DOMINATING ORDERS, VERTEX PURSUIT GAMES AND COMPUTABILITY THEORY

LEIGH EVRON, REED SOLOMON, AND RACHEL D. STAHL

Abstract. In the vertex pursuit game of cops and robbers on finite graphs, Player C has a winning strategy if and only if the graph admits a dominating order. Such graphs are called constructible in the graph theory literature. This equivalence breaks down for infinite graphs and variants of the game have been proposed to reestablish partial connections between constructibility and being C-win. We answer an open question of Lehner about one of these variants by giving examples of weak C-win graphs which are not constructible. We show that the index set of computable constructible graphs is Π^1_1 hard and the index set of computable constructible locally finite graphs is Π^0_4 hard. Finally, we give an example of a computable constructible graph for which every dominating order computes $0''$.

1. Introduction

The game of cops and robbers is played on a fixed graph G by two players. To start the game, Player C (traditionally a cop, but perhaps a cat instead) chooses an initial vertex to occupy, then Player R (a robber, or perhaps alternately, a rat) chooses an initial vertex. In subsequent rounds, Player C moves to a vertex adjacent to her current position, followed by Player R moving to a vertex adjacent to his current position. If the players are ever on the same vertex, then the game ends and Player C wins. Otherwise, if the game continues through ω many rounds, Player R wins. This game is an open game, so for any fixed graph, one of the two players has a winning strategy. We say G is *C-win* if Player C has a winning strategy, and G is *R-win* otherwise.

There are many variations on this game, but the standard version uses reflexive graphs (which allow each player the option to remain on their current vertex during their turn) and assumes the graph G is connected (otherwise Player R can win trivially by starting in a separate component from Player C). Throughout, we will assume our graphs are reflexive, connected and countable.

Nowakowski and Winkler [6], and independently Quilliot [7], characterized the C-win finite graphs using the following observation. Imagine the position of the players at the start of a round in which Player C wins. Player R cannot avoid capture, so every vertex connected to his current position must also be connected to Player C's position. That is, Player R's position is dominated by Player C's position. (A vertex y *dominates* x if $y \neq x$ and each vertex connected to x is also connected to y.)

They proved that a finite graph G is C-win if and only if G can be constructed in such a way that when each vertex is added to the graph, it is dominated by a previously added vertex. More formally, G is C-win if and only if there is an

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enumeration of the vertices $v_0, \dots, v_k$ such that for each $i > 0$, v_i is dominated by some vertex in the induced subgraph on $\{v_0, \dots, v_i\}$. An order of the vertices with this property is called a *dominating order* (or sometimes a *construction order*) and a finite graph G is *constructible* if it admits such an order.

The notion of a dominating order extends naturally to infinite graphs using well orders. A *dominating order* of G is a well order of the vertices, often written as v_α, such that for each $\alpha > 0$, v_α is dominated in the induced subgraph on $\{v_\beta \mid \beta \leq \alpha\}$. Unfortunately, the equivalence between being C-win and being constructible breaks down for infinite graphs. The infinite chain graph

$$v_0 \text{ --- } v_1 \text{ --- } v_2 \text{ --- } v_3 \cdots\cdots$$

is constructible (and $v_0 \prec v_1 \prec v_2 \prec \cdots$ is a dominating order), but clearly Player R can win by "running down the chain" ahead of Player C. (We omit drawing the reflexive edges in this and future graphs.)

Player R can win on many infinite graphs by simply "running away" from Player C (e.g. on any locally finite infinite graph, see Stahl [10]). To compensate for this deficiency, Lehner [5] (building on work of Chastand, Laviolette and Polat [3]) introduced the notion of a weak C-win graph. Player C gets a *weak win* either if she captures Player R in finitely many rounds or if the game continues for infinitely many rounds and Player R only visits each vertex finitely often. G is *weak C-win* if Player C has a strategy to weak win on G. By Borel determinacy, one of the two players has a winning strategy for any fixed graph under the weak win criterion.

For finite G, being C-win and being weak C-win are equivalent, and so a finite graph is weak C-win if and only if it is constructible. Lehner [5] proved that if an infinite graph is constructible, then it is weak C-win. He asked whether the converse holds.

We answer Lehner's question in Section 2 by giving an effective method to transform trees $T \subseteq \omega^{<\omega}$ into graphs G_T such that G_T admits a dominating order if and only if T is well founded. We prove that G_T is always weak C-win, and therefore, for any non-well founded tree T, G_T provides a negative answer to Lehner's question.

This transformation also gives information about the possible ranks of constructible graphs. The *rank* (or *construction time*) of a constructible graph G is the least order type of a dominating order for G. Using the transformation T_G, we show in Section 2 that the ranks of constructible graphs are unbounded in the countable ordinals. By way of contrast, Lehner [5] showed that the rank of a constructible locally finite graph is at most ω.

Ivan, Leader and Walters [4] recently posted a paper extending these two results of ours. They give a clever construction of a graph G that is C-win (not just weak C-win) but not constructible. They build G using a finite graph that is closely related to one of our coding graphs in Section 3. By carefully gluing together copies of this graph, they create an infinite graph in which Player C can "get ahead of" Player R on an infinite path, but which does not allow Player R to use the same shortcut to escape as he is chased towards a root node and eventually captured in finite time. They also show how to combine copies of this finite graph in an inductive manner to realize each countable ordinal as the rank of a graph.

Our transformation from trees to graphs gives a lower bound for the complexity of the index set of constructible computable graphs (which is Σ^1_2 by its definition). Because the transformation is effective and being well founded is Π^1_1-complete, it follows that this index set is Π^1_1-hard. In Section 2, we use the fact that the rank of a locally finite graph is bounded by ω to show the index set for constructible locally finite computable graphs is Σ^1_1. While we do not know the precise complexity of these two index sets in the arithmetic or analytic hierarchy, these results show their complexities are not the same.

Our original motivation for this project was to understand the complexity of dominating orders on constructible computable graphs. We turn to this question in Section 3, where we describe a framework for building such graphs that is suitable for coding a single jump. Using this framework, we build a locally finite constructible computable graph G such that every dominating order on G computes $0'$ and we investigate some subtleties in giving an effective version of Lehner's result that the rank of a locally finite graph is bounded by ω.

In Section 4, we give a second framework, this time designed to code two jumps. We use this method to build a locally finite constructible computable graph G for which every dominating order computes $0''$, and to prove that the index set of locally finite constructible computable graphs is Π^0_4-hard. Although these results seems to exhaust our particular construction method, we do not see any reason to suppose they are optimal.

We end this section with a summary of notation, conventions and formal definitions. A *graph* consists of a vertex set V and a symmetric reflexive edge relation $E \subseteq V \times V$. We assume our graphs are connected, countable and $V \subseteq \mathbb{N}$. We say $x, y \in V$ are *neighbors* if $E(x, y)$ and we define $N_G[x] = \{v \in G \mid E(x, v)\}$. The vertex y *dominates* x if $x \neq y$ and $N_G[x] \subseteq N_G[y]$. Because our graphs are reflexive, if y dominates x, then x and y are connected by an edge. A graph is *locally finite* if $N_G[x]$ is finite for all x.

For an order $\prec$ of V and $x \in V$, let $V_{\preceq x} = \{v \in G \mid v \preceq x\}$, and let $G_{\preceq x}$ be the induced subgraph on $V_{\preceq x}$. A *dominating order of* G is a well order $\prec$ of V such that for all $x \in V$, if x is not the $\prec$-least element of V, then x is dominated in $G_{\preceq x}$. G is called *constructible* if it admits a dominating order. (Note that this notion of constructibility from the graph theory literature is unrelated to the notion of constructibility in set theory.) For a constructible graph G, the *rank of* G is the least ordinal α such that G has a dominating order of type α.

A *tree* $T \subseteq \omega^{<\omega}$ is a set of finite strings which is closed under initial segments. For finite strings $\sigma, \tau \in \omega^{<\omega}$, we write $\sigma \sqsubseteq \tau$ to denote that σ is an initial segment of τ, we let $|\sigma|$ denote the length of σ and we use λ for the empty string. We say τ is an *immediate successor* of σ if $\sigma \sqsubseteq \tau$ and $|\tau| = |\sigma| + 1$. A node $\sigma \in T$ is a *leaf* if it has no immediate extensions. A tree T is *well founded* if it does not contain an infinite path.

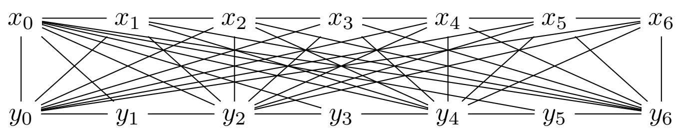
For an excellent introduction to the game of cops of robbers, see Bonato and Nowakowski [2]. Our notation and terminology in computability theory follows Ash and Knight [1], Sacks [8] and Soare [9].

2. Trees and graphs

The main result of this section is a transformation from tree to graphs. Whenever we describe the edge relation for a graph, we implicitly assume we take the reflexive and symmetric closure of the described relation.

Theorem 2.1. *There is a computable functional that uniformly transforms trees $T \subseteq \omega^{<\omega}$ into graphs G_T such that G_T is constructible if and only if T is well founded.*

Proof. Let H denote the graph with vertices $V_H = \{x_i \mid i \leq 6\} \cup \{y_i \mid i \leq 6\}$ and with the edge relation defined by $E_H(x_0, y_i)$ for $i \leq 6$, $E_H(x_i, y_j)$ for $1 \leq i \leq 6$ and j even, and both $E(x_i, x_{i+1})$ and $E(y_i, y_{i+1})$ for $i \leq 5$. See the picture below.



This graph H will be the key building block in several constructions.

