Roots of polynomials in fields of generalized power series

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June 12, 2019

Abstract

We consider roots of polynomials in fields of generalized power series. Newton [6] and Puiseux [7], [8] showed that if K is an algebraically closed field of characteristic 0, then the field $K\{\{t\}\}$ of *Puiseux series* over K is algebraically closed. Maclane [5] showed that if K is an algebraically closed field of characteristic 0 and G is a divisible ordered Abelian group, then the *Hahn field* $K((G))$ is algebraically closed. Our goal is to measure the complexity of the roots and the root-taking process in these fields. For a polynomial over $K\{\{t\}\}$, the roots are computable in K and the coefficients. In fact, knowing that a polynomial is non-constant, and given K and the coefficients, we can apply a uniform effective procedure to find a root. Puiseux series have length at most ω. Hahn series may be longer, and the complexity of the roots goes up with the length. In [3] and [4], there are results bounding the lengths of roots of a polynomial over $K((G))$ in terms of the lengths of the coefficients. Using these results, we show that the generalized Newton-Puiseux Theorem holds in any admissible set. We set bounds on the complexity of initial segments of a root. The bounds are sharp for initial segments of length less than $\omega + \omega$.

1 Introduction

The *Puiseux series* over a field K are formal power series of the form $\sum_{i\in\omega} a_i t^{q_i}$, where $a_i \in K$ and $(q_i)_{i\in\omega}$ is an increasing sequence of rationals with an upper bound on the denominators. Newton [6], in 1676, and Puiseux [7], [8], in 1850-1851, showed that if K is algebraically closed field of characteristic 0, then the set $K\{\{t\}\}$ of Puiseux series over K is an algebraically closed field. For a divisible ordered Abelian group G and a field K, the *Hahn series* have the form $\sum_{i<\alpha} a_i t^{g_i}$, where $a_i \in K$ and $(g_i)_{i\in\alpha}$ is an increasing sequence of elements of G indexed by an ordinal α. If G has cardinality κ, the length of the sequence may be any ordinal $\alpha < \kappa^+$. Maclane proved the analogue of the Newton-Puiseux

This work was done partially while the first and third authors were visiting the Institute for Mathematical Sciences, National University of Singapore in 2017. Their visit was supported by the Institute.

Theorem for Hahn fields, showing that if K is an algebraically closed field of characteristic 0, and G is a divisible ordered Abelian group, then the set $K((G))$ of Hahn series is a field that is again algebraically closed.

We are interested in complexity in these fields. In Section 2, we give some background on fields of Puiseux series. In Section 3, we consider complexity in Puiseux series. We show that if K is algebraically closed of characteristic 0, with universe a subset of ω, and $p(x) = A_0 + A_1x + \ldots + A_nx^n$, where $A_i \in K\{\{t\}\}$, then the roots of $p(x)$ are all computable in K and the A_i's. Moreover, we have a uniform effective procedure that, given K with universe a subset of ω, and the coefficients of a non-constant polynomial $p(x)$, produces a root. In Section 4, we give some background on Hahn fields. In Section 5, we consider complexity. We show that Maclane's Theorem holds in any admissible set M. That is, if G and K are in M, then for any polynomial over $K((G))$ with coefficients in M, the roots are all in M. For this, we use results from [3], [4] bounding the length of roots of polynomials over $K((G))$ in terms of the lengths of the coefficients. In the case where G and K have universe a subset of ω, we consider the complexity of initial segments of the roots of polynomials.

2 Puiseux series

Definition 2.1. *The* Puiseux series *over a field K are the formal sums of the form $s = \sum_{i \geq k} a_i t^{\frac{i}{n}}$, where n is a positive integer, k is an integer, and for each integer $i \geq k$, $a_i \in K$. The* support *of s, denoted by $Supp(s)$, is the set of rationals $\frac{i}{n}$ such that $a_i \neq 0$.*

It is helpful to think of t as infinitesimal. Thus, t^q is infinite if $q < 0$, and t^q is infinitesimal if $q > 0$. We write $K\{\{t\}\}$ for the set of Puiseux series with coefficients in K. We define addition and multiplication on $K\{\{t\}\}$ in a natural way.

- In the sum $s + s'$, the coefficient of t^q is the sum of the coefficients of t^q in s and in s' (if there is no term corresponding to q, we take the coefficient to be 0).
- In the product $s \cdot s'$, the coefficient of t^q is the sum of the products $b \cdot b'$ such that for some $r, r' \in \mathbb{Q}$ with $r + r' = q$, b is the coefficient of t^r in s and b' is the coefficient of $t^{r'}$ in s'.

There is a natural valuation w on $K\{\{t\}\}$ defined as follows.

$$w(s) = \begin{cases} \min(Supp(s)) & \text{if } s \neq 0 \\ \infty & \text{if } s = 0 \end{cases}$$

Note that if $s \neq 0$, then $Supp(s) \neq \emptyset$ and so has a least element. We sometimes refer to $w(s)$ as the weight of s. If K is ordered, there is also a natural order on $K\{\{t\}\}$ defined by s is positive if $s \neq 0$ and the coefficient of $t^{w(s)}$ is positive.

Newton [6] and Puiseux [7], [8] showed the following.

Theorem 2.2 (Newton-Puiseux)**.** *If K is algebraically closed of characteristic* 0*, then $K\{\{t\}\}$ is also algebraically closed.*

For a proof, see Basu, Pollack, and Roy [2], or Walker [10]. Below, we say a little about the root-taking process.

Lemma 2.3. *Let $p(x)$ be a non-constant polynomial over $K\{\{t\}\}$. For suitably chosen g, the coefficients of $t^g p(x)$ all have non-negative valuation, and the least valuation is* 0*. The roots are the same as for $p(x)$.*

Proof. For $p(x) = A_0 + A_1 x + \ldots + A_n x^n$, take the least $w(A_i)$, and let $g = -w(A_i)$. □

Lemma 2.4. *Let $p(x)$ be a non-constant polynomial over $K\{\{t\}\}$. Assuming that $p(x)$ has a non-zero root, then for suitably chosen g, $p(t^{-g}x)$ has a root with positive valuation.*

Proof. Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$, where $w(A_i) \geq 0$. If r is a non-zero root of $p(x)$ and $-g < w(r)$, then $t^g r$ is a root of $p(t^{-g}x)$ with positive valuation. We will see soon how to choose g just by looking at the coefficients A_i. □

Partial proof of Newton-Puiseux Theorem. Consider a polynomial $p(x) = A_0 + A_1 x + \ldots + A_n x^n$, where $w(A_i) \geq 0$ for all i. If $A_0 = 0$, then 0 is a root of $p(x)$. We suppose that $A_0 \neq 0$. We draw the *Newton polygon*. We start by graphing the points $(i, w(A_i))$, for $0 \leq i \leq n$, omitting any points such that $A_i = 0$. The polygon is convex, with certain of the points $(i, w(A_i))$ lying on the edges, and other points lying above. Since $A_0 \neq 0$, the first vertex is $(0, w(A_0))$. We rotate a ray through the vertex $(0, w(A_0))$ counter-clockwise from a starting position pointing downwards until it touches another point $(j, w(A_j))$. It may hit more than one point at the same time, in which case, we let $(j, w(A_j))$ be the point with the largest index that it hits. The segment from $(0, w(A_0))$ to $(j, w(A_j))$ is the first side of the Newton polygon. We repeat this process with A_j and continue until every point is either on or above a side of the Newton polygon.

To find the valuation of a root r of $p(x)$, we consider a side of the Newton polygon. For a side with first point $(i, w(A_i))$ and last point $(j, w(A_j))$, $\nu = \frac{w(A_i) - w(A_j)}{j-i}$ is the valuation of at least one root. (We will not prove this.) The negatives of the slopes of the sides of the polygon are exactly the valuations of the roots. Convexity means that the slopes are increasing. Thus, the greatest valuation of a root is the one we get from the first side, the one with least slope.

