0} is the $\prec$-least element of G and a_{i_0} is the $\prec$-least element of G_{i_0}. Third, if $i \neq i_0$, then $x_i \prec a_i$ and a_i is the $\prec$-least element of G_i. By the third observation, a_i is the $\prec_i$-least element of G_i for all i except possibly i_0.

The fact that $\prec_i$ is a dominating order on G_i follows from the observations above plus the facts that $N_G[a_i] = N_{G_i}[a_i] \cup \{x_i\}$ and that $N_{G_i}[v] = N_G[v]$ for all $v \in G_i$ with $v \neq a_i$. □

Lemma 3.4. *Let G be a comb graph with teeth graphs G_i. For any sequence of dominating orders $\prec_i$ on the graphs G_i with least element a_i, there is a dominating order $\prec$ of G such that for every i, the restriction of $\prec$ to G_i is $\prec_i$.*

Proof. Define an order $\prec$ on G by setting $u \prec v$ if and only if (i) $u \in G_i \cup \{x_i\}$, $v \in G_j \cup \{x_j\}$ and $i < j$, (ii) $u = x_i$ and $v \in G_i$, or (iii) $u, v \in G_i$ and $u \prec_i v$. The order $\prec$ can be visualized as

$$x_0 \prec (G_0, \prec_0) \prec x_1 \prec (G_1, \prec_1) \prec \cdots \prec x_i \prec (G_i, \prec_i) \prec x_{i+1} \prec \cdots$$

where $x_i \prec (G_i, \prec_i) \prec x_{i+1}$ denotes that x_i comes before all the elements of G_i, that all the elements of G_i come before x_{i+1}, and that the elements of G_i are ordered among themselves by $\prec_i$. It is straightforward to verify that $\prec$ is the desired dominating order on G. □

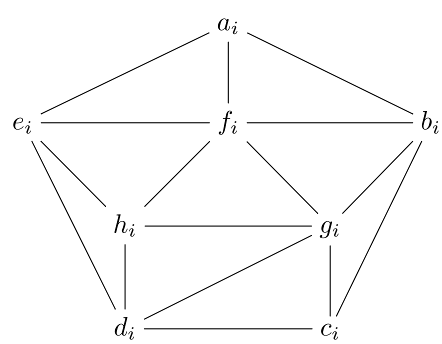
Note that if the graphs G_i in Lemma 3.4 are finite, then the dominating order given in the proof has order type ω. We end this section with two results using this framework for constructing graphs. The first application, given in the following theorem, will be improved in Theorem 4.11.

Theorem 3.5. *There is a computable graph G such that G is constructible and every dominating order computes $0'$.*

Proof. We build G as a comb graph in which each G_i will have one of two isomorphism types. Let X_i have domain $\{a_i, b_i, c_i, d_i\}$ and edges E_{X_i} given by

$$a_i \;\text{———}\; b_i \;\text{———}\; c_i \;\text{———}\; d_i$$

and let Y_i have domain $\{a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i\}$ and edges E_{Y_i} given by



Because X_i is an induced subgraph of Y_i, there is a uniformly computable sequence of graphs G_i such that if $i \notin 0'$, then $G_i \cong X_i$, and if $i \in 0'$, then $G_i \cong Y_i$. Therefore, there is a computable comb graph G with teeth graphs G_i.

The only dominating order of X_i starting with a_i is $a_i \prec b_i \prec c_i \prec d_i$. Therefore, in every dominating order of X_i starting with a_i, we have $c_i \prec d_i$.

There are several dominating orders of Y_i that start with a_i, for example $a_i \prec f_i \prec g_i \prec h_i \prec b_i \prec e_i \prec d_i \prec c_i$. However, we claim that $d_i \prec c_i$ in each such order. To see why, note that since Y_i is finite, the last element in any dominating order must be dominated in the full graph Y_i. Therefore, because only a_i and c_i

are dominated in Y_i, every dominating order on Y_i that starts with a_i must end with c_i. In particular, $d_i \prec c_i$ in such an order.