Fix a tree $T \subseteq \omega^\omega$. The following construction uses the graph H if you visualize the tree growing downward (i.e. the immediate successors of σ sit below σ). The vertices of G_T are $V_T = \{v_i^\sigma \mid i \leq 6 \text{ and } \sigma \in T\}$. For each $\sigma \in T$, we define $E_T(v_i^\sigma, v_{i+1}^\sigma)$ for $\sigma \in T$ and $i \leq 5$. When τ is an immediate successor of σ, we connect the nodes v_i^σ and v_j^τ as the nodes x_i and y_j are connected in H. That is, $E_T(v_0^\sigma, v_i^\tau)$ holds for $i \leq 6$, and $E_T(v_i^\sigma, v_j^\tau)$ holds for $1 \leq i \leq 6$ and j even. This completes the description of G_T.

It is clear that G_T is uniformly computable from T. We prove that G_T is constructible if and only if T is well founded in the following series of lemmas. To simplify the notation, we fix T and drop the subscript T from G_T.

Lemma 2.2. *For any dominating order $\prec$ on G, any immediate successor pair $\sigma \sqsubseteq \tau$ in T, and any $k \leq 3$, there can be at most three nodes of the form v_i^σ such that $v_i^\sigma \prec v_{2k}^\tau$.*

Proof. Suppose four nodes $v_{j_0}^\sigma$, $v_{j_1}^\sigma$, $v_{j_2}^\sigma$ and $v_{j_3}^\sigma$ satisfy $v_{j_\ell}^\sigma \prec v_{2k}^\tau$. Consider which node dominates v_{2k}^τ in $G_{\preceq v_{2k}^\tau}$. We have $\{v_{j_0}^\sigma, v_{j_1}^\sigma, v_{j_2}^\sigma, v_{j_3}^\sigma, v_{2k}^\tau\} \subseteq N_{\preceq v_{2k}^\tau}[v_{2k}^\tau]$. By the construction of G, the only nodes which connect to four v_i^σ nodes are of the form:

(1) v_{2u}^τ, or
(2) v_w^ρ for the unique node ρ such that $\rho \sqsubseteq \sigma$ is an immediate successor pair in T, or
(3) v_{2u}^μ for any node $\mu \neq \tau$ such that $\sigma \sqsubseteq \mu$ is an immediate successor pair in T.

For (1), v_{2u}^τ cannot dominate v_{2k}^τ in $G_{\preceq v_{2k}^\tau}$ because v_{2u}^τ is connected to v_{2k}^τ if and only if $u = k$. For (2), v_w^ρ is not connected to v_{2k}^τ because $\rho \sqsubset \sigma \sqsubset \tau$ and hence τ is not an immediate successor of ρ in T. For (3), v_{2u}^μ is not connected to v_{2k}^τ because $|\mu| = |\tau|$ and $\mu \neq \tau$. Therefore, none of these nodes can dominate v_{2k}^τ in $G_{\preceq v_{2k}^\tau}$, contradicting the fact that $\prec$ is a dominating order. □

For a dominating order $\prec$ on G and a node $\sigma \in T$, let $m_\prec^\sigma$ denote the $\prec$-greatest vertex in $\{v_{2\ell}^\sigma \mid \ell \leq 3\}$. That is, $m_\prec^\sigma$ is the greatest even index vertex in G with superscript σ.

Lemma 2.3. *For any dominating order $\prec$ on G and any immediate successor pair $\sigma \sqsubseteq \tau$, $v_{2k}^\tau \prec m_\prec^\sigma$ for all $k \leq 3$. In particular, $m_\prec^\tau \prec m_\prec^\sigma$.*

Proof. Fix a dominating order $\prec$ on G and a successor pair $\sigma \sqsubseteq \tau$. Suppose there is a node v_{2k}^τ such that $m_\prec^\sigma \prec v_{2k}^\tau$. Since $v_{2\ell}^\sigma \preceq m_\prec^\sigma \prec v_{2k}^\tau$ for all $\ell \leq 3$, there are four nodes of the form v_i^σ such that $v_i^\sigma \prec v_{2k}^\tau$, contradicting Lemma 2.2. □

Lemma 2.4. *If T has an infinite path, then G does not have a dominating order.*

Proof. Let f be an infinite path in T. Assume for a contradiction that G has a dominating order $\prec$. Let $\sigma_n = f \upharpoonright n$. For each n, $\sigma_n \sqsubseteq \sigma_{n+1}$ is an immediate successor pair. By Lemma 2.3, $m_{\prec}^{\sigma_{n+1}} \prec m_{\prec}^{\sigma_n}$ for all n, so $\prec$ contains an infinite descending chain, contradicting the fact that $\prec$ is a well order. □

It remains to show that if T is well-founded, then G has a dominating order. We construct the dominating order using two different decompositions of T. The first decomposition of T is by levels, where the level of a node $\sigma \in T$ is the finite ordinal $|\sigma|$. For $n \in \omega$, let $L_n = \{\sigma \in T \mid |\sigma| = n\}$. For $\sigma, \tau \in L_n$, $\sigma <_{L_n} \tau$ (σ is *left of* τ) if $\sigma \neq \tau$ and $\sigma(i) < \tau(i)$ for the least $i < n$ such that $\sigma(i) \neq \tau(i)$. For each n, $(L_n, <_{L_n})$ is a well order. We combine these orders on L_n to get a well order $<_L^*$ on T defined by

$$\sigma <_L^* \tau \text{ if and only if } |\sigma| < |\tau| \text{ or } \big(|\sigma| = |\tau| \text{ and } \sigma <_{L_{|\sigma|}} \tau\big).$$

The second decomposition of T uses the standard notion of ordinal rank on a well-founded tree. For a leaf $\sigma \in T$, $\text{rank}_T(\sigma) = 0$. For a non-leaf node $\sigma \in T$,

$$\text{rank}_T(\sigma) = \sup\{\text{rank}_T(\tau) + 1 \mid \tau \text{ is an immediate successor of } \sigma\}.$$

Because T is well-founded, every node in T is assigned an ordinal rank by transfinite recursion and the largest rank is assigned to the root node λ. Let $R_\alpha = \{v \in T \mid \text{rank}_T(v) = \alpha\}$ be the set of nodes in T of rank α. Each set R_α countable, so we can fix well orders $<_\alpha$ such that $(R_\alpha, <_\alpha)$ has order type $\leq \omega$ for each $\alpha \leq \text{rank}_T(\lambda)$. We combine these orders on R_α to get a well order $<_r^*$ on T defined by

$$v_i^\sigma <_r^* v_j^\tau \text{ if and only if } \text{rank}_T(\sigma) < \text{rank}_T(\tau) \text{ or } \big(\text{rank}_T(\sigma) = \text{rank}_T(\tau) = \alpha \text{ and } \sigma <_\alpha \tau\big).$$

We make use of the both the well orders $<_L^*$ and $<_r^*$ in the next lemma.

Lemma 2.5. *If T does not contain an infinite path, then G has a dominating order.*

Proof. Fix T with no infinite paths. We define a dominating order $\prec$ on G. The nodes of the form v_0^σ will form an initial segment of the dominating order, so $v_0^\sigma \prec v_i^\tau$ for all $\sigma, \tau \in T$ and $i \geq 1$. For nodes of the form v_0^x, set $v_0^\sigma \prec v_0^\tau$ if and only if $\sigma <_L^* \tau$. Before defining the order on the remaining elements, we verify that $\prec$ has the dominating property on this initial segment. The least element under $\prec$ is v_0^λ since λ is the only node with length 0. For $\tau \neq \lambda$, let σ be such that $\sigma \sqsubseteq \tau$ is an immediate successor pair. Since $|\sigma| = |\tau| - 1$, we have $\sigma <_L^* \tau$ and hence $v_0^\sigma \prec v_0^\tau$.

We claim that v_0^σ dominates v_0^τ in $G_{\preceq v_0^\tau}$. To prove this claim, it suffices to show that the only vertices connected to v_0^τ in $G_{\preceq v_0^\tau}$ are v_0^τ and v_0^σ. Let v_0^μ be a vertex such that $v_0^\mu \prec v_0^\tau$ and $\mu \neq \sigma, \tau$. Since $v_0^\mu \prec v_0^\tau$, we know that $|\mu| \leq |\tau|$. We split into two cases.

First, suppose $|\mu| = |\tau|$. In this case, neither μ nor τ is an immediate successor of the other. Since $\mu \neq \tau$, it follows that there is no edge between v_0^μ and v_0^τ in G.

Second, suppose $|\mu| < |\tau|$. Obviously, in this case, μ is not an immediate successor of τ. Because $\mu \neq \sigma$, τ is also not an immediate successor of μ. Therefore, again, there is no edge in G between v_0^μ and v_0^τ, completing the proof of the claim.

We order the remaining elements v_i^σ with $\sigma \in T$ and $1 \leq i \leq 6$ as follows. Set $v_0^\tau \prec v_i^\sigma$ for all $\tau, \sigma \in T$ and $i \geq 1$. For $\tau, \sigma \in T$ and $i, j \geq 1$, set

$$v_i^\sigma \prec v_j^\tau \text{ if and only if } \big(\sigma = \tau \text{ and } i < j\big) \text{ or } \big(\sigma \neq \tau \text{ and } \sigma <_r^* \tau\big).$$

If the order type of $(T, <_r^*)$ is β, then the order type of $(\{v_i^\sigma \mid \sigma \in T \text{ and } 1 \leq i \leq 6\}, \prec)$ is $6 \cdot \beta$ because each node $\sigma \in T$ corresponds to an element of β, and the node σ splits into vertices $v_1^\sigma, \ldots, v_6^\sigma$ which are ordered as a discrete interval $v_1^\sigma \prec \ldots \prec v_6^\sigma$ of length 6. In particular, $\prec$ well orders $\{v_i^\sigma \mid \sigma \in T \text{ and } 1 \leq i \leq 6\}$. Since $\prec$ also well orders the initial segment $\{v_0^\sigma \mid \sigma \in T\}$, it follows that $\prec$ well orders G.