Fix a side L of the Newton polygon, and let ν be the negative of the slope. The *carrier* Δ_ν is the set of points $(k, w(A_k))$ that lie on the side L. The *ν-principal part* is the polynomial $\sum_{k \in \Delta_\nu} c_k z^{k-i}$, where c_k is the first non-zero coefficient in A_k. This is a polynomial over K. If b is a root of the ν-principal part, then b is the coefficient of t^ν in a root of $p(x)$. Any root of the ν-principal part serves. Then bt^ν is the first term of a root. Note that if the constant term c_i in the ν-principal part is non-zero, then the root b is also not zero. Considering the ν's corresponding to all sides of the Newton polygon, and all roots b of the

ν-principal parts counted with multiplicity, we have found the first non-zero terms of all non-zero roots of $p(x)$.

Let $r_1 = b_0 t^{\nu_0}$ be a possible first term of a root, where $\nu_0 = \nu$ and $b_0 = b$ are determined above. To find the second term of a root with r_1 as the first term, we consider the new polynomial $q(x) = p(r_1 + x) = B_0 + B_1 x + \ldots + B_n x^n$. If $B_0 = 0$, then r_1 is a root of $p(x)$ because $q(0) = 0$. Otherwise, we find the first term of a root of $q(x)$, using the method above. Say this is $b_1^{\nu_1}$, and let $r_2 = b_0 t^{\nu_0} + b_1 t^{\nu_1}$. Given r_n, the sum of the first n terms of a root r, we consider $p_n(x) = p(r_n + x)$. If the constant term is 0, then r_n is a root of $p(x)$. Otherwise, we find the first term of a root $b_{n+1} t^{\nu_{n+1}}$ of $p_n(x)$. Adding this to r_n, we have r_{n+1}. Continuing, we get a root of form r_n or r_ω. □

3 Complexity in fields of Puiseux series

In this section, our goal is to measure complexity in a field of Puiseux series $K\{\{t\}\}$. We must say exactly how we are representing Puiseux series.

3.1 Representing Puiseux series

We suppose that the field K has universe ω, and we fix a computable copy of $\mathbb{Q}$ with universe ω. We represent elements of $K\{\{t\}\}$ by functions $f : \omega \to K \times \mathbb{Q}$ such that if $f(n) = (a_n, q_n)$, then

- q_n increases with n, and
- there is a uniform bound on the denominators of the q_n terms.

To be explicit, the function $f : \omega \to K \times \mathbb{Q}$ such that $f(n) = (a_n, q_n)$ corresponds to the series $\sum_{n\in\omega} a_n t^{q_n}$. Throughout this section, we assume Puiseux series are representing this way. Note that for $f(n) = (a_n, q_n)$ representing a Puiseux series, q_n is defined for all n. This fact, together with the fact that there is a bound on the denominators of the q_n, implies that $\lim_{n\to\infty} q_n = \infty$.

3.2 Complexity of basic operations

Lemma 3.1. *Given K and $s, s' \in K\{\{t\}\}$, we can effectively compute $s + s'$ and $s \cdot s'$.*

Proof. Say that $s(n) = (a_n, q_n)$ and $s'(n) = (a'_n, q'_n)$. We compute the value $(s + s')(n) = (b_n, r_n)$ as follows. Let r_0 be the smaller of q_0, q'_0. If $q_0 < q'_0$, then $b_0 = a_0$, if $q'_0 < q_0$, then $b_0 = a'_0$, and if $q_0 = q'_0$, then $b_0 = a_0 + a'_0$. Given (b_n, r_n), we find the least m such that $r_n < q_m$ and the least m' such that $r_n < q'_{m'}$. Then r_{n+1} is the smaller of $q_m, q'_{m'}$. If $q_m < q'_{m'}$, then $b_{n+1} = a_m$, if $q'_{m'} < q_m$, then $b_{n+1} = a'_{m'}$, and if $q_m = q'_{m'}$, then $b_{n+1} = a_m + a'_{m'}$.

For simplicity, we suppose that $q_0, q'_0 \geq 0$ and we compute $(s \cdot s')(n) = (b_n, r_n)$ and the finite set $S_n = \{(m, m') : q_m + q'_{m'} = r_n\}$ as follows. Let $r_0 = q_0 + q'_0$, and let $b_0 = a_0 a'_0$. We have $S_0 = \{(0, 0)\}$. Given (b_n, r_n) and S_n, we compute (b_{n+1}, r_{n+1}) and S_{n+1} as follows. Let M be least integer greater than all m for $(m, m') \in S_n$, and let M' be the least integer greater than all m' for $(m, m') \in S_n$. We take r_{n+1} to be the least rational greater than r_n such that for some $m \leq M$ and $m' \leq M'$, $r_{n+1} = q_m + q'_{m'}$. Let S_{n+1} be the set of pairs (m, m') such that $q_m + q'_{m'} = r_{n+1}$. (For all of these pairs, $m \leq M$ and $m' \leq M'$.) The coefficient b_{n+1} is equal to $\sum_{(m,m') \in S_{n+1}} a_m a'_{m'}$. □

Using similar arguments, one can show the following lemma.

Lemma 3.2. *Given K and s, it is Π^0_1, but not computable, to say that $s = 0$. Given that $s \neq 0$, we can effectively find $w(s)$. Moreover, regardless of whether s is non-zero, we can effectively determine the ordering between $w(s)$ and q for any $q \in \mathbb{Q}$.*

Here is an effective version of Lemma 2.3.

Lemma 3.3. *Given a non-constant polynomial $p(x)$ over $K\{\{t\}\}$, we can effectively find a polynomial $q(x) = t^g p(x)$ with the same roots, such that the coefficients of $q(x)$ all have non-negative valuation and the sides of the Newton polynomials for $p(x)$ and $q(x)$ have the same slope.*

Proof. Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$. Since $p(x)$ is not constant, there is some $i > 0$ such that $A_i \neq 0$. We can find $w(A_i)$ for some such i. We can effectively tell whether $w(A_j) < w(A_i)$ for each $j \neq i$ by Lemma 3.2. Thus, we can find the least value of $w(A_i)$, say $-g$, and we can find the least k such that $w(A_k) = -g$. Then $q(x) = t^g p(x)$ has the same roots as $p(x)$, and if $q(x) = B_0 + B_1 x + \ldots + B_n x^n$, then $w(B_i) = w(A_i) + g$. Therefore, its coefficients have non-negative valuations and the Newton polygon for $q(x)$ is the result of shifting the one for $p(x)$ vertically by g. It follows that the slopes of the sides of the polygons for $p(x)$ and $q(x)$ are the same. □

Lemma 3.4. *Given a non-constant polynomial $p(x)$ such that 0 is not a root and all coefficients have non-negative valuation, we can effectively find the first side of the Newton polygon, including identifying all of the points $(i, w(A_i))$ on the first side of the polygon.*

Proof. Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$. Since 0 is not a root, we can find $w(A_0)$. Since $p(x)$ is non-constant, there exists $i > 0$ such that $A_i \neq 0$. Therefore, we can find $w(A_i)$ and the equation of the line through $(0, w(A_0))$ and $(i, w(A_i))$. For each j, we can determine whether $(j, w(A_j))$ lies strictly below or on this line by checking whether $w(A_j) < q$ or $w(A_j) = q$ for the point (j, q) on the line. If we find $(j, w(A_j))$ lies strictly below this line, we try the line through $(0, w(A_0))$ and $(j, w(A_j))$. Again we look for a point below this line. Continuing in this way, we eventually find the first side of the Newton polygon and all of the points on this side. □

The next lemma is an effective version of Lemma 2.4

Lemma 3.5. *Given a non-constant polynomial $p(x)$ such that 0 is not a root and all coefficients have non-negative valuation, we can effectively find g and h such that the first side of the Newton polygon for $t^h p(t^{-g}x)$ has negative slope, and all coefficients have non-negative valuation.*

Proof. Suppose the first side of the Newton polynomial for $p(x)$ joins $(0, w(A_0))$ to $(i, w(A_i))$. If the slope is positive, we consider

$$q(x) = p(t^{-g}x) = B_0 + B_1 x + \ldots + B_n x^n,$$

where $B_i = t^{-gi}A_i$. The Newton polygon for $q(x)$ will also join $(0, w(B_0))$ to $(i, w(B_i))$. The slope of the first side is at most $\frac{w(B_i)-w(B_0)}{i} = \frac{w(A_i)-ig-w(A_0)}{i} \leq \frac{w(A_i)}{i} - g$. If $gi > w(A_i)$, then this is negative. The first side may have slope that is even more negative.