Each G_i has a dominating order starting with a_i, so by Lemma 3.4, G admits a dominating order. Fix any dominating order $\prec$ on G and let $\prec_i$ be the restriction to G_i. By Lemma 3.3, $\prec_i$ is a dominating order of G_i starting with a_i, with the possible exception of one index i_0. Let $K_\prec = \{i \in \mathbb{N} \mid d_i \prec c_i\}$. For any $i \neq i_0$, $i \in K_\prec$ if and only if $i \in 0'$, and therefore, $0'$ is computable from an arbitrary dominating order of G. □

For the second application of this method, we distinguish two ways that a computable dominating order on a locally finite graph could have order type ω.

Definition 3.6. Let G be a computable locally finite graph and let $\prec$ be a computable dominating order of G. We say $\prec$ is a *computable dominating order of type* ω if the classical order type of $(G, \prec)$ is ω. We say $\prec$ is a *computable dominating order of strong type* ω if there is a computable order preserving bijection $f : (\omega, \leq) \to (G, \prec)$.

Theorem 3.7. *There is a computable locally finite graph G such that G has a computable dominating order of type ω but not a computable dominating order of strong type ω.*

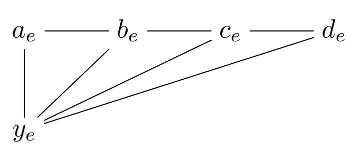
Proof. We build our computable graph G with domain ω. For each index e, if φ_e is a bijection of ω onto G, then we let $\prec_e$ be the binary relation on G defined by $v \prec_e w$ if and only if $\varphi_e^{-1}(v) < \varphi_e^{-1}(w)$. To ensure that G does not have a computable order of strong type ω, it suffices to meet the following requirements.

$$R_e : \text{ If } \varphi_e : \omega \to G \text{ is a bijection, then } \prec_e \text{ is not a dominating order on } G$$

We build G as a comb graph and we use G_e to satisfy the requirement R_e.

As in the proof of Theorem 3.5, G_e will have one of two isomorphism types. We start with G_e equal to the graph X_e from Theorem 3.5. Our construction of G_e then proceeds in stages.

(1) Wait for a stage s_0 such that $\{a_e, b_e, c_e, d_e\} \subseteq \text{range}(\varphi_{e,s_0})$. If there is no such stage, we stay in (1) forever and $G_e = X_e$.
(2) Set $n_e = \max\{x \mid \varphi_{e,s_0}(x) \in \{a_e, b_e, c_e, d_e\}\}$. Wait for a stage $s_1 > s_0$ such that $\varphi_{e,s_1}(x)$ converges for all $x \leq n_e$. If there is no such stage (or if we see φ_e is not injective), we stay in (2) forever and $G_e = X_e$.
(3) At stage $s_1 + 1$, we add a vertex y_e to G_e to form the following graph Z_e.



We choose y_e so that $\varphi_e(x) \neq y_e$ for all $x \leq n_e$. We retain $G_e = Z_e$ at all future stages.

Since X_e is an induced subgraph of Z_e and the switch from $G_e = X_e$ to $G_e = Z_e$ is determined by a Σ^0_1 event, the sequence of graphs G_e is uniformly computable. Therefore, there is a computable comb graph G with teeth graphs G_e.

We have already seen that X_e has a dominating order starting with a_e. Z_e has several dominating orders starting with a_e, for example, $a_e \prec b_e \prec y_e \prec c_e \prec d_e$.

However, no dominating order of Z_e can end with y_e because y_e is not dominated in the full graph Z_e.