It remains to prove that each vertex v_i^σ with $i \geq 1$ is dominated in $G_{\preceq v_i^\sigma}$. Fix v_i^σ with $i \geq 1$. If $\sigma \neq \lambda$, let $\mu = \sigma \upharpoonright (|\sigma| - 1)$, so σ is the immediate successor of μ. Note that $\text{rank}_T(\sigma) < \text{rank}_T(\mu)$, so $v_i^\sigma \prec v_j^\mu$ for $j \geq 1$, although $v_0^\mu \prec v_i^\sigma$. If τ is an immediate successor of σ, then $\text{rank}_T(\tau) < \text{rank}_T(\sigma)$ and so $v_j^\tau \prec v_i^\sigma$ for all $j \leq 6$. With these observations in mind, the neighbors of v_i^σ in $G_{\preceq v_i^\sigma}$ are

(1) v_{i-1}^σ,
(2) v_{2k}^τ for each $\tau \in T$ which is an immediate successor of σ and each $k \leq 3$, and
(3) v_0^μ (if $\sigma \neq \lambda$).

We claim that v_{i-1}^σ dominates v_i^σ in $G_{\preceq v_i^\sigma}$. (1) is handled because v_{i-1}^σ is connected to itself. (2) is handled because τ is an immediate predecessor of σ, so v_{i-1}^σ either is connected to all v_j^τ vertices (if $i - 1 = 0$) or is connected to all v_{2k}^τ vertices (if $i - 1 > 0$). In either case, it is connected to all the vertices in (2). Finally, if $\sigma \neq \lambda$, then since σ is an immediate successor of μ in T, v_0^μ is connected to every vertex of the form v_j^σ. In particular, v_0^μ is connected to v_{i-1}^σ, so (3) is handled, completing the proof that v_{i-1}^σ dominates v_i^σ in $G_{\preceq v_i^\sigma}$. □

This completes the proof of Theorem 2.1. □

Theorem 2.1 gives us information about the index set of computable constructible graphs. From its definition, this index set is Σ_2^1 since G is constructible if and only if there is a binary relation $\prec$ on G such that $\prec$ is a well order that satisfies the domination condition. The domination condition is arithmetical, but to say $\prec$ is a well order is Π_1^1, and hence the definition is Σ_2^1. Since the index set of well founded computable trees in $\omega^{<\omega}$ is Π_1^1-complete and the functional in Theorem 2.1 is computable, we get the following corollary.

Corollary 2.6. *The index set of computable constructible graphs is* Π_1^1*-hard.*

Using the following proposition, we can contrast this situation with the index set of computable locally finite graphs that are constructible.

Proposition 2.7 (Lehner [5])**.** *A countable locally finite graph is constructible if and only if admits a dominating order of type* $\leq \omega$.

Proposition 2.8. *The index set of computable locally finite constructible graphs is* Σ_1^1.

Proof. By Proposition 2.7, a locally finite graph is constructible if and only if it has a dominating order of type $\leq \omega$. A linear order has order type $\leq \omega$ if and only if every element has finitely many predecessors. Therefore, a locally finite graph G is constructible if and only if there is a binary relation $\prec$ on G such that $\prec$ is a linear order in which every vertex has finitely many predecessors and which satisfies the domination condition. Saying $\prec$ is a linear order in which each vertex has finite many predecessors is arithmetical, so the entire statement is Σ_1^1. □

We return to the question of proving a lower bound on the complexity of this index set in Section 4. The next theorem shows there is no analog of Proposition 2.7 for countable constructible graphs.

Theorem 2.9. *Let $T \subseteq \omega^{<\omega}$ be a well-founded tree. The rank of G_T (as a constructible graph) is greater than or equal to rank$_T(\lambda)$.*

Proof. Fix T and let $\alpha = \text{rank}_T(\lambda)$. Let $\prec$ be an arbitrary dominating order on G. It suffices to show the order type of $(G_T, \prec)$ is at least α. Recall that for $\sigma \in T$, $m^\sigma_\prec$ is the $\prec$-greatest element of $\{v^\sigma_{2k} \mid k \leq 3\}$. We drop the subscripts on G_T and $m^\sigma_\prec$. By Lemma 2.3, if $\sigma \sqsubset \tau$, then $m^\tau \prec m^\sigma$. For $\sigma \in T$, let β_σ be the order type of $(G_{\preceq m^\sigma}, \prec)$.

Claim. For every $\sigma \in T$, $\text{rank}_T(\sigma) \leq \beta_\sigma$.

The theorem follows from the claim because $\alpha = \text{rank}_T(\lambda) \leq \beta_\lambda \leq$ order-type$(G, \prec)$. Therefore, to complete the theorem, it suffices to prove the claim.

We prove the claim by induction on $\text{rank}_T(\sigma)$. When $\text{rank}_T(\sigma) = 0$, the claim follows because $0 \leq \beta_\sigma$ trivially. For the induction case, assume $\text{rank}_T(\sigma) = \gamma$. Let τ be an immediate successor of σ. By Lemma 2.3, $m^\tau \prec m^\sigma$ and hence $G_{\preceq m^\tau} \subsetneq G_{\preceq m^\sigma}$ and $\beta_\tau < \beta_\sigma$. Furthermore, since τ is a proper extension of σ, $\text{rank}_T(\tau) < \text{rank}_T(\sigma)$, and so $\text{rank}_T(\tau) \leq \beta_\tau$ by the induction hypothesis. Therefore,

$$\begin{aligned} \text{rank}_T(\sigma) &= \sup\{\,\text{rank}_T(\tau) + 1 \mid \tau \text{ is immediate successor of } \sigma\} \\ &\leq \sup\{\,\beta_\tau + 1 \mid \tau \text{ is immediate successor of } \sigma\} \\ &\leq \beta_\sigma \end{aligned}$$

completing the proof of the claim. □

Corollary 2.10. *The set of ranks of countable constructible graphs is cofinal in ω_1.*

The last theorem in this section shows that the graphs G_T for non-well founded trees T provide examples of weak C-win graphs that are not constructible.

Theorem 2.11. *For every tree $T \subseteq \omega^{<\omega}$, G_T is weak C-win.*

Proof. We describe Player C's strategy and then verify the strategy is weak winning. When describing the strategy, we let v^σ_i denote Player C's position and v^τ_j denote Player R's position. We maintain a list of inductive hypotheses depending on which player's turn it is to move.

- If it is Player R's turn, then $\sigma \sqsubseteq \tau$. If $\sigma = \tau$, then $i < j$, and if $|\sigma| < |\tau|$, then $i = 0$.
- If it is Player C's turn, then σ and τ are comparable. If $\sigma = \tau$, then $i < j$, and if $|\tau| < |\sigma|$, then $|\tau| = |\sigma| - 1$.

Player C starts at vertex v^λ_0. For subsequent rounds, assume Player C is at v^σ_i and Player R is at v^τ_j. Player C moves as follows.

(1) If $\tau \sqsubset \sigma$, then by the inductive hypothesis, $|\tau| = |\sigma| - 1$. If $j = 0$ or i is even, then there is an edge from v^σ_i to v^τ_j. In this case, she moves to v^τ_j and wins. Otherwise, $j > 0$ and i is odd. In this case, she moves to v^τ_0.
(2) If $\sigma = \tau$, then by the inductive hypothesis, $i < j$. She moves to v^σ_{i+1} (and wins if $j = i + 1$).

(3) If $|\sigma| + 1 = |\tau|$, we break into two cases. If $i = 0$ or j is even, there is an edge between v_i^σ and v_j^τ. In this case, she moves to v_j^τ and wins. Otherwise, $i > 0$ and j is odd. In this case, she moves to v_{j-1}^τ.

(4) If $|\sigma| + 1 < |\tau|$, then she moves to $v_0^{\sigma^\frown\tau(|\sigma|)}$.

This completes the description of Player C's strategy. It is straightforward to check that the induction hypotheses for Player R's move hold in each case. Next, we verify that the inductive hypotheses for Player C's move hold after Player R moves. Assume it is Player R's turn to move and we break into two cases.

First, suppose $\sigma = \tau$. By the inductive hypothesis, $i < j$, so in particular, $j \neq 0$. Player R's options are: (i) move to v_{j-1}^τ, v_j^τ or v_{j+1}^τ (assuming $j < 6$); (ii) move to a vertex of the form $v_k^{\tau\upharpoonright(|\tau|-1)}$ (assuming $\tau \neq \lambda$); or (iii) move to a vertex of the form v_{2k}^μ where μ is an immediate successor of τ on T. In each case, the induction hypothesis for Player C's move holds.

Second, suppose $|\sigma| < |\tau|$. By the inductive hypothesis, $\sigma \sqsubseteq \tau$ and $i = 0$. Player R's options are: (iv) move to v_{j-1}^τ (assuming $j > 0$), v_j^τ or v_{j+1}^τ (assuming $j < 6$); (v) move to a vertex of the form $v_k^{\tau\upharpoonright(|\tau|-1)}$; or (vi) move to a vertex of the form v_k^μ where μ is an immediate successor of τ in T (where the index k must be even if $j > 0$). Again, the induction hypothesis for Player C's move holds in each case.