This transformation may yield coefficients B_i some of which have negative valuation. We can make another transformation as in Lemma 3.3 to determine h so that the coefficients of the polynomial have non-negative valuation, and the first side of the Newton polynomial has negative slope. □

Note that if the first side of the Newton polygon has a negative slope, then the term in the root constructed from this side (as described above in the sketch of the proof of the Newton-Puiseux theorem) will have positive weight. Furthermore, the Newton polygon is convex and therefore the term corresponding to the first side has larger weight than the terms determined by any other side. In particular, if we start with a polynomial with only non-zero roots and we use the first side of the Newton polygon to compute the first term of a root, then this root will have the largest valuation of all the roots. This fact provides the key step in the following lemma.

Lemma 3.6. *If $p(x)$ is a non-constant polynomial and $n \in \omega$, then we can effectively tell whether there is a root r with $w(r) > n$. If there is no such root, then we know that $A_0 \neq 0$.*

Proof. Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$. We may suppose that $w(A_i) \geq 0$ for all i. Not knowing whether $A_0 = 0$, we do not know whether 0 is a root. Since $p(x)$ is non-constant, there is some $i > 0$ such that $A_i \neq 0$. We can find such an i and then effectively check whether $w(A_0) > ni + w(A_i)$.

If $w(A_0) \leq ni + w(A_i)$, then we know $A_0 \neq 0$. By Lemma 3.4, we can find the first side of the Newton polygon. Using this side, we can calculate the greatest valuation of a root and hence determine whether there is a root r with $w(r) > n$.

Otherwise, suppose $w(A_0) > ni + w(A_i)$. If $A_0 \neq 0$, then the first side of the Newton polygon has slope at most $\frac{w(A_i)-w(A_0)}{i} \leq -\frac{ni}{i} < -n$. For any root r corresponding to this side, $w(r) > n$. Of course, if $A_0 = 0$, then 0 is a root with valuation greater than n. □

Lemma 3.7. *Let $p(x) = B_0 + B_1 x + \cdots + B_n x^n$ and let $0 < r < n$ be such that $w(B_i) > 0$ for $i < r$, $w(B_r) = 0$, and $w(B_i) \geq 0$ for $i > r$. If $w(B_0) = q$ and the slope of the first side of the Newton polygon is δ, then $\delta \leq -q/r$.*

Proof. Assume $w(B_0) = q$. By the assumptions on the weights of the coefficients, the first side of the Newton polygon connects $(0, B_0)$ to (i, B_i) for some $0 < i \leq r$. If it ends at (r, B_r), then the slope is $-q/r$. If it connects to a point (i, B_i) for $0 < i < r$, then the slope will be more steeply negative. □

3.3 Complexity of the root-taking process

Let K be an algebraically closed field of characteristic 0 with universe a subset of ω and let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$ be a non-constant polynomial with coefficients in $K\{\{t\}\}$. Following the sketched proof of the Newton-Puiseux Theorem, we want to uniformly compute a root r of $p(x)$ of the form

$$r = b_0 t^{\nu_0} + b_1 t^{\nu_1} + b_2 t^{\nu_2} + \cdots$$

by inductively computing b_i and ν_i with $b_i \in K$ and $\nu_i \in \mathbb{Q}$ such that $\nu_0 < \nu_1 < \cdots$ and the denominators of the ν_i bounded.

There are two main problems to doing this computation uniformly in the field K and the coefficients A_i. The first problem is that we cannot effectively tell whether a coefficient A_i is 0. In particular, this means we cannot determine whether 0 is a root of a given polynomial by uniformly checking whether the constant term is 0. Also, we cannot uniformly determine the valuations $w(A_i)$ of the coefficients and hence cannot uniformly compute the Newton polygon.

The second problem in giving a uniform version of the standard proof is that the root r may be given by a finite summation. This problem is related to the previous one in the sense that the sum will be finite when 0 is the root of a certain polynomial in the inductive construction. In the classical setting, there is no problem with viewing a finite sum as an element of $K\{\{t\}\}$ because we can think about appending additional terms with coefficient 0. However, we have to represent our root as a total function and hence we need to explicitly append these terms with coefficient 0. The danger is that we must append these terms while checking whether the constant term of a given polynomial is 0. If we discover that the given polynomial does not have 0 as a root, we have to construct the next term of r in a way that is consistent with the terms we appended during our search process.

In the proof of Theorem 3.8 below, we will follow the outline and the notation of the proof of the Newton-Puiseux Theorem in Walker [10]. Our focus will be on overcoming the two problems noted above. We will note various algebraic properties of the defined parameters, but we leave it to the reader to consult [10] for the proofs of these properties.

Theorem 3.8. *There is a uniform effective procedure that, given K and the sequence of coefficients for a non-constant polynomial over $K\{\{t\}\}$, yields a root.*

Proof. Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$ be a non-constant polynomial over $K\{\{t\}\}$. By Lemma 3.3, we may suppose that for all i, $w(A_i) \geq 0$. Our initial plan is to start approximating a root of $p(x)$ that looks like 0 until (if ever) we see evidence that $A_0 \neq 0$. At that point, we switch to the general case of producing a root of the form

$$c_1 t^{\gamma_1} + c_2 t^{\gamma_1+\gamma_2} + c_3 t^{\gamma_1+\gamma_2+\gamma_3} + \dots$$

where each $c_i \in K$ and each $\gamma_i \in \mathbb{Q}$, with $\gamma_i > 0$, whenever $i > 1$.

Not knowing whether $A_0 = 0$, we use Lemma 3.6 to test whether there is a root r with $w(r) > 0$. If so, we give our root r the first term $0t^0$. We test whether there is a root r with $w(r) > 1$, and, if so, we add the term $0t^1$. We continue, testing whether there is a root r with $w(r) > n$, and, if so, adding the term $0t^n$. If the answer is always positive, then $A_0 = 0$ and we have produced the root $0 = 0t^0 + 0t^1 + \cdots$ of $p(x)$. If for some positive n, we find that there is no root r with $w(r) > n$, then we know that $A_0 \neq 0$. In this case, we switch to trying to find a root of the form above.

Suppose $p(x)$ has no root r with $w(r) > n$ and hence $A_0 \neq 0$. By Lemma 3.5, we can effectively find g and h such that the first side of the Newton polygon for $p_0(x) = t^h p(t^{-g}x)$ has negative slope and all the coefficients of $p_0(x)$ have non-negative valuations. We switch to finding a root y of $p_0(x)$ and then translating y back to a root r of $p(x)$. We need to check that this process is consistent with having already determined that $0t^0 + \cdots + 0t^{n-1}$ is an initial segment of our root r for $p(x)$. (We can assume $n > 0$ since if $n = 0$ then we have not specified any initial terms of our root r for $p(x)$.)