G has a computable dominating order of type ω constructed as in the proof of Lemma 3.4. At stage 0, we start with the empty order. At stage $s+1$, we attach $x_s \prec a_s \prec b_s \prec c_s \prec d_s$ to the end of the order determined at stage s. Next, we check if any G_e (for $e \leq s$) changed from X_e to Z_e at stage s. If so, then we add y_e to the order so that $b_e \prec y_e \prec c_e$. Although the addition of the element y_e is delayed until we see G_e change to Z_e, the order $\prec$ (in the end) has the form

$$x_0 \prec G_0 \prec x_1 \prec G_1 \prec \cdots$$

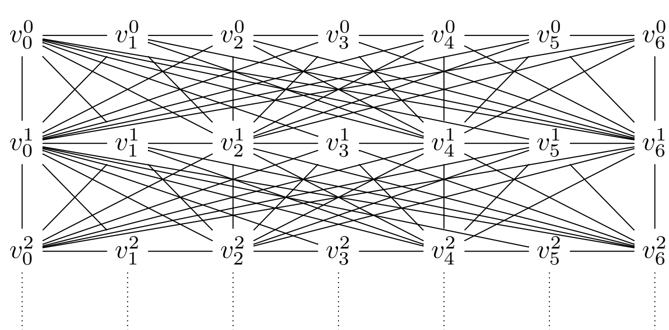
Therefore, $\prec$ is a computable dominating order of type ω.

To finish the proof, suppose for a contradiction that there is a computable dominating order $\prec$ of G of strong type ω. Fix an index e such that $\varphi_e : \omega \to G$ is a bijection with $i < j \Leftrightarrow \varphi_e(i) \prec \varphi_e(j)$.

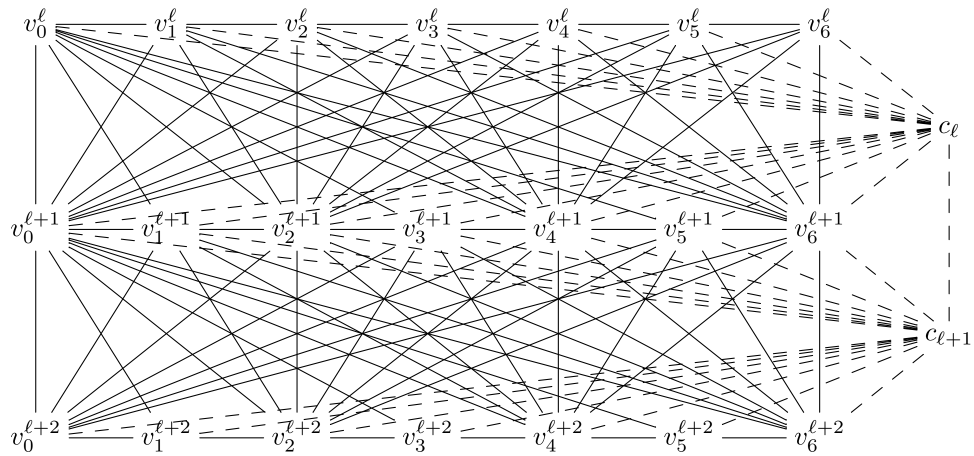
Consider the construction of G_e. Since φ_e is a bijection, we find stages $s_0 < s_1$ and define the parameter n_e in Steps (1) and (2). Therefore, G_e is isomorphic to Z_e. Fix m such that $\varphi_e(m) = y_e$. By construction, $n_e < m$. It follows that for all $v \in \{a_e, b_e, c_e, d_e\}$, $v \prec y_e$. Therefore, $N_{\preceq y_e}[y_e] = \{a_e, b_e, c_e, d_e, y_e\}$. However, no other node in G contains this set within its neighbors, and so no node can dominate y_e in $G_{\preceq y_e}$. This contradicts the fact that $\prec$ is a dominating order. □

4. Second computability construction

To prove additional computability theoretic results about dominating orders, we use a family of graphs $K(X)$, parameterized by a set $X \subseteq \omega$. We begin with the graph formed by applying Theorem 2.1 to a tree which consists of a single infinite path. The resulting graph consists of ω many rows v_i^ℓ, $0 \leq i \leq 6$, in which v_i^ℓ is connected to v_{i+1}^ℓ. The rows are connected by adding edges from v_0^ℓ to every $v_j^{\ell+1}$, and from v_i^ℓ to $v_j^{\ell+1}$ when $i > 0$ and j even.