It remains to prove Player C's strategy is weak winning. Assume for a contradiction that Player R has a strategy that allows him to avoid losing in a finite round and to occupy a fixed vertex v_n^ν infinitely often. Let $v_{i_m}^{\sigma_m}$ denote Player C's position after her move in the m-th round, and let $v_{j_m}^{\tau_m}$ denote Player R's position after his move in the m-th round. The inductive relationship between σ_s and τ_s is given by the hypotheses for Player C's move, and the inductive relationship between σ_{s+1} and τ_s is given by the hypotheses for Player R's move.

Claim. There is a value u such that either $\sigma_u = \tau_u$ or $\sigma_{u+1} = \tau_u$.

Assume for a contradiction there is no such value u. The proof proceeds in several small steps. First, we claim $|\sigma_{s+1}| < |\tau_s|$ for all s. By the induction hypothesis, $\sigma_{s+1} \sqsubseteq \tau_s$ for all s. By the assumption that $\sigma_{s+1} \neq \tau_s$, we must have $|\sigma_{s+1}| < |\tau_s|$ for all s.

Second, we claim $|\sigma_s| \leq |\tau_s|$ for all s. Since $\sigma_0 = \lambda$, this holds trivially for $s = 0$. For $s > 0$, $|\sigma_s| < |\tau_{s-1}|$ by the first claim. Since the difference in the values of $|\tau_s|$ and $|\tau_{s-1}|$ is at most 1, we have $|\sigma_s| \leq |\tau_s|$.

Third, we claim $|\sigma_s| < |\tau_s|$ for all s. By the induction hypothesis, σ_s and τ_s are comparable. By the second claim, $|\sigma_s| \leq |\tau_s|$, so $\sigma_s \sqsubseteq \tau_s$. Since $\sigma_s \neq \tau_s$ by assumption, it follows that $|\sigma_s| < |\tau_s|$.

Fourth, we claim $|\sigma_s| + 1 < |\tau_s|$ for all s. Suppose this inequality fails for a fixed s. Since $|\sigma_s| < |\tau_s|$, it follows that $|\sigma_s| + 1 = |\tau_s|$. Player C acts in (3) to set $\sigma_{s+1} = \tau_s$ (since we have assumed she cannot win in a finite round), contradicting our assumption the claim is false.

Finally, having established that $|\sigma_s| + 1 < |\tau_s|$ for all s, we know Player C acts in (4) at every round. In particular, $|\sigma_{s+1}| = |\sigma_s| + 1$. Since $|\tau_s| > |\sigma_s|$, it follows that $\lim_s |\tau_s| = \infty$, contradicting our assumption that Player R occupies the vertex v_n^ν infinitely often.

Claim. There is a value u for which $\sigma_{u+1} = \tau_u$.

Fix u from the previous claim. If $\sigma_{u+1} = \tau_u$, then we are done. Otherwise, $\sigma_u = \tau_u$ and it is Player C's turn. She acts in (2) to keep $\sigma_{u+1} = \tau_u$ since we assume she cannot win in a finite round.

Claim. For each $t \geq u$, $\sigma_{t+1} = \tau_t$.

We prove this claim by induction on t. When $t = u$, it follows from the previous claim. For the induction case, assume that $\sigma_{t+1} = \tau_t$. By the inductive hypotheses on Player R's turn to move, we know $i_{t+1} < j_t$ and hence $j_t \neq 0$. Player R's possible moves are described in (i)-(iii) above. In (i), Player R maintains $\tau_{t+1} = \tau_t$, so Player C acts in (2) to set $\sigma_{t+2} = \sigma_{t+1} = \tau_t = \tau_{t+1}$. In (ii), Player R sets $\tau_{t+1} = \tau_t \upharpoonright (|\tau_t| - 1)$, so Player C acts in (1) to set $\sigma_{t+2} = \tau_{t+1}$ (since we assume she does not win in a finite round).

In (iii), Player R would set $\tau_{t+1} = \mu$ for some immediate successor μ of τ_t in T. We prove this case cannot occur. Since $j_t \neq 0$, the value of j_{t+1} must be even. Therefore, there is an edge between $v_{i_{t+1}}^{\sigma_{t+1}}$ and $v_{j_{t+1}}^{\tau_{t+1}}$. Player C acts in (3) to win by moving to $v_{j_{t+1}}^{\tau_{t+1}}$, contradicting our assumption that she does not win in a finite round.

Claim. At each round $t \geq u$, Player R sets $\tau_{t+1} = \tau_t$ or $\tau_{t+1} = \tau_t \upharpoonright (|\tau_t| - 1)$. Moreover, he can only set $\tau_{t+1} = \tau_t$ for finitely many rounds before setting $\tau_{t+1} = \tau_t \upharpoonright (|\tau_t| - 1)$.

The first sentence follows directly from the proof of the previous claim. To prove the second sentence, notice that if Player R sets $\tau_{t+1} = \tau_t$, then $\sigma_{t+1} = \tau_{t+1}$ by the previous claim. Player C acts in (2) to set $\sigma_{t+2} = \tau_{t+1}$ and $i_{t+2} = i_{t+1} + 1$. That is, she chases Player R down the finite chain $v_0^{\tau_{t+1}}, \ldots, v_6^{\tau_{t+1}}$. Since we assume he does not lose in a finite round, Player R must eventually set $\tau_{t+1} = \tau_t \upharpoonright (|\tau_t| - 1)$. This completes the proof of the claim.

By the last claim, once we reach the stage u, Player R can only maintain a given value of τ_t for finitely many rounds before setting $|\tau_{t+1}| < |\tau_t|$. Therefore, there is a round s at which $\tau_s = \lambda$. Player C sets $\sigma_{s+1} = \lambda$ and wins in finite many more moves by chasing Player R down the finite chain $v_0^\lambda, \ldots, v_6^\lambda$, contradicting our assumption that she doesn't win in a finite round and completing the verification that Player C's strategy is weak winning. □

Corollary 2.12. *For any tree $T \subseteq \omega^{<\omega}$ that is not well-founded, the graph G_T is weak C-win but not constructible.*

Corollary 2.13. *There is a locally finite graph G that is weak C-win but not constructible.*

Proof. Let T consist of a single infinite path, such as $T = \{0^n \mid n \in \omega\}$. G_T is weak C-win by Theorem 2.11, is locally finite because T is finitely branching, and is not constructible by Theorem 2.1. □

3. Tree-form graphs and coding one jump

In this section, we develop a general framework for constructing graphs which we can use to code information into dominating orders.

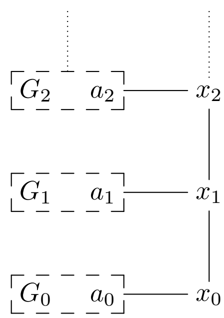
Lemma 3.1. *Let $\prec$ be a dominating order on G. For every $v \in G$, the induced subgraph $G_{\preceq v}$ is connected.*

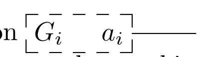
Proof. For a contradiction, suppose v is $\prec$-least such that $G_{\preceq v}$ is not connected. Since any singleton subgraph is connected, v is not the $\prec$-least element of G. Fix $u \prec v$ such that u dominates v in $G_{\preceq v}$, and so $E(u,v)$ holds. $G_{\preceq v} = (\cup_{w \prec v} G_{\preceq w}) \cup \{v\}$ and $u \in \cup_{w \prec v} G_{\preceq w}$, so it suffices to prove $\cup_{w \prec v} G_{\preceq w}$ is connected. By the minimality of v, $\cup_{w \prec v} G_{\preceq w}$ is the union of a chain of connected subgraphs and hence is connected. $\square$

Definition 3.2. Let $G_i = (V_i, E_i)$, $i \in \omega$, be a sequence of disjoint connected graphs, each with a designated node a_i. The *tree-form graph* $G = (V, E)$ *with spine* x_i, *connectors* a_i *and graph branches* G_i is the graph defined by

$$V = \bigcup_{i \in \omega} (V_i \cup \{x_i\}) \quad \text{and} \quad E = \bigcup_{i \in \omega} (E_i \cup \{\langle x_i, x_{i+1} \rangle, \langle x_{i+1}, x_i \rangle, \langle x_i, a_i \rangle, \langle a_i, x_i \rangle\}).$$

A tree-form graph looks like



where the notation  indicates that the graph G_i is attached to G by connecting x_i to a_i, but making no other connections between x_i and nodes in G_i.

Lemma 3.3. *Let G be a tree-form graph with graph branches G_i and let $\prec$ be a dominating order on G. Let $\prec_i$ be the restriction of $\prec$ to G_i. For all i, $\prec_i$ is a dominating order on G_i, and for all i except possibly one, $\prec_i$ has least element a_i.*

Proof. Fix a dominating order $\prec$ of G. Let v_0 be the $\prec$-least element of G. Let i_0 be such that either $v_0 = x_{i_0}$ or $v_0 \in G_{i_0}$. We claim x_{i_0} is the $\prec$-least element of the form x_j for $j \in \omega$. The claim is clear if $v_0 = x_{i_0}$, so suppose $v_0 \in G_{i_0}$. Let w be the $\prec$-least vertex such that $w \notin G_{i_0}$. By Lemma 3.1, $G_{\preceq w}$ is connected, so the node w must be connected to some node in G_{i_0}. The only node with this property is x_{i_0}, so we must have $w = x_{i_0}$.

Fix an index $i \in \omega$ such that $i \neq i_0$ and let w_i be the $\prec$-least element of G_i. We claim that $w_i = a_i$ and $x_i \prec a_i$. Since $i \neq i_0$ and $G_{\preceq w_i}$ is connected, there must be an edge between w_i and some node $v \in G \setminus G_i$ with $v \prec w_i$. The only edge connecting an element of G_i with an element of $G \setminus G_i$ is $E(a_i, x_i)$. Therefore, $w_i = a_i$, $v = x_i$ and $x_i \prec a_i$.