Note that y is a root of $p_0(x)$ if and only if $r = t^{-g}y$ is a root of $p(x)$. Using the first side of the Newton polygon for $p_0(x)$ to find the first term in a root y, we will eventually obtain a root y of $p_0(x)$ with maximum weight. Therefore, $t^{-g}y$ will be a root of $p(x)$ with maximum weight, and hence $w(t^{-g}y) > n - 1$ because $p(x)$ has a root with weight $> n - 1$. It follows that we can put the terms in the root $t^{-g}y$ after our declared initial segment $0t^0 + \cdots + 0t^{n-1}$ of the root of $p(x)$ without conflict. (More formally, each time we add a term ct^γ to y below, we add the corresponding term $ct^{\gamma-g}$ to r. To simplify the notation, we focus only on the root y of $p_0(x)$.)

We have now reduced our algorithm to determining the parameters c_i and γ_i in a root y of $p_0(x)$ of the form

$$y = c_1 t^{\gamma_1} + c_2 t^{\gamma_1+\gamma_2} + c_3 t^{\gamma_1+\gamma_2+\gamma_3} + \dots$$

where each $c_i \in K$ and each $\gamma_i \in \mathbb{Q}$, with $\gamma_i > 0$ whenever $i > 1$. Furthermore, we need to use the first side of the Newton polygon to determine γ_1 as noted in the previous paragraph. We will determine γ_i and c_i inductively, with c_i and γ_i (as well as the additional parameters L_i, β_i and r_i) determined in the i^{th} round of computation.

Having performed this transformation from $p(x)$ to $p_0(x)$ using Lemma 3.5, we have fixed a rational g such that $r = t^{-g}y$ is our desired root of $p(x)$. It is important that we do not need to perform similar transformations in later rounds of computation in this algorithm. After describing the first round of computation, we list properties that hold inductively at the end of each round of computation and which ensure that we do not need to apply Lemma 3.5 again in this process. In describing the computation, we ignore this transformation from $p(x)$ to $p_0(x)$ and we abuse notation by letting $p_0(x) = A_0 + A_1 x + \ldots + A_n x^n$.

1^{st} round of computation

1. Find γ_1 and the parameters L_1 and β_1:

   - By Lemma 3.4, we can find the negative slope of the first side of the Newton polygon of $p_0(x)$ and the points $P_i = (i, w(A_i))$ that lie on this side. Since $A_0 \neq 0$, we know $(0, w(A_0))$ is one endpoint of this line segment.
   - Let L be the line forming the first side of the Newton polygon. Viewing the polygon as lying in the u-v plane, the equation for L has the form $v + \gamma u = \beta$ with $\gamma, \beta \in \mathbb{Q}$. Note that $\beta = w(A_0)$ and that $\gamma > 0$ because it is the negative of the slope of L.
   - Set $L_1 = L$, $\gamma_1 = \gamma$ and $\beta_1 = \beta$.

2. Find c_1 and the parameter r_1.

   - Let I be the set of indices i such that P_i is on L_1 and let $k > 0$ be the greatest index in I. For each $i \in I$, let $a_i \in K$ be the coefficient of the leading term in A_i. Since we know the finite value of $w(A_0)$, we can assume $a_0 \neq 0$ by ignoring any initial terms in A_0 with 0 coefficients.
   - Let $\psi(z) \in K[x]$ be the polynomial $\psi(z) = \sum_{i \in I} a_i z^i$ of degree k. Since $a_0 \neq 0$, $\psi(z)$ has a nonzero constant term, so $\psi(0) \neq 0$. Because K is algebraically closed, $\psi(z)$ has a nonzero root in K.
   - Let c_1 be a root of $\psi(z)$ of minimal multiplicity and let r_1 be the multiplicity of the root c_1.

This completes the first round of computation to determine c_1 and γ_1. Recall that our goal is to compute a root of the form

$$y = c_1 t^{\gamma_1} + c_2 t^{\gamma_1+\gamma_2} + c_3 t^{\gamma_1+\gamma_2+\gamma_3} + \ldots = t^{\gamma_1}(c_1 + c_2 t^{\gamma_2} + c_3 t^{\gamma_2+\gamma_3} + \ldots)$$

If we think of y_1 as the undetermined part $y_1 = c_2 t^{\gamma_2} + c_3 t^{\gamma_2+\gamma_3} + \ldots$, then y_1 is a root of the polynomial $p_1(x) = t^{-\beta_1} p_0(t^{\gamma_1}(c_1 + x))$. If $p_1(0) = 0$, then $c_1 t^{\gamma_1}$ is already a root of $p_0(x)$. Otherwise, we need to determine c_2 and γ_2. The key property of $p_1(x)$ that will allow us to continue uniformly is that when we write $p_1(x)$ as $p_1(x) = B_0 + B_1 x + \ldots + B_n x^n$, we have the following three properties (see [10]).

(P1) $w(B_i) \geq 0$ for all $i \leq n$,

(P2) $w(B_i) > 0$ for all $i \leq r_1 - 1$, and

(P3) $w(B_{r_1}) = 0$.

By these properties and Lemma 3.7, if $B_0 \neq 0$, then the slope of the first side of the Newton polygon for $p_1(x)$ is at most $-w(B_0)/r_1$, and in particular, is negative.

The second round of computation forms the basis for the continuing induction. We do not know which of the new coefficients $B_0, \ldots, B_n$ are non-zero, so we can only determine if their valuations are finite in a c.e. manner. In particular, at the start of this round of computation, we need to expand our root by adding zero terms $c_1t^{\gamma_1} + 0t^{2\gamma_1} + 0t^{3\gamma_1} + \cdots$ until we see evidence that $B_0 \neq 0$. If $B_0 = 0$, then we obtain a representation of the root $c_1t^{\gamma_1}$ of $p_0(x)$. If $B_0 \neq 0$, then once we see this fact, we need to compute the next parameters γ_2 and c_2. However, we need to add these zero terms in a careful manner so that we do not add a term $0t^{k\gamma_1}$ such that our eventual value γ_2 satisfies $\gamma_1 + \gamma_2 \leq k\gamma_1$. Under such circumstances, we cannot include a term of the form $c_2t^{\gamma_1+\gamma_2}$ in a computable fashion after appending $0t^{k\gamma_1}$. To handle this difficulty, we use Lemma 3.7 and the fact that if $w(B_0)$ is finite, the γ_2 will be the absolute value of the (negative) slope of the first side of the Newton polygon for $p_1(x)$.

2$^{\text{nd}}$ round of computation

1. Expand y with terms of the form $0t^{k\gamma_1}$ as we check whether $B_0 \neq 0$.

   - Set $k = 2$ and start the following loop.
   - Determine $q \in \mathbb{Q}$ such that $q/r_1 \geq k\gamma_1$ and check whether $w(B_0) > q$.
   - If $w(B_0) > q$, then by Lemma 3.7, we know that if $w(B_0)$ is finite, then the absolute value of the slope of the first side of the Newton polygon is greater than $k\gamma_1$. We add the term $0t^{k\gamma_1}$ to y, increment k and repeat the loop.
   - If $w(B_0) \leq q$, then we know the exact value of $w(B_0)$. We move on to Step 2 to start our calculation of the next term $c_2t^{\gamma_1+\gamma_2}$ of y.

2. Find γ_2 and the parameters L_2 and β_2:

   - By Lemma 3.4, we can find the line L forming the first side of the Newton polygon for $p_1(x)$ and the points P_i that lie on this side. The equation for L has the form $v + \gamma u = \beta$ with $\beta = w(A_0)$ and $\gamma > 0$.
   - Set $L_2 = L$, $\gamma_2 = \gamma$ and $\beta_2 = \beta$.

3. Find c_2 and the parameter r_2.

   - Let I be the set of indices i such that P_i is on L_2, let k be the greatest index in I and for each $i \in I$, let $a_i \in K$ be the coefficient of the leading term in B_i. Note that $0 < k \leq r_1$ by (P1)-(P3). Because $w(B_0)$ is finite, we can assume $a_0 \neq 0$.
   - Let $\psi(z) \in K[x]$ be the polynomial $\psi(z) = \sum_{i \in I} a_i z^i$ of degree k. Since $a_0 \neq 0$ and K is algebraically closed, $\psi(z)$ has a nonzero root in K.