We refer to this graph as K. It will be an induced subgraph of a larger graph $\widehat{K}$. To form $\widehat{K}$, for each ℓ, we add an auxiliary node c_ℓ connected to all the elements of rows ℓ and $\ell+1$, and we connect c_ℓ and $c_{\ell+1}$ as shown below.



$\widehat{K}$ is the graph consisting of v_i^ℓ and c_ℓ for all $\ell \in \omega$ and $0 \le i \le 6$. The nodes v_i^ℓ, $0 \le i \le 6$ form the ℓ-th row of this graph, and the nodes c_ℓ are called auxiliary nodes. Because of these auxiliary nodes, $\widehat{K}$ is constructible, although the induced subgraph K is not constructible (by Theorem 2.1).

Lemma 4.1. *$\widehat{K}$ has a dominating order with least element v_0^0.*

Proof. The dominating order starts with the initial segment

$$v_0^0 \prec c_0 \prec v_1^0 \prec v_2^0 \prec v_3^0 \prec v_4^0 \prec v_5^0 \prec v_6^0.$$

Each of these elements is dominated by the preceding element in the appropriate initial subgraph. We continue to construct the dominating order row by row, with row ℓ and the auxiliary element c_ℓ ordered as

$$v_0^\ell \prec c_\ell \prec v_1^\ell \prec v_2^\ell \prec v_3^\ell \prec v_4^\ell \prec v_5^\ell \prec v_6^\ell.$$

These elements are dominated by $c_{\ell-1}$ in the appropriate initial subgraph because $c_{\ell-1}$ is connected to each of them and to all the elements in row $\ell-1$, and because none of the elements in row $\ell+1$ have entered the dominating order yet. □

Definition 4.2. For $X \subseteq \omega$, $K(X)$ is the induced subgraph of $\widehat{K}$ containing the nodes v_i^ℓ for $\ell \in \omega$ and $0 \le i \le 6$ and the auxiliary nodes c_k for $k \in X$.

The set X specifies the nodes c_k to add to K to form $K(X)$. We will generalize the fact that $\widehat{K} = K(\omega)$ is constructible, while $K = K(\emptyset)$ is not, by showing that $K(X)$ is constructible if and only if X is cofinite.

Definition 4.3. Let $\prec$ be a dominating order on $K(X)$. For each row r, let $m_\prec^r$ (or m^r when $\prec$ is clear from context) denote the $\prec$-greatest even indexed node v_{2i}^r in row r. We say row r is $\prec$-*special* (or *special* when $\prec$ is clear) if $m^r \prec m^{r+1}$.

If $\prec$ is a dominating order on $K(X)$, there must be a special row since otherwise $m^0 \succ m^1 \succ m^2 \succ \cdots$ would be an infinite descending sequence. For the next several lemmas, assume $\prec$ is a dominating order on $K(X)$.

Lemma 4.4. *If $r \notin X$, then $m^{r+1} \prec m^r$. That is, if r is special, then $r \in X$.*

Proof. If $r \notin X$, then the auxiliary node c_r is not in $K(X)$. It follows that $m^{r+1} \prec m^r$ by the same argument given in Lemmas 2.2 and 2.3. □

Lemma 4.5. *For all rows r, $m^{r+1} \prec \max\{m^r, m^{r+2}\}$.*

Proof. Suppose $m^r, m^{r+2} \prec m^{r+1}$. The neighbors of m^{r+1} in $K(X)_{\preceq m^{r+1}}$ include all the even indexed nodes in rows r and $r+2$. Only vertices of the form v_{2j}^{r+1} are connected to all of these nodes, regardless of which auxiliary nodes are in $K(X)$. However, no node v_{2j}^{r+1} can dominate m^{r+1} in $K(X)_{\preceq m^{r+1}}$. □

Lemma 4.6. *If row r is special, then every row $\ell \geq r$ is special, and hence $\ell \in X$ for all $\ell \geq k$.*