Continuing with our fixed index $i \neq i_0$, let $\prec_i$ be the restriction of $\prec$ to G_i. We show $\prec_i$ is a dominating order on G_i with least element a_i. The relation $\prec_i$ is a well-order of G_i, and by the second claim, a_i is the least element of G_i. Let $v \in G_i$ with $v \neq a_i$. It suffices to show there is a node $u \in G_i$ such that $u \prec_i v$ and u dominates v in $G_{i,\preceq v}$. Since $\prec$ is a dominating order of G, we fix $u \in G$ such that

$u \prec v$ and u dominates v in $G_{\preceq v}$. Therefore, $u \in N_{G_{\preceq v}}[v] \subseteq N_{G_{\preceq v}}[u]$. Because $v \in G_i$ and $v \neq a_i$, $N_G[v] \subseteq G_i$. Therefore, $u \in G_i$ and

$$N_{G_{i,\preceq v}}[v] = N_{G_{\preceq v}}[v] \cap G_i \subseteq N_{G_{\preceq v}}[u] \cap G_i \subseteq N_{G_{i,\preceq v}}[u]$$

showing that u dominates v in $G_{i,\preceq v}$.

It remains to show $\prec_{i_0}$ is a dominating order of G_{i_0}. When $v_0 = x_{i_0}$ or $v_0 = a_{i_0}$, this fact follows from the argument in the preceding paragraph. Assume $v_0 \in G_{i_0}$ and $v_0 \neq a_{i_0}$. We need to show each $v \in G_{i_0}$ with $v \neq v_0$ is dominated in $G_{i_0,\preceq v}$ by some $u \prec_{i_0} v$. For $v \neq a_{i_0}$, this follows as in the preceding paragraph. However, in this case, we need an argument for a_{i_0}. Suppose $b \prec a_{i_0}$ dominates a_{i_0} in $G_{\preceq a_{i_0}}$, so $N_{G_{\preceq a_{i_0}}}[a_{i_0}] \subseteq N_{G_{\preceq a_{i_0}}}[b]$. By the first claim above, each $v \prec x_{i_0}$ is in G_{i_0}, and $a_{i_0} \prec x_{i_0}$. Therefore, $b \in G_{i_0}$, $N_{G_{i_0,\preceq a_{i_0}}}[a_{i_0}] = N_{G_{\preceq a_{i_0}}}[a_{i_0}]$ and $N_{G_{i_0,\preceq a_{i_0}}}[b] = N_{G_{\preceq a_{i_0}}}[b]$. It follows that $N_{G_{i_0,\preceq a_{i_0}}}[a_{i_0}] \subseteq N_{G_{i_0,\preceq a_{i_0}}}[b]$, and hence b dominates a_{i_0} in $G_{i_0,\preceq a_{i_0}}$. □

Lemma 3.4. *Let G be a tree-form graph with graph branches G_i such that each G_i has a dominating order with least element a_i. Let $\prec_i$ be a sequence of dominating orders for the graphs G_i with least element a_i. There is a dominating order $\prec$ of G such that for every i, the restriction of $\prec$ to G_i is $\prec_i$. In particular, if each G_i admits a dominating order with least element a_i, then G admits a dominating order.*

Proof. Fix the sequence of dominating orders $\prec_i$ with least elements a_i. Define an order $\prec$ on G by setting $u \prec v$ if and only if (i) $u \in G_i \cup \{x_i\}$, $v \in G_j \cup \{x_j\}$ and $i < j$, (ii) $u = x_i$ and $v \in G_i$, or (iii) $u, v \in G_i$ and $u \prec_i v$. The order $\prec$ can be visualized as

$$x_0 \prec (G_0, \prec_0) \prec x_1 \prec (G_1, \prec_1) \prec \cdots \prec x_i \prec (G_i, \prec_i) \prec x_{i+1} \prec \cdots$$

where $x_i \prec (G_i, \prec_i) \prec x_{i+1}$ denotes that x_i comes before all the elements of G_i, that all the elements of G_i come before x_{i+1}, and that the elements of G_i are ordered among themselves by $\prec_i$. It is straightforward to verify that $\prec$ is a well order of G using the fact that each $\prec_i$ is a well order of G_i. Furthermore, it is clear that the restriction of $\prec$ to G_i is $\prec_i$.

To show that $\prec$ is a dominating order, it suffices to show that for each $v \neq x_0$, there is a $u \prec v$ such that u dominates v in $G_{\preceq v}$. We break into three cases.

First, suppose that $v = x_i$ for some $i > 0$. By the definition of a tree-form graph, $N_G[x_i] = \{x_{i-1}, x_i, a_i, x_{i+1}\}$. Since $x_{i-1} \prec x_i \prec a_i \prec x_{i+1}$, it follows that $N_{G_{\preceq x_i}}[x_i] = \{x_{i-1}, x_i\}$, and therefore, $x_{i-1} \prec x_i$ dominates x_i in $G_{\preceq x_i}$.

Second, suppose that $v = a_i$ for some i. Since a_i is the $\prec_i$-least element of G_i and $N_G[a_i] \subseteq G_i \cup \{x_i\}$, we have that $N_{G_{\preceq a_i}}[a_i] = \{x_i, a_i\}$. Therefore, $x_i \prec a_i$ dominates a_i in $G_{\preceq a_i}$.

Third, suppose that $v \in G_i$ and $v \neq a_i$. In this case, $N_G[v] = N_{G_i}[v]$. Let $u \in G_i$ be such that $u \prec_i v$ and u dominates v in $G_{i,\preceq_i v}$. By the definition of $\prec$, we have $u \prec v$. Furthermore, $N_{G_{\preceq v}}[v] = N_{G_{i,\preceq v}}[v] \subseteq N_{G_{i,\preceq v}}[u] \subseteq N_{G_{\preceq v}}[u]$, so u dominates v in $G_{\preceq v}$ as required. □

Lemma 3.5. *Let G be a tree-form graph with finite graph branches G_i such that each G_i has a dominating order with least element a_i. G has a dominating of order of order type ω.*

Proof. Since each G_i is finite, the dominating order defined in the proof of Lemma 3.4 has order type ω. Alternately, since G is finitely branching, it has a dominating order of type ω by Proposition 2.7. □

Lemma 3.6. *For any uniform computable sequence of connected graphs G_i with distinguished elements a_i, the tree-form graph G with graph branches G_i is computable.*

Proof. Fix a uniform computable construction of the sequence G_i and let $G_{i,s}$ be the finite portion of G_i built at the end of stage s. Without loss of generality, we can assume that the first element to appear in $G_{i,s}$ is a_i. We build G in stages with the approximation at stage s consisting of the spine nodes $x_0, \dots, x_s$ with attached graph branches $G_{0,s}, \dots, G_{s,s}$. □

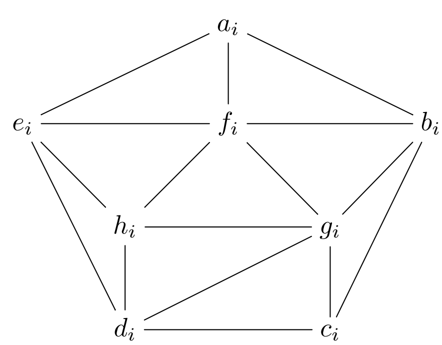
We end this section with two results using this framework for constructing graphs. The first application, given in the following theorem, will be improved in Theorem 4.9.

Theorem 3.7. *There is a computable graph G such that G is constructible and every dominating order computes $0'$.*

Proof. We build G as a tree-form graph in which each branch graph G_i will have one of two isomorphism types. Let X_i have domain $\{a_i, b_i, c_i, d_i\}$ and edges E_{X_i} given by

$$a_i \text{ --- } b_i \text{ --- } c_i \text{ --- } d_i$$

and let Y_i have domain $\{a_i, b_i, c_i, d_i, e_i, f_i, g_i, h_i\}$ and edges E_{Y_i} given by



Because $X_i \subseteq Y_i$ and $E_{X_i} = E_{Y_i} \cap X_i^2$, there is a uniformly computable sequence of graphs G_i such that if $i \notin 0'$, then $G_i \cong X_i$, and if $i \in 0'$, then $G_i \cong Y_i$. Therefore, there is a computable tree-form graph G with branches G_i.

By Lemma 3.1, the only dominating order of X_i starting with a_i is $a_i \prec b_i \prec c_i \prec d_i$. Therefore, in every dominating order of X_i starting with a_i, we have $c_i \prec d_i$.

There are several dominating orders of Y_i that start with a_i, for example $a_i \prec f_i \prec g_i \prec h_i \prec b_i \prec e_i \prec d_i \prec c_i$. However, we claim that $d_i \prec c_i$ in each such dominating order. Since Y_i is finite, the last element in any dominating order must be dominated in the full graph Y_i. Therefore, because only a_i and c_i are dominated in Y_i, every dominating order on Y_i that starts with a_i must end with c_i. In particular, $d_i \prec c_i$.

Each G_i has a dominating order starting with a_i, so by Lemma 3.4, G admits a dominating order. Fix any dominating order $\prec$ on G and let $\prec_i$ be the restriction to G_i. By Lemma 3.3, $\prec_i$ is a dominating order of G_i starting with a_i, with the possible exception of one index i_0. Let $K_\prec = \{i \in \mathbb{N} \mid d_i \prec c_i\}$. For any $i \neq i_0$, $i \in K_\prec$ if and only if $i \in 0'$, and therefore, $0'$ is computable from an arbitrary dominating order of G. □

For the second application of this method, we distinguish two ways that a computable dominating order on a locally finite graph could have order type ω.