- Let c_2 be a root of $\psi(z)$ of minimal multiplicity and let r_2 be the multiplicity of the root c_2. Since $\psi(z)$ has degree $k \leq r_1$, we have $0 < r_2 \leq r_1$.

This completes the second round of the computation. We are now in a position to iterate the construction by repeating the second round of computation (and adjusting the indices accordingly) with the polynomial

$$p_2(x) = t^{-\beta_2} p_1(t^{\gamma_2}(c_2 + x)) = t^{-\beta_2-\beta_1} p_0(c_1 t^{\gamma_1} + c_2 t^{\gamma_1+\gamma_2} + t^{\gamma_1+\gamma_2} x).$$

Writing $p_2(x)$ in the form $p_2(x) = B_0 + B_1 x + \cdots + B_n x^n$ (with new coefficients that more formally ought to be labeled $B_{i,2}$), the properties (P1)-(P3) hold.

In general, to determine the value of c_{n+1} and γ_{n+1}, we start by adding terms of the form $0t^{\gamma_1+\gamma_2+\cdots+\gamma_{n-1}+k\gamma_n}$ while we run a search procedure to see if the constant term of $p_{n+1}(x)$ satisfies $B_0 \neq 0$. We use Lemma 3.7 to make sure that we do not get into a situation in which we have added such a term for which we might have $\gamma_{n+1} \leq k\gamma_n$.

If this search does not halt, then $B_0 = 0$ and hence $p_{n+1}(0) = 0$. In this case, we have computed a representation of the root $c_1 t^{\gamma_1} + \ldots + c_n t^{\gamma_1+\cdots+\gamma_n}$ of $p_0(x)$. Otherwise, we eventually determine that $B_0 \neq 0$ and we find the finite value of $w(B_0)$. We use Lemma 3.4 to determine the first side of the Newton polygon, and let γ_{n+1} be the absolute value of the slope of this line and c_{n+1} be the root of the appropriate polynomial in $K[x]$.

This completes the construction of the root y of $p_0(x)$ and hence the root of $p(x)$. The proof that y is an element of $K\{\{t\}\}$ (i.e. there is a bound on the denominators of the γ_i terms) and that y is a root of $p_0(x)$ is exactly as in [10] because we have produced the same root. □

Corollary 3.9. *For a polynomial $p(x) = A_0 + A_1 x + \ldots + A_n x^n$ over $K\{\{t\}\}$, all roots are computable in K and the coefficients A_i.*

Proof, from Theorem 3.8. The polynomial $p(x)$ has at most n distinct roots. Some of these may be finite, so they are computable. For an infinite root r, fix a finite initial segment $r_k = b_0 t^{\nu_0} + \cdots b_k t^{\nu_k}$ of r that does not extend to any other root of $p(x)$. Let $r = r_k + y$ with $w(y) > \nu_k$. Consider the polynomial $q(x) = p(r_k + x)$ and note that y is a root of $q(x)$. We claim that y is the root of $q(x)$ with greatest valuation. Suppose $s \neq y$ were a root of $q(x)$ with $w(s) > w(y)$. Then $w(s) > \nu_k$, so $r_k + s$ is a root of $p(x)$ that extends r_k, contradicting the fact that r is the unique root of $p(x)$ extending r_k.

Applying the procedure in Theorem 3.8 to $q(x)$ gives us the root of $q(x)$ with maximal weight, and so gives us the root y such that $r = r_k + y$. Therefore, y is computable from K and the coefficients of $q(x)$, or equivalently, it is computable from K, the coefficients of $p(x)$ and the finite (non-uniform) sum r_k. It follows that r is (non-uniformly) computable from K and the coefficients of $p(x)$. □

Theorem 3.8 yields the following.

Theorem 3.10. *Let I be a family of subsets of ω such that if $X_1, \ldots, X_k \in I$ and Y is computable from $X_1, \ldots, X_n$, then $Y \in I$. Then the Newton-Puiseux Theorem holds in I.*

Proof. Suppose K is an algebraically closed field of characteristic 0 computable from some $X \in I$, and let R be the set of Puiseux series over K represented in I. If $p(x) = A_0 + \ldots + A_n x^n$ is a non-constant polynomial for which all $A_i \in R$, then $p(x)$ has a root in R. □

4 Hahn series

Let K be a field, and let G be a divisible ordered Abelian group.

Definition 4.1. *The* Hahn series *(obtained from K and G) are formal power series of the form $s = \sum_{g \in S} a_g t^g$, where S is a well-ordered subset of G and $a_g \in K$. The* support *of s is $Supp(s) = \{g \in S : a_g \neq 0\}$ and the* length *of s is the order type of $Supp(s)$.*

We write $K((G))$ for the set of Hahn series with coefficients in K and terms corresponding to elements of G. The operations and the valuation on $K((G))$ are defined in the natural way.

Definition 4.2. *Let $s = \sum_{g \in S} a_g t^g$, $s' = \sum_{g \in S'} a'_g t^g$, where S, S' are well ordered.*

- *$s + s' = \sum_{g \in S \cup S'} (a_g + a'_g) t^g$, where $a_g = 0$ if $g \notin S$, and $a'_g = 0$ if $g \notin S'$.*
- *$s \cdot s' = \sum_{g \in T} b_g t^g$, where $T = \{g_1 + g_2 : g_1 \in S \;\&\; g_2 \in S'\}$, and for each $g \in T$, we define $B_g = \{(g_1, g_2) \in S \times S' : g_1 + g_2 = g\}$ and set $b_g = \sum_{(g_1, g_2) \in B_g} b_{g_1} \cdot b_{g_2}$.*

Definition 4.3. *For $s \in K((G))$, the valuation (or weight) $w(s)$ is the least $g \in Supp(s)$, if $s \neq 0$, and $w(0) = \infty$.*

Here is the result of Maclane [5].

Theorem 4.4 (Generalized Newton-Puiseux Theorem)**.** *Let G be a divisible ordered Abelian group, and let K be a field that is algebraically closed of characteristic 0. Then $K((G))$ is also algebraically closed.*

We sketch the root taking process, which is essentially a transfinite version of the Newton-Puiseux process.

Partial proof. Let $p(x) = A_0 + A_1 x + \ldots + A_n x^n$ be a polynomial over $K((G))$, where $p(x)$ is non-constant. We may suppose that for all i, $w(A_i) \geq 0$—if this is not so initially, we replace $p(x)$ by $t^g p(x)$ for suitable positive g. We may also suppose that there is an infinitesimal root—if this is not so initially, we replace $p(x)$ by $p(t^{-g} x)$ for suitable positive g. We define a sequence of initial segments of a root. Let $r_0 = 0$ and let $p_0(x) = p(r_0 + x)$. If the constant term is 0, then r_0 is a root. Otherwise, we consider the Newton polygon, and find the first term of a root as in the case of Puiseux series. Let r_1 consist of just this first term and form $p_1(x) = p(r_1 + x)$. Given r_α an initial segment of a root having length α, we form the polynomial $p_\alpha(x)$. If the constant term is 0, then 0 is a root of $p_\alpha(x)$, which means that r_α is a root of $p(x)$. Otherwise, we consider the Newton polygon for $p_\alpha(x)$ and we find the first term of a root. We let $r_{\alpha+1}$ be the result of adding this to r_α. For limit α, having determined r_β for $\beta < \alpha$, we let r_α consist of all of the terms of all r_β. Eventually, the process stops. The length must be less than κ^+, where κ is the cardinality of G. □

We can bound the lengths of roots of a polynomial in terms of the lengths of the coefficients. The result below is proved in [3], [4].