Proof. If row r is special, then $m^r \prec m^{r+1}$. It follows from Lemma 4.5 that $m^{r+1} \prec m^{r+2}$, and hence row $r+1$ is special. By induction, we get every row $\ell \geq r$ is special, and hence by Lemma 4.4, $\ell \in X$ for all $\ell \geq r$. □

Lemma 4.7. *$K(X)$ is constructible if and only if X is cofinite. Furthermore, if X is cofinite, then $K(X)$ admits a dominating order with least element v_0^0.*

Proof. For the forward direction, fix a dominating order on $K(X)$. As observed above, $K(X)$ has a special row r, and so by Lemma 4.6, $\ell \in X$ for all $\ell \geq r$.

For the other direction, assume X is cofinite. The case when $X = \omega$ follows from Lemma 4.1, so assume $\overline{X} = \omega - X$ is nonempty. Let k be the largest element $\overline{X}$ and let $y_0 < y_1 < \cdots < y_i$ be the numbers $y < k$ that are in X. The auxiliary nodes in $K(X)$ are $\{c_{y_0}, \ldots, c_{y_i}\} \cup \{c_\ell : \ell > k\}$. We construct a dominating order starting with the initial segment

$$v_0^0 \prec v_0^1 \prec \cdots \prec v_0^{k+1} \prec c_{y_0} \prec c_{y_1} \prec \cdots \prec c_{y_i} \prec c_{k+1}.$$

This initial segment satisfies the dominating conditions because $v_0^{\ell+1}$ is dominated by v_0^ℓ in $K(X)_{\preceq v_0^{\ell+1}}$, c_{y_j} is dominated by $v_0^{y_j}$ in $K(X)_{\preceq c_{y_j}}$, and c_{k+1} is dominated by v_0^{k+1} in $K(X)_{\preceq c_{k+1}}$.

We next add the remaining elements from rows 0 through $k+1$, starting with row $k+1$ and working down to row 0.

$$v_1^{k+1} \prec v_2^{k+1} \prec v_3^{k+1} \prec v_4^{k+1} \prec v_5^{k+1} \prec v_6^{k+1} \prec v_1^k \prec v_2^k \prec \cdots$$

The domination property is satisfied because each of these vertices v_j^ℓ is dominated by v_{j-1}^ℓ in $K(X)_{\preceq v_j^\ell}$. We order the rest of $K(X)$ row by row starting with the remainder of row $k+1$.

$$v_1^{k+1} \prec v_2^{k+1} \prec v_3^{k+1} \prec v_4^{k+1} \prec v_5^{k+1} \prec v_6^{k+1} \prec c_{k+2} \prec v_0^{k+2} \prec v_1^{k+2} \prec \cdots$$

The remaining elements of row $k+1$ and c_{k+2} are dominated by c_{k+1} in the appropriate subgraph. Following this pattern, the elements of each row ℓ for $\ell > k+1$, as well as the vertex $c_{\ell+1}$ are dominated by c_ℓ in the appropriate subgraph. □

Lemma 4.8. *Let A_k, $k \in \omega$, be a uniformly c.e. sequence of sets. There is a computable presentation of the comb graph G with teeth graphs $G_k = K(A_k)$ in which the designated element a_k is the v_0^0 node. Furthermore, G is constructible if and only if every set A_k is cofinite.*

Proof. We can uniformly construct a computable copy of $K(A_k)$ from an enumeration of A_k by building a computable copy of the graph K and adding auxiliary nodes c_n as n is enumerated into A_k. By Lemma 4.7, if A_k is not cofinite, then $K(A_k)$ is not constructible, and hence by Lemma 3.3, G is not constructible. If each A_k is cofinite, then each graph $K(A_k)$ has a dominating order with least element v_0^0, and hence G has a dominating order. □

Our first application of these graphs is to show the index set of computable locally finite constructible graphs is Π_4^0-hard.