Definition 3.8. Let G be a computable locally finite graph and let $\prec$ be a computable dominating order of G. We say $\prec$ is a *computable dominating order of type* ω if the classical order type of $(G, \prec)$ is ω. We say $\prec$ is a *computable dominating order of strong type* ω if there is a computable order preserving bijection $f : (\omega, \leq) \to (G, \prec)$.

Theorem 3.9. *There is a computable locally finite graph G such that G has a computable dominating order of type ω but not a computable dominating order of strong type ω.*

Proof. We build our computable graph G with domain ω. For each index e, if φ_e is a permutation of ω (and hence could be a bijection $\omega \to G$), then we let $\prec_e$ be the binary relation defined on G by $v \prec_e w$ if and only if $\varphi_e^{-1}(v) < \varphi_e^{-1}(w)$. To ensure that G does not have a computable order of strong type ω, it suffices to meet the following requirements.

$$R_e : \text{ If } \varphi_e : \omega \to G \text{ is a bijection, then } \prec_e \text{ is not a dominating order on } G$$

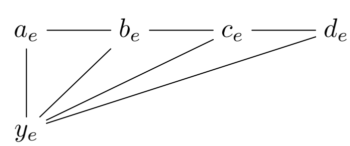
We build G as a tree-form graph with graph branches G_e and we use G_e to satisfy the requirement R_e.

As in the proof of Theorem 3.7, G_e will have one of two isomorphism types. We start with G_e equal to the graph X_e with vertices $\{a_e, b_e, c_e, d_e\}$ and edge relation E_{X_e} given by

$$a_e \;\text{———}\; b_e \;\text{———}\; c_e \;\text{———}\; d_e$$

Our construction of G_e proceeds in stages.

(1) Wait for a stage s_0 such that $\{a_e, b_e, c_e, d_e\} \subseteq \text{range}(\varphi_{e,s_0})$. If there is no such stage, we stay in (1) forever and $G_e = X_e$.
(2) Set $n_e = \max\{x \mid \varphi_{e,s_0}(x) \in \{a_e, b_e, c_e, d_e\}\}$. Wait for a stage $s_1 > s_0$ such that $\varphi_{e,s_1}(x)$ converges for all $x \leq n_e$. If there is no such stage (or if we see φ_e is not injective), we stay in (2) forever and $G_e = X_e$.
(3) At stage $s_1 + 1$, we add one vertex y_e to G_e to form the graph Z_e with vertices $\{a_e, b_e, c_e, d_e, y_e\}$ and edge relation E_{Z_e} given by



G_e is not changed again, so the final value of G_e is Z_e.

Since $X_e \subseteq Z_e$, $E_{X_e} = E_{Z_e} \cap X_e^2$ and the switch from $G_e = X_e$ to $G_e = Z_e$ is determined by a Σ^0_1 event, the sequence of graphs G_e is uniformly computable. Therefore, there is a computable tree-form graph G with graph branches G_e. Furthermore, we can assume that G is constructed with domain ω and such that $\varphi_e(x) \neq y_e$ for all $x \leq n_e$ in (3).

Since each G_e is finite, G is locally finite. We have already seen that X_e has a unique dominate order starting with a_e given by $a_e \prec b_e \prec c_e \prec d_e$. Z_e has several dominating orders starting with a_e, for example, $a_e \prec b_e \prec y_e \prec c_e \prec d_e$. However, no dominating order of Z_e can end with y_e because y_e is not dominated in the graph Z_e.

By Lemma 3.5, G has a dominating order of type ω. In fact, we can define a computable dominating order of type ω for G in stages as follows. At stage 0, we start with the empty order. At stage $s+1$, we first attach $x_s \prec a_s \prec b_s \prec c_s \prec d_s$ to the end of the order determined at stage s. Second, we check if for any $e \leq s$, the graph G_e changed from X_e to Z_e at stage s. If so, then for each such e, we add y_e to the order so that $b_e \prec y_e \prec c_e$. Although the addition of the element y_e is delayed until we see G_e change from X_e to Z_e, the order $\prec$ (in the end) has the form

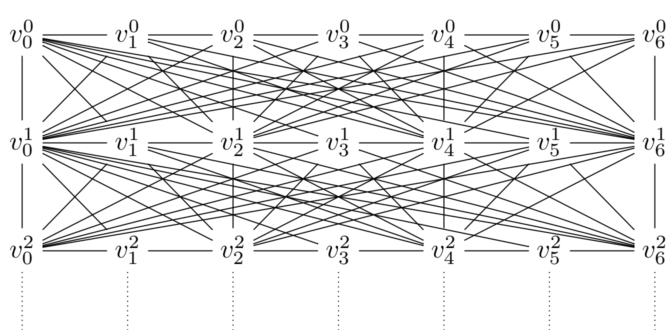
$$x_0 \prec G_0 \prec x_1 \prec G_1 \prec \cdots$$

as in the proof of Lemma 3.4. Therefore, $\prec$ is a computable dominating order of type ω.

To finish the proof of the theorem, we need to show that G does not have a computable dominating order of strong type ω. For a contradiction, suppose $\prec$ is a computable dominating order on G of strong type ω. Fix an index e such that $\varphi_e : \omega \to G$ is a bijection with $i < j \Leftrightarrow \varphi_e(i) \prec \varphi_e(j)$. Consider the construction of G_e. Since φ_e is a bijection, we will find stages $s_0 < s_1$ and define the parameter n_e in Steps (1) and (2). Therefore, G_e is isomorphic to Z_e. Fix m such that $\varphi_e(m) = y_e$. By construction, we know $\varphi_e(x) \neq y_e$ for all $x \leq n_e$, so $n_e < m$. It follows that for all $v \in \{a_e, b_e, c_e, d_e\}$, $v \prec y_e$. Therefore, $N_{\preceq y_e}[y_e] = \{a_e, b_e, c_e, d_e, y_e\}$. However, no other node in G contains this set within its neighbors, and so no node can dominate y_e in $G_{\preceq y_e}$, giving the desired contradiction. □

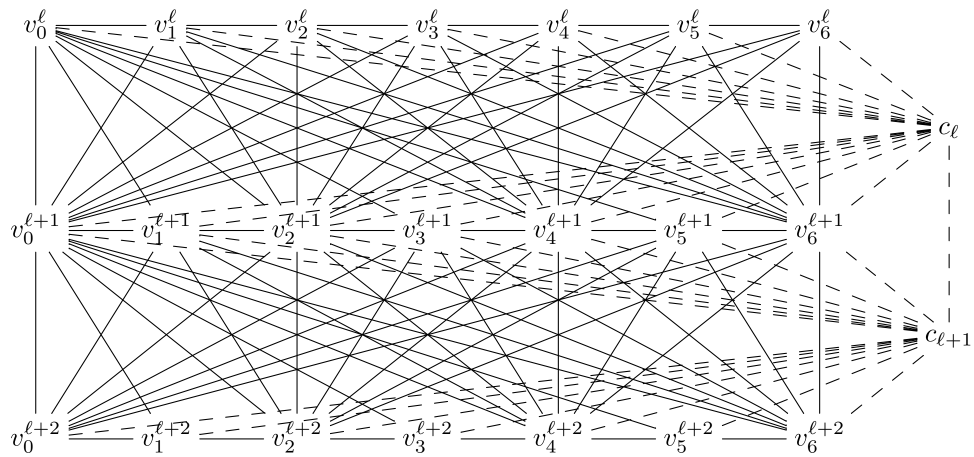
4. Second computability construction

To prove additional computability theoretic results about dominating orders, we use a family of graphs $K(X)$, parameterized by a set $X \subseteq \omega$. We begin with the graph formed by applying Theorem 2.1 to a tree which consists of a single infinite path. The resulting graph consists of ω many rows v_i^ℓ, $0 \leq i \leq 6$, in which v_i^ℓ is connected to v_{i+1}^ℓ. The rows are connected by adding edges from v_0^ℓ to every $v_j^{\ell+1}$, and from v_i^ℓ to $v_j^{\ell+1}$ when $i > 0$ and j even.



We refer to this graph as L. It consists of the rows in a larger graph K. To form K, for each ℓ, we add an auxiliary node c_ℓ connecting the elements of row ℓ and

$\ell+1$, and we connect c_ℓ and $c_{\ell+1}$ as shown below.



K is the graph consisting of v_i^ℓ and c_ℓ for all $\ell \in \omega$ and $0 \leq i \leq 6$. The nodes v_i^ℓ, $0 \leq i \leq 6$ form the ℓ-th row of the graph, and the nodes c_ℓ are called auxiliary nodes. The induced subgraph L is not constructible by Theorem 2.1. However, because of the auxiliary nodes, K is constructible.

Lemma 4.1. *K has a dominating order with least element v_0^0.*

Proof. The dominating order starts with the initial segment

$$v_0^0 \prec c_0 \prec v_1^0 \prec v_2^0 \prec v_3^0 \prec v_4^0 \prec v_5^0 \prec v_6^0.$$

Each of these elements is dominated by the preceding element in the appropriate initial subgraph. We continue to construct the dominating order row by row, with row ℓ and the auxiliary element c_ℓ ordered as

$$v_0^\ell \prec c_\ell \prec v_1^\ell \prec v_2^\ell \prec v_3^\ell \prec v_4^\ell \prec v_5^\ell \prec v_6^\ell.$$

Each of these elements is dominated by $c_{\ell-1}$ in the appropriate initial subgraph because $c_{\ell-1}$ is connected to all of these elements and to all the elements in row $\ell-1$, and because none of the elements in row $\ell+1$ have entered the dominating order yet. $\square$

Definition 4.2. For $X \subseteq \omega$, $K(X)$ is the induced subgraph of K containing the nodes v_i^ℓ for $\ell \in \omega$ and $0 \leq i \leq 6$ and the auxiliary nodes c_k for $k \notin X$.