Theorem 4.5. *Let K be an algebraically closed field of characteristic 0, and let G be a divisible ordered Abelian group. Let $p(x) = A_0 + \ldots + A_n x^n$ be a polynomial over $K((G))$. We suppose that the coefficients all have countable length. If γ is a countable limit ordinal greater than the lengths of all A_i, then the roots of $p(x)$ all have length less than ω^{ω^γ}.*

We can extend Theorem 4.5 to the case where γ is uncountable.

Corollary 4.6. *Let K be an algebraically closed field of characteristic 0, and let G be a divisible ordered Abelian group. Let $p(x) = A_0 + \ldots + A_n x^n$ be a polynomial over $K((G))$. If γ is a a limit ordinal (possibly uncountable) greater than the lengths of all A_i, then the roots of $p(x)$ all have length less than ω^{ω^γ}.*

Proof. Take a transitive set M satisfying some set theory (enough to prove Maclane's Theorem) and containing the field K, the group G, and the coefficients A_i. Say that α_i is the length of A_i, these may be uncountable ordinals. Let γ be the first limit ordinal greater than all α_i. Let $r_1, \ldots, r_n$ be the roots of $p(x)$ in M (listed with multiplicity). Say that β_i is the length of r_i. The coefficients A_i and the roots r_i are each identified with a function defined on all of G, giving the coefficients in K. For each coefficient A_i and each root r_i, there is an isomorphism between the support (with the ordering from G) and the ordinal length.

Let M' be a countable elementary substructure of M that contains the following elements: K, G, the coefficients A_i, the roots r_i, the ordinals α_i, γ, and β_i, and the functions mapping the supports isomorphically onto the lengths. Note that since M' is countable, the elements of G may not all be present. Similarly, the ordinals α_i, γ, and β_i may not have all of their predecessors in M'. Applying a Mostowski collapse f, we get M'' such that $M' \cong_f M''$. M'' is again a transitive set, and the ordinals are actual ordinals. If G and γ were uncountable in M, then $f(G)$ and $f(\gamma)$ are countable (viewed in the real world) even though they are uncountable in M''.

Being a root, and having a specific ordinal length are absolute between M'' and the real world. In M'', and in the real world, $f(r_i)$ is a root of $f(p)(x)$. The coefficients $f(A_i)$ have length $f(\alpha_i) < f(\gamma)$, where $f(\gamma)$ is a limit ordinal. The roots $f(r_i)$ have length $f(\beta_i)$. Since the ordinals are countable in the real world, we can see in the real world that the roots $f(r_i)$ of $f(p)$ have length $f(\beta_i)$ which must be less than $\omega^{\omega^{f(\gamma)}}$. By absoluteness, in M'', $f(r_i)$ has length $f(\beta_i)$, which is less than $\omega^{\omega^{f(\gamma)}}$. Then in M' and M, r_i has length β_i, which is less than $\omega^{\omega^{\gamma}}$. $\square$

5 Complexity in Hahn fields

We want to measure complexity of basic operations and root-taking in a Hahn field $K((G))$. We first say how we are representing Hahn series. To represent $s \in K((G))$, we could use a function f from an ordinal α to $K \times G$ such that the second component of $f(\beta)$ increases with $\beta < \alpha$. Alternatively, we could use a function σ from G to K such that $\{g \in G : \sigma(g) \neq 0\}$ is well ordered. We use the first approach to get a result on Maclane's Theorem in admissible sets. We then use the second approach to say more precisely how complicated it is to compute the basic operations on Hahn series, and to set bounds on the complexity of the root-taking process.

5.1 Maclane's Theorem in admissible sets

An *admissible* set A is a transitive set that satisfies the axioms of Kripke-Platek set theory. We consider admissible sets that contain ω. What is important for us is that we can define functions F by induction on the ordinals, provided that we have a Σ_1 formula describing the way we pass from $F|\alpha$ to $F(\alpha)$. We will show that the generalized Newton-Puiseux Theorem holds in admissible sets. We consider an admissible set A that contains the algebraically closed field K and the divisible ordered Abelian group G. We consider those elements of $K((G))$ that are represented by a function s in A from an ordinal α to $K \times G$ such that if $s(\beta) = (b_\beta, g_\beta)$, then $\beta < \gamma < \alpha$ implies $g_\beta < g_\gamma$.

Lemma 5.1. *Let A be an admissible set containing the algebraically closed field K and the divisible ordered Abelian group G. If s, s' are elements of $K((G))$ in A, then $s + s'$, $s \cdot s'$, Supp(s) and the length of s are all in A.*

Lemma 5.2. *Let A be an admissible set. The function $\alpha \to \omega^\alpha$ is Σ_1-definable on A.*

Theorem 5.3. *Let A be an admissible set. Then the generalized Newton-Puiseux Theorem holds in A.*

Proof sketch. We sketch the proof which follows the standard root-taking process, indicating where the bounds in Corollary 4.6 are used. Let K be an algebraically closed field of characteristic 0, and let G be a divisible ordered Abelian group, both in A. Let $p(x)$ be a polynomial over $K((G))$, with coefficients A_i in A. Let α_i be the length of A_i. We must show that $p(x)$ has a root r in A. We define, by Σ_1-induction on ordinals α, a sequence of initial segments r_α of a root r, where r_α has length α until/unless there is some $\beta < \alpha$ such that r_β is a root.

We let $r_0 = 0$. Given r_α, we form $p_\alpha(x) = p(r_\alpha + x) = B_0 + B_1 + \ldots + B_n x^n$. By Taylor's Theorem, $B_i = \frac{p^{(i)}(r_\alpha)}{i!}$. These coefficients are in A. If $B_0 = 0$, then r_α is a root of $p(x)$. Otherwise, we consider the Newton polygon for $p_\alpha(x)$. This polygon will have at least one side such that if ν is the negative of the slope of this side, then ν is greater than the elements in $\mathrm{Supp}(p_\alpha(x))$. Fix such a value ν and let b be a root of the ν-principal part. Then $r_{\alpha+1} = r_\alpha + bt^\nu$. For limit α, we let r_α be the element of $K((G))$ whose initial segments are the r_β for $\beta < \alpha$.

We have Σ_1 formulas saying how $r_\alpha + 1$ is obtained from r_α and, for limit α, how r_α is obtained from the r_β's for $\beta < \alpha$. Thus, we have a Σ_1 formula defining the function F on ordinals α in A, such that $F(\alpha) = r_\alpha$ if no proper truncation of r_α is a root, and $F(\alpha) = r_\beta$ if $\beta < \alpha$ and r_β is a root. Using our bounds on lengths of roots, we can show that some r_β is a root. Let γ be the maximum of $\alpha_i + \omega$. Then γ is a limit ordinal in A, and the ordinal ω^{ω^γ} is also in A. By the results above, the roots of $p(x)$ all have length less than ω^{ω^γ}. So, for some $\beta \in A$, r_β is a root of $p(x)$. □

5.2 More precise results

Let K be an algebraically closed field and G be a divisible ordered Abelian group, both with universe a subset of ω. To represent an element of $K((G))$, we take a function s from G to K such that $Supp(s) = \{g \in G : s(g) \neq 0\}$ is well ordered. We want to know how many jumps it takes to compute the basic operations on $K((G))$, and to find initial segments of a root of a polynomial over $K((G))$.

5.3 Complexity of basic operations

We begin with the complexity of the basic operations.