Theorem 4.9. *The index set of computable locally finite constructible graphs is* Π_4^0*-hard.*

Proof. Let R be an arbitrary Π_4^0 relation on ω. It suffices to build a uniform computable sequence of locally finite graphs G_k such that for all k, $R(k)$ holds if and only if G_k admits a dominating order.

The index set $\mathrm{Cof} = \{e : W_e \text{ is cofinite}\}$ is Σ_3^0-complete, so we can fix a uniform c.e. sequence of sets A_e^k for $e, k \in \omega$ such that for all k

$$R(k) \text{ holds } \Leftrightarrow \forall e\,(A_e^k \text{ is cofinite}).$$

For each k, let G_k be the comb graph with teeth graphs $K(A_e^k)$ for $e \in \omega$. We can uniformly construct the sequence of computable graphs G_k. By Lemma 4.7, G_k is constructible if and only if for all e, A_e^k is cofinite. Therefore, $R(k)$ holds if and only if G_k is constructuble. □

Our second application improves Theorem 3.5. It uses the following lemma, which is a straightforward exercise to verify.

Lemma 4.10. *If a function* $f : \omega \to \omega$ *satisfies* $f(e) \geq \max W_e$ *for each finite c.e. set* W_e*, then* $0'' \leq_T f$*.*

Theorem 4.11. *There is a computable locally finite constructible graph* G *such that every dominating order on* G *computes* $0''$*.*

Proof. It suffices to build G such that for every dominating order computes the index set $\mathrm{Inf} = \{e : W_e \text{ is infinite}\}$. G will be a comb graph with teeth graphs $K(A_e)$, where A_e is a uniformly c.e. family of cofinite sets.

We enumerate A_e in stages with $A_{e,s}$ denoting the set at the end of stage s. We simultaneously define markers $m_{e,s}$ and the finite sets $A_{e,s}$ by recursion. Set $m_{e,0} = 0$ and $A_{e,0} = \emptyset$. Define

$$m_{e,s+1} = \begin{cases} m_{e,s} & \text{if } W_{e,s+1} = W_{e,s} \\ s+1 & \text{otherwise} \end{cases}$$

and $A_{e,s+1} = \{0, \ldots, s+1\} - \{m_{e,s+1}\}$. That is, if an element enters W_e at stage $s+1$, then we enumerate $m_{e,s}$ into A_e and reset our marker $m_{e,s+1} = s+1$. If no element enters $W_{e,s+1}$, then leave our marker fixed and enumerate $s+1$ into A_e.

If W_e is infinite, then $\lim_s m_{e,s} = \infty$ and $A_e = \omega$. If W_e is finite, then $\lim_s m_{e,s} = m_e$ and $A_e = \omega - \{m_e\}$ where m_e is the least stage such that $W_{e,m_e+1} = W_e$. In this case, by the usual conventions on uses, $\max W_e \leq m_e$.

Let G be a computable copy of the comb graph with teeth graphs $K(A_e)$. Fix a dominating order $\prec$ on G. For each e, using $\prec$, we can find a special row in $K(A_e)$ by searching. Let $f(e) =$ the row number of the first special row we find in $K(A_e)$ and note that f is computable from $\prec$. If W_e is finite, then $A_e = \omega - \{m_e\}$, and hence by Lemmas 4.4 and 4.6, $f(e) > m_e$. Therefore, $f(e) > \max W_e$ when W_e is finite. It follows by Lemma 4.10 that $0'' \leq_T f$ and hence $\prec$ computes $0''$. □

5. Open questions

By Theorems 2.1 and 2.12, the ranks of computable constructible graphs are cofinal in ω_1^{CK}. Is there a computable constructible graph without a hyperarithmetic dominating order?

There are large gaps in the index set results. It remains open to close the gap between Π^1_1 and Σ^1_2 for the index set of computable constructible graphs, and to close the gap between Π^0_4 and Σ^1_1 for the index set of computable locally finite constructible graphs.

Finally, we see no reason to believe Theorem 4.11 is optimal. For which computable ordinals α is it possible to construct a computable graph G such that every dominating order computes $0^{(\alpha)}$?

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