The set X specifies the nodes c_k to remove from K to form $K(X)$. For example, $K(\emptyset) = K$, and $K(\omega) = L$. We will generalize the fact that K is constructible, while L is not, by showing that $K(X)$ has a dominating order if and only if X is finite.

If X is co-c.e., then $\overline{X}$ is the c.e. set of auxiliary nodes we need to add to L to form $K(X)$, so we can build a computable copy of $K(X)$ uniformly in a c.e. index for $\overline{X}$. In fact, $K(X)$ has a computable copy if and only if X is co-c.e.

Definition 4.3. Let $\prec$ be a dominating order on $K(X)$. For each row r, let g_r be such that $v_{g_r}^r$ is the $\prec$-greatest element of row r. Row r is $\prec$-*special* (or *special* if $\prec$ is clear from context) if there is an index $j \neq 0$ such that $v_0^r \prec v_{g_{r+1}}^{r+1}$ and $v_j^r \prec v_{g_{r+1}}^{r+1}$. That is, row r is special if both v_0^r and another element of row r enter the dominating order before the last element of row $r+1$.

Lemma 4.4. *For every dominating order $\prec$ on $K(X)$, there is a $\prec$-special row.*

Proof. Let r be the $\prec$-least element of $\{v^\ell_{g_\ell} : \ell \in \omega\}$. All elements of row r enter the dominating order before $v^{r+1}_{g_{r+1}}$, so row r is special. □

The following lemma will be the main tool we use to classify when $K(X)$ is constructible and to recover information that is coded in a dominating order for $K(X)$.

Lemma 4.5. *Let $X \subseteq \omega$ be nonempty. For every dominating order $\prec$ on $K(X)$ and for every $k \in X$, if row r is $\prec$-special, then $r > k$.*

Proof. Fix $\prec$ and k. We prove by downward induction that each row $\ell \leq k$ is not special. Suppose for a contradiction that row k is special. Fix $j \neq 0$ such that $v^k_0, v^k_j \prec v^{k+1}_{g_{k+1}}$.

We claim that either $v^{k+2}_0 \prec v^{k+1}_{g_{k+1}}$ or $c_{k+1} \prec v^{k+1}_{g_{k+1}}$. Note that c_{k+1} may not be in $K(X)$, in which case, this claim should be read as stating that $v^{k+2}_0 \prec v^{k+1}_{g_{k+1}}$. References below to other auxiliary vertices which may not be in $K(X)$ should be read in the same manner.

To prove this claim, suppose neither inequality holds and suppose $v^{k+1}_{g_{k+1}} \prec v^{k+2}_0 \prec c_{k+1}$. Consider which node dominates v^{k+2}_0 in $K(X)_{\preceq v^{k+2}_0}$. Because $v^{k+1}_{g_{k+1}} \prec v^{k+2}_0$, v^{k+2}_0 is connected to every node in row $k+1$ in $K(X)_{\preceq v^{k+2}_0}$. The only other nodes in $K(X)$ (or that could be in $K(X)$) that are connected to every node in row $k+1$ are v^k_0 and c_{k+1}. However, v^k_0 is not connected to v^{k+2}_0 in $K(X)$, and c_{k+1} is not in $K(X)_{\preceq v^{k+2}_0}$ by our assumption that $v^{k+2}_0 \prec c_{k+1}$. Therefore, neither of these nodes dominates v^{k+2}_0 in $K(X)_{\preceq v^{k+2}_0}$. The argument when $v^{k+1}_{g_{k+1}} \prec c_{k+1} \prec v^{k+2}_0$ is similar.

Having established the claim, we derive a contradiction by considering which node dominates $v^{k+1}_{g_{k+1}}$ in $K(X)_{\preceq v^{k+1}_{g_{k+1}}}$. All references to domination in the next two paragraphs are relative to the subgraph $K(X)_{\preceq v^{k+1}_{g_{k+1}}}$. By the claim, at least one of c_{k+1} and v^{k+2}_0 is in this subgraph. Therefore, no vertex in row k dominates $v^{k+1}_{g_{k+1}}$. Since v^k_0 is in this subgraph, neither c_{k+1} nor any vertex in row $k+2$ dominates $v^{k+1}_{g_{k+1}}$. Finally, since c_k is not in the subgraph (because $k \in X$, so c_k is not even in $K(X)$), the only vertices left that could dominate $v^{k+1}_{g_{k+1}}$ are in row $k+1$.

We eliminate the vertices in row $k+1$ in two cases. First, if $1 \leq g_{k+1} \leq 5$, then $v^{k+1}_{g_{k+1}}$ is connected to both $v^{k+1}_{g_{k+1}-1}$ and $v^{k+1}_{g_{k+1}+1}$, so neither of these nodes dominates it, ruling out this case. Second, if $g_{k+1} = 0$ or $g_{k+1} = 6$, then $v^{k+1}_{g_{k+1}}$ is connected to v^k_j, while neither v^{k+1}_1 nor v^{k+1}_5 are connected to v^k_j. Therefore, v^{k+1}_1 does not dominate $v^{k+1}_{g_{k+1}}$ when $g_{k+1} = 0$, and v^{k+1}_5 does not dominate $v^{k+1}_{g_{k+1}}$ when $g_{k+1} = 6$. This completes the proof that row k is not special.

We proceed by downward induction. Assume $0 < \ell \leq k$ and row ℓ is not special. We show row $\ell-1$ is not special. Assume for a contradiction that row $\ell-1$ is special and fix $j \neq 0$ such that $v^{\ell-1}_0, v^{\ell-1}_j \prec v^\ell_{g_\ell}$. Since row ℓ is not special, every vertex in row $\ell+1$ is in $K(X)_{\preceq v^\ell_{g_\ell}}$. We derive a contradiction by considering which vertex dominates $v^\ell_{g_\ell}$ in $K(X)_{\preceq v^\ell_{g_\ell}}$. The references to domination in the next paragraph are relative to this subgraph.

Since $v^\ell_{g_\ell}$ is connected to $v^{\ell+1}_0$, it is not dominated by $c_{\ell-1}$ or by a vertex in row $\ell-1$. Also, because $v^\ell_{g_\ell}$ is connected to $v^{\ell-1}_0$, it is not dominated by c_ℓ or by a vertex in row $\ell+1$. Therefore, the only vertices left that could dominate $v^\ell_{g_\ell}$ are in row ℓ. These vertices do not dominate $v^\ell_{g_\ell}$ by the same argument given in the proof that row k is not special. □

Lemma 4.6. *$K(X)$ is constructible if and only if X is finite. Furthermore, if X is finite, $K(X)$ admits a dominating order with least element v^0_0.*

Proof. For a contradiction, assume X is infinite and $\prec$ is a dominating order on $K(X)$. By Lemma 4.5, there is no $\prec$-special row since if row r is $\prec$-special, then $r > k$ for all $k \in X$. Since every dominating order on $K(X)$ has a special row, we have a contradiction.

For the other direction, assume X is finite. The case when $X = \emptyset$ follows from Lemma 4.1, so assume X is nonempty. Let k be the largest element X and let $y_0 < y_1 < \cdots < y_i$ be the numbers $y < k$ that are in $\overline{X}$. The auxiliary nodes in $K(X)$ are $\{c_{y_0}, \ldots, c_{y_i}\} \cup \{c_\ell : \ell > k\}$. We construct a dominating order starting with the initial segment

$$v^0_0 \prec v^1_0 \prec \cdots \prec v^{k+1}_0 \prec c_{y_0} \prec c_{y_1} \prec \cdots \prec c_{y_i} \prec c_{k+1}.$$

This initial segment satisfies the dominating conditions because $v^{\ell+1}_0$ is dominated by v^ℓ_0 in $K(X)_{\preceq v^{\ell+1}_0}$, c_{y_j} is dominated by $v^{y_j}_0$ in $K(X)_{\preceq c_{y_j}}$, and c_{k+1} is dominated by v^{k+1}_0 in $K(X)_{\preceq c_{k+1}}$.

We next add the remaining elements from rows 0 through $k+1$, starting with row $k+1$ and working down to row 0.

$$v^{k+1}_1 \prec v^{k+1}_2 \prec v^{k+1}_3 \prec v^{k+1}_4 \prec v^{k+1}_5 \prec v^{k+1}_6 \prec v^k_1 \prec v^k_2 \prec \cdots$$

The domination property is satisfied because each of these vertices v^ℓ_j is dominated by v^ℓ_{j-1} in $K(X)_{\preceq v^\ell_j}$. We order the rest of $K(X)$ row by row starting with the remainder of row $k+1$.

$$v^{k+1}_1 \prec v^{k+1}_2 \prec v^{k+1}_3 \prec v^{k+1}_4 \prec v^{k+1}_5 \prec v^{k+1}_6 \prec c_{k+2} \prec v^{k+2}_0 \prec v^{k+2}_1 \prec \cdots$$

The remaining elements of row $k+1$ and c_{k+2} are dominated by c_{k+1} in the appropriate subgraph. Following this pattern, the elements of each row ℓ for $\ell > k+1$, as well as the vertex $c_{\ell+1}$ are dominated by c_ℓ in the appropriate subgraph. □

Lemma 4.7. *Let A_k, $k \in \omega$, be a uniformly c.e. sequence of sets. There is a computable presentation of the tree-form graph G with graph branches $G_k = K(\overline{A_k})$ and distinguished elements $a_k = v^0_0$. Furthermore, G is constructible if and only if every set A_k is cofinite.*

Proof. We can uniformly construct a computable copy of $K(\overline{A_k})$ from an enumeration of A_k by building a computable copy of the graph L and adding auxiliary nodes c_n as n is enumerated into A_k. Therefore, by Lemma 3.6, we can build a computable copy of G.