Lemma 5.4 (Sums)**.**

1. *There is a uniform effective procedure that, given K, G, and elements s, s' of $K((G))$, yields $s + s'$.*

2. *There is a uniform effective procedure that, given n, K, G, and elements $s_1, \ldots, s_n$ of $K((G))$, yields $s_1 + \ldots + s_n$.*

Proof. Both parts follow since $(s_1 + \ldots + s_n)(g) = s_1(g) + \ldots + s_n(g)$. □

Lemma 5.5 (Zero)**.** *We have a Δ^0_2 procedure that, given K, G, and $s \in K((G))$ determines whether $s = 0$. (In fact, the set of representations of 0 is effective Π^0_1.)*

Proof. We suppose that G has universe ω. Given s, we ask whether there is some g such that $s(g) \neq 0$. □

Lemma 5.6 (Valuation)**.** *We have a Δ^0_2 procedure that, given K, G, and $s \in K((G))$, determines $w(s)$.*

Proof. If X computes K, G, and s, then using X', we find $w(s)$ as follows. We first ask whether $Supp(s) = \emptyset$. If so, then $w(s) = \infty$. If not, we can effectively find some $g_1 \in Supp(s)$. We then ask whether $Supp(s) \cap pred(g_1) = \emptyset$. If so, then $w(s) = g_1$. If not, we find $g_2 \in Supp(s)$ with $g_2 < g_1$. We continue until we find $g_k \in Supp(s)$ such that $Supp(s) \cap pred(g_k) = \emptyset$. Then $w(s) = g_k$. □

Lemma 5.7 (Products)**.**

1. *There is a uniform Δ_2^0 procedure that, given K, G, and elements s_1, s_2 of $K((G))$, yields $s_1 \cdot s_2$.*
2. *There is a uniform Δ_2^0 procedure that, given n, K, G, and elements $s_1, \ldots, s_n$ of $K((G))$, yields $s_1 \cdot \cdots \cdot s_n$.*

Proof. For a given $g \in G$, we ask whether there exist g_1 and g_2 such that $g_1 + g_2 = g$ and $s_i(g_i) \neq 0$. If we have one pair, we ask whether there is another. Because $Supp(s_i)$ are well ordered, there are only finitely many such pairs. Therefore, after finitely many Δ_2^0 questions, we have the full set S of pairs (g_1, g_2) such that $g_1 + g_2 = g$ and $s_i(g_i) \neq 0$. The coefficient of g is the sum over the pairs $(g_1, g_2) \in S$ of the products $s_1(g_1)s_2(g_2)$. The uniform process for multiplying n terms is similar. □

Lemma 5.8. *There is a Δ_2^0 procedure that, given K, G, a polynomial $p(x_1, \ldots, x_n) \in K[x_1, \ldots, x_n]$, and elements $s_1, \ldots, s_n \in K((G))$, yields $p(s_1, \ldots, s_n)$.*

Proof. The polynomial p is a sum of monomials $ax_1^{i_1} \cdot \ldots \cdot x_n^{i_n}$. Given $s_1, \ldots, s_n \in K((G))$, we can find the value of any monomial $as_1^{i_1} \cdot \ldots \cdot s_n^{i_n}$, for $a \in K$, in the same way that we found the product of two elements s, s'. We can then take the sum effectively. □

Taking products becomes simpler when one of the sums is finite. This observation will help lower the complexity of the initial segments in the root taking process later.

Lemma 5.9. *Let $s \in K((G))$ and let $r = b_0 t^{\nu_0} + \cdots b_k t^{\nu_k}$ with $b_i \in K$, $\nu_i \in G$ and $\nu_0 < \nu_1 < \cdots < \nu_k$. There is a uniform procedure that, given s and r, yields $s \cdot r$.*

Proof. First, consider the case of $s \cdot b_0 t^{\nu_0}$ where $s = \sum_{g \in S} a_g t^g$. We have

$$\left(\sum_{g \in S} a_g t^g \right) \cdot b_0 t^{\nu_0} = \sum_{g \in S} a_g b_0 t^{g + \nu_0} = \sum_{g \in \widehat{S}} \widehat{a}_g t^g$$

where $g \in \widehat{S}$ if and only if $g - \nu_0 \in S$ and $\widehat{a}_g = a_{g - \nu_0} b_0$. For the general case, we take the sum of the uniformly computed terms $s \cdot b_i t^{\nu_i}$. □

Lemma 5.10. *Let $p(x) = A_0 + \cdots + A_n x^n$ be a polynomial over $K((G))$ and let $r = b_0 t^{\nu_0} + \cdots + b_k t^{\nu_k}$ be a finite sum as above. There is a uniform procedure that, given $p(x)$ and r, yields $p(r)$.*

Proof. Expanding r^i, we can uniformly calculate $A_i r^i$ by Lemma 5.9 and then take a sum to obtain $p(r)$. □

Lemma 5.11. *Given K, G, the coefficients of a polynomial $p(x)$ over $K((G))$, and a finite sum $r = b_0 t^{\nu_0} + \cdots + b_k t^{\nu_k}$ as above, we can effectively compute $p(r + x)$.*

Proof. To find the coefficients of $p(r + x)$, it suffices, by Taylor's Theorem, to be able to compute $p^{(i)}(r)$ for $i \leq n$. We can do these computations uniformly by Lemma 5.10. □

5.4 Initial segments of roots

We consider the complexity of a procedure that, given K, G, and a polynomial $p(x) = A_0 + A_1 x + \ldots + A_n x^n$ over $K((G))$, at step α determines an initial segment r_α of a root of $p(x)$, where

1. $r_0 = 0$
2. for $\alpha > 0$, either r_α has length α and extends r_β for all $\beta < \alpha$ or else there is some $\beta < \alpha$ such that r_β is already root and $r_\alpha = r_\beta$.

Note that r_α is an element of $K((G))$ and hence is a function $r_\alpha : G \to K$ with well ordered support. In the case when none of the r_β for $\beta < \alpha$ are a root, $Supp(r_\alpha)$ has order type α. The totality of r_α will be important in determining the complexity at limit levels because, to present r_α as a total function that has each r_β as an initial segment, we need to be able to determine the sequence r_β uniformly and we need to be able to determine for any element $g \in G$ whether there is a $\beta < \alpha$ such that $g \in Supp(r_\beta)$.

We describe a function f such that for computable ordinals α, it is $\Delta^0_{f(\alpha)}$ in K, G, and p to carry out step α of the procedure. Here are the first few values of f.

1. At Step 0, we write 0. This is computable, so $f(0) = 1$.
2. At Step 1, we must decide whether $r_0 = 0$ is a root, and if not, we must give r_1 of length 1. To determine whether 0 is a root of $p(x) = A_0 + \ldots + A_n x^n$, it is enough to check whether $A_0 = 0$. This is Δ^0_2. If $A_0 \neq 0$, then we find $w(A_0)$. This is also Δ^0_2. In fact, we have a uniform procedure Δ^0_2 in K, G, and $p(x)$, for determining which A_i are non-zero, and finding $w(A_i)$ for each such i. Then, proceeding as Newton did, we can effectively compute (b_0, ν_0) such that $r_1 = b_0 t^{\nu_0}$. So, $f(1) = 2$.
3. At Step $n + 1$, having found r_n of length n, we must decide whether r_n is a root, and if not, we must extend to r_{n+1}. By Lemma 5.11, we can compute effectively the new polynomial $p_n(x) = p(r_n + x)$. As in Step 1, deciding whether r_n is a root of $p(x)$, or finding r_{n+1} is Δ^0_2 in K, G, and p. So, $f(n) = 2$ for all finite $n \geq 1$.

4. At Step ω, we must decide whether r_n is a root for some finite n, and if not, we must give r_ω as a function. To determine whether there is some finite n such that r_n is a root of $p(x)$ is Δ^0_3. Assuming that this is not the case, the sequence $(r_n)_{n\in\omega}$ is Δ^0_2 in K, G, and $p(x)$, but giving r_ω as a total function is Δ^0_3. So, $f(\omega) = 3$.