By Lemma 4.6, if A_k is infinite, then $K(\overline{A_k})$ is not constructible, and hence by Lemma 3.3, G is not constructible. If each A_i is finite, then each graph branch $K(\overline{A_k})$ has a dominating order with least element v^0_0, and hence G has a dominating order. □

Our first application of these graphs is to show the index set of computable locally finite constructible graphs is Π^0_4-hard. Recall from Proposition 2.8 that this index set is Σ^1_1.

Theorem 4.8. *The index set of computable locally finite constructible graphs is Π^0_4-hard.*

Proof. Let R be an arbitrary Π^0_4 relation on ω. It suffices to build a uniform computable sequence of locally finite graphs G_k such that for all k, $R(k)$ holds if and only if G_k admits a dominating order.

The index set $\text{Cof} = \{e : W_e \text{ is cofinite}\}$ is Σ^0_3-complete, so we can fix a uniform c.e. sequence of sets A^k_e for $e, k \in \omega$ such that for all k

$$R(k) \text{ holds } \Leftrightarrow \forall e\,(A^k_e \text{ is cofinite}).$$

For each k, let G_k be the tree-form graph with graph branches $K(\overline{A^k_e})$ for $e \in \omega$. By Lemma 4.7, we can uniformly construct the sequence of computable graphs G_k. G_k is constructible if and only if for all e, $\overline{A^k_e}$ is finite. Therefore, $R(k)$ holds if and only if G_k is constructuble. □

Our second application of these graphs improves the result in Theorem 3.7.

Theorem 4.9. *There is a computable locally finite constructible graph G such that every dominating order on G computes $0''$.*

Proof. It suffices to build G such that for every dominating order computes the index set $\text{Inf} = \{e : W_e \text{ is infinite}\}$. G will be a tree-form graph with graph branches $K(\overline{A_k})$, where A_k is a uniformly c.e. family of sets.

We enumerate the family A_k in stages with $A_{k,s}$ denoting the set at the end of stage s. While $s < k$, we set $A_{k,s} = \emptyset$. For each k, we keep a parameter $m_{k,s}$ such that for $s < k$, $m_{k,s}$ is undefined, $m_{k,k} = k$, and for $s \geq k$, $m_{k,s} \leq s$ and $m_{k,s} \leq m_{k,s+1}$. The limit $\lim_s m_{k,s}$ may be infinite, but we let $m_k = \lim_s m_{k,s}$ when the limit is finite. For each index k and stage $s \geq k$, we define $A_s = \{0, \ldots, s\} - \{m_{k,s}\}$. It follows that if $\lim_s m_{k,s} = \infty$, then $A_k = \omega$, and if $\lim_s m_{k,s} = m_k$, then $A_k = \omega - \{m_k\}$.

It remains to describe the definition of the markers $m_{k,s}$. Although $A_{k,s}$ is completely determined by $m_{k,s}$, we will specify $A_{k,s}$ for clarity. At stage 0, set $m_{0,0} = 0$ and $A_{0,0} = \emptyset$. At stage $s > 0$, set $m_{s,s} = s$ and $A_{s,s} = \{0, \ldots, s-1\}$. For $k < s$, break into two cases.

For $k = 2e$, if $e \in 0'_s - 0'_{s-1}$, then set $m_{2e,s} = s$ and enumerate $m_{2e,s-1}$, into A_{2e} so $A_{2e,s} = \{0, \ldots, s-1\}$. Otherwise, set $m_{2e,s} = m_{2e,s-1}$ and enumerate s into A_{2e}, so $A_{2e,s} = \{0, \ldots, s\} - \{m_{2e,s}\}$.

For $k = 2e+1$, if there is an $x > m_{2e+1,s-1}$ such that $x \in W_{e,s}$, then set $m_{2e+1,s} = s$ and enumerate $m_{2e+1,s-1}$ into A_{2e+1}, so $A_{2e+1,s} = \{0, \ldots, s-1\}$. Otherwise, set $m_{2e+1,s} = m_{2e+1,s-1}$ and enumerate s into A_{2e+1}, so $A_{2e+1,s} = \{0, \ldots, s\} - \{m_{2e+1,s}\}$.

This completes the construction of the uniform c.e. sequence A_k. By Lemma 4.7, let G be a computable copy of the tree-form graph with graph branches $K(\overline{A_k})$.

Lemma 4.10. *For $k = 2e$, $\lim_s m_{k,s} = m_k$ exists, and $e \in 0'$ if and only if $e \in 0'_{m_k}$.*

Proof. After the initial definition $m_{k,k} = k$, the value of $m_{k,s}$ can change at most once. This change occurs if e enters $0'_{s+1}$ at a stage $s+1 \geq k$. Therefore, the limit m_k exists. To prove the second property, assume e enters $0'$ at stage $s+1$. If $s+1 \leq k$, then $s+1 \leq m_{k,k} = m_k$. If $k < s+1$, then at stage $s+1$, we set $m_{k,s+1} = s+1$ and hence $s+1 = m_{k,s+1} = m_k$. In either case, we have $s+1 \leq m_k$ as required. □

Lemma 4.11. *For $k = 2e+1$, $\lim_s m_{k,s} = m_k$ exists if and only if W_e is finite. Furthermore, if W_e is finite, then for all $x \in W_e$, $x < m_k$.*

Proof. Suppose W_e is infinite. We definite a sequence of stages $t_0 < t_1 < \cdots$ such that $m_{k,t_i} = t_i$, and hence $\lim_s m_{k,s} = \infty$. Let $t_0 = k$ and note that $m_{k,k} = k$ as required. Assume $m_{k,t_i} = t_i$. Since W_e is infinite, there is a least stage $t_{i+1} > t_i$ at which an element $x > t_i$ enters W_e. At stage t_{i+1}, we set $m_{k,t_{i+1}} = t_{i+1}$.

Suppose W_e is finite. Since we only change the value of $m_{k,s}$ when a new element enters $W_{e,s}$, the parameter $m_{k,s}$ must reach a finite limit m_k. Let t be the stage at which we set $m_{k,t} = t = m_k$. No element $x \geq t$ can enter W_e after stage t or else we would increase the value of $m_{k,s}$. Since $x \in W_{e,t}$ implies $x < t$, we have $x < m_k$ for all $x \in W_e$. □

Lemma 4.12. *G is constructible.*

Proof. If $\lim_s m_{k,s} = m_k$, then $A_k = \omega - \{m_k\}$, and if $\lim_s m_{k,s} = \infty$, then $A_k = \omega$. In either case, $\overline{A_k}$ is finite, and hence by Lemma 4.6, each graph branch $K(\overline{A_k})$ admits a dominating order with least element v_0^0. Therefore, by Lemma 3.4, G is constructible. □

Lemma 4.13. *For any dominating order $\prec$ on G, $0'' \leq_T \prec$.*

Proof. Let $f_\prec(k) =$ the least ℓ such that row ℓ is $\prec$-special in G_k. Note that $f_\prec$ is computable from $\prec$ since each row is finite. To show $0'' \leq_T f_\prec$, we prove two claims.

First, we claim $0' \leq_T f_\prec$. It suffices to show that $e \in 0'$ if and only if $e \in 0'_{f_\prec(2e)}$. By Lemma 4.10, $e \in 0'$ if and only if $e \in 0'_{m_{2e}}$. However, since $G_{2e} = K(\{c_{m_{2e}}\})$, it follows from Lemma 4.5 that $m_{2e} \leq f_\prec(2e)$. This inequality proves the claim.

Second, we claim W_e is infinite if and only if there is an $x \in W_e$ such that $x \geq f_\prec(2e+1)$. The forward direction is obviously true. For the backward direction, assume W_e is finite. By Lemma 4.11, W_e does not contain an element $x \geq m_{2e+1}$. Furthermore, since $G_{2e+1} = K(\{c_{m_{2e+1}}\})$, it follows from Lemma 4.5 that $m_{2e+1} \leq f_\prec(2e+1)$. Therefore, W_e cannot contain an element $x \geq f_\prec(2e+1)$.

To see that $0''$ is computable from the dominating order $\prec$, it suffices to show that $f_\prec$ can compute the index set Inf. Note that $f_\prec$ can determine if $e \in \text{Inf}$ by using the oracle $0'$ to decide whether there is an $x \in W_e$ with $x \geq f_\prec(2e+1)$. Since $0' \leq_T f_\prec$, this process is computable in $f_\prec$. □

This completes the proof of Theorem 4.9. □

5. Open questions

By Theorems 2.1 and 2.9, the ranks of computable constructible graphs are cofinal in ω_1^{CK}. Is there a computable constructible graph without a hyperarthmetic dominating order?

There are large gaps in the index set results. It remains open to close the gap between Π^1_1 and Σ^1_2 for the index set of computable constructible graphs, and to close the gap between Π^0_4 and Σ^1_1 for the index set of computable locally finite constructible graphs.

Finally, we see no reason to believe Theorem 4.9 is optimal. For which computable ordinals α is it possible to construct a computable graph G such that every dominating order computes $0^{(\alpha)}$?

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Department of Mathematics, University of Connecticut, Storrs, CT 06269
Email address: leigh.evron@uconn.edu

Department of Mathematics, University of Connecticut, Storrs, CT 06269
Email address: david.solomon@uconn.edu

Department of Mathematics, Bridgewater State University, Bridgewater, MA 02325
Email address: rstahl@bridgew.edu