5. At Step $\omega + 1$, we must determine whether r_ω is a root, and if not, we must give $r_{\omega+1}$. To determine whether r_ω is a root of $p(x)$, we could either determine directly whether $p(r_\omega) = 0$, or find $p_\omega(x) = p(r_\omega + x)$ and then check whether the constant term is 0. If we find $p_\omega(x)$, and the constant term is not 0, then we could continue to find $r_{\omega+1}$. To find p_ω, we use the sequence version of r_ω, which is Δ^0_2. The new coefficients have the form $B_i = \frac{p^{(i)}(r_\omega)}{i!}$. Each B_i is a finite sum of products of a coefficient A_j and a power of r_ω. For $g \in G$, to find the coefficient of t^g in B_i, we ask whether there exist $h_0 \in Supp(A_j)$ and some n, with a tuple of h's in $Supp(r_n)$ for which the sum is g. If we find one such tuple, we ask whether there is another We continue until we have the full finite set S of tuples. For each tuple in S, we take the product of the coefficients. The sum of these products is the coefficient of t^g in B_i. Thus, finding $p_\omega(x)$ is Δ^0_3 in K, G, and p. Then it is Δ^0_4 to say whether the constant term is 0, and if not, to find the first term of a root. So, $f(\omega + 1) = 4$.

6. At Step $\omega + 2$, we must determine if $r_{\omega+1}$ is a root, and if not, we must find the next term to give $r_{\omega+2}$. Writing $r_{\omega+1} = r_\omega + b_\omega t^{\nu_\omega}$, we have $p_{\omega+1}(x) = p(r_{\omega+1} + x) = p_\omega(b_\omega t^{\nu_\omega} + x)$. Since $p_\omega(x)$ is Δ^0_3, it follows from Lemma 5.11 that $p_{\omega+1}(x)$ is also Δ^0_3. Therefore, it is Δ^0_4 to determine whether the constant term is 0, and if not, to find the next term. So, $f(\omega + 2) = 4$. Following the same pattern, $f(\omega + n) = 4$ for all $n \geq 1$.

These results are sharp.

Proposition 5.12. *For finite $n \geq 1$, Step n is Δ^0_2-hard, Step ω is Δ^0_3-hard, and for finite $n \geq 1$, Step $\omega + n$ is Δ^0_4-hard.*

Proof. At Step 1, we must decide whether r_0 is a root, and if not, we must give r_1. We focus on the first part of Step 1, deciding whether 0 is a root. Now, 0 is a root iff the constant term is 0, and this is Π^0_1. Let K be a computable copy of the field of algebraic numbers, and let G be a computable copy of the additive group of rationals. For Step 1, we show that for an arbitrary c.e. set S, there is a uniformly computable sequence of polynomials $p_e(x) = B_e - x$ such that $e \in S$ iff 0 is not a root of $p_e(x)$. We let $Supp(B_e) = \emptyset$ if $e \notin S$, and if for some first s, $e \in S_s$, then we put the pair $\langle e, s\rangle$ into $Supp(B_e)$. We have $e \in S$ iff 0 is not a root of $p_e(x)$.

Similarly, for any finite $n \geq 1$, we show that for an arbitrary c.e. set S, there is a uniformly computable sequence of polynomials $p_e(x) = B_e - x$ such that $e \in S$ iff $Supp(B_e)$ has more than n elements. We start by including $0, 1, \ldots, n-1$. At step s, we define more of χ_{B_e}, keeping elements out of B_e, until/unless e appears in S. If this happens, then we add one more element into $Supp(B_e)$. We have $e \in S$ iff r_n is not a root of $p_e(x)$ because $|Supp(r_n)| = n$.

At Step ω, we must decide whether r_n is a root for some finite n, and if not, then we must combine the r_n's to form r_ω. Consider the m-complete Σ^0_2 set $Fin = \{e : W_e \text{ is finite}\}$. We produce a uniformly computable sequence of polynomials $p_e(x) = B_e - x$ such that $e \in Fin$ iff $Supp(B_e)$ is finite. At step s, we keep the next element g of G (in the $\leq_\omega$ order) out of $Supp(B_e)$ unless s is first such that some $k \in W_{e,s}$, and in this case, we put g into $Supp(B_e)$. Then $e \in Fin$ iff some r_n is a root.

At Step $\omega+1$, we must decide whether r_ω is a root, and if not, then we must find $r_{\omega+1}$. Consider an arbitrary Π^0_3 set S. We define a uniformly computable sequence of polynomials $p_e(x) = B_e - x$ such that if $e \in S$, then $Supp(B_e)$ has order type ω, and otherwise, $Supp(B_e)$ has order type at least $\omega + \omega$. We have uniformly c.e. sets $W_{f(e,k)}$ such that $e \in S$ iff for all k, $W_{f(e,k)}$ is finite. We have a computable sequence of rationals $s_0 < s_1 < \ldots s_n < \ldots$. We start with the plan to have $Supp(B_e)$ consist of these s_i. At step s, the characteristic function of $Supp(B_e)$ is defined on all s_i plus the first s elements of G (in the $<_\omega$-ordering). Whenever a new element enters $W_{f(e,n)}$, we choose a new q in the interval (r_n, r_{n+1}) to the right of any we have chosen earlier, and we add it to the support. Consider the polynomial $B_e - x$, with the unique root $r = B_e$. If $e \in S$, then $r = r_\omega$. If $e \notin S$, then r has length at least $\omega + \omega$, so r_ω is not a root.

Similarly, we can show that for finite $n > 1$, for any Π^0_3 set S, there is a uniformly computable sequence of polynomials $p_e(x) = B_e - x$ such that $e \in S$ iff $Supp(B_e)$ has order type $\omega + n - 1$. We let B_e be as above, except that we add to the support of B_e an extra $(n-1)$-tuple, greater than all s_i. If $e \in S$, then B_e has length $\omega + n - 1$, and if $e \notin S$, then B_e has length at least $\omega + \omega$. As above, we take polynomials $B_e - x$. If $e \in S$, then $r_{\omega+n-1}$ is a root. If $e \notin S$, then $r_{\omega+n-1}$ is not a root. □

Continuing the complexity pattern, we have the following general result about the values of f.

Proposition 5.13. *Let α be a computable limit ordinal.*

1. *$f(\alpha) = sup_{\beta<\alpha} f(\beta) + 1$.*
2. *for $n \geq 1$, $f(\alpha + n) = f(\alpha) + 1$.*

Determining $r_{\omega+\omega}$ as a function is Δ^0_5, but we do not know whether this upper bound is sharp. However, there is a weak hardness result showing that the complexity continues to go up with length.

Proposition 5.14. *For each computable ordinal α, Step ω^α is $\Pi^0_{2\alpha}$-hard.*

Proof. Let S be a $\Pi^0_{2\alpha}$ set. We can build a uniformly computable sequence of orderings $\mathcal{A}_n$ such that $\mathcal{A}_n$ has order type ω^α if $n \in S$ and some $\gamma < \omega^\alpha$ otherwise. We get a uniformly computable sequence C_n of subsets of $\mathbb{Q}$, all contained in the interval $(0, 1)$, such that $C_n \cong \mathcal{A}_n$. Let B_n be the sum of t^q for $q \in C_n$, and consider the polynomial $B_n - x$, with the unique root $r = B_n$. For these polynomials, if $n \in S$, then $r = r_{\omega^\alpha}$, and if $n \notin S$, then $r = r_\gamma$ for some $\gamma < \omega^\alpha$. Thus, S is reducible to Step ω^α, applied to these polynomials. □

References

[1] J. Barwise, *Admissible Sets and Structures: An Approach to Definability*, Perspectives in Math. Logic, vol. 7, Springer-Verlag, 1975.

[2] S. Basu, R. Pollack, and M.-F. Roy, *Algorithms in Real Algebraic Geometry*, Springer, 2011.

[3] J. F. Knight and K. Lange, "Lengths of developments in $K((G))$", *Selecta Mathematica*, vol. 25(2019).

[4] J. F. Knight and K. Lange, "Truncation-closed subfields of a Hahn field," pre-print.

[5] S. Maclane, "The universality of formal power series fields", *Bull. Amer. Math. Soc.*, vol. 45(1939), pp. 888-890.

[6] I. Newton, "Letter to Oldenburg dated 1676 Oct 24", *The Correspondence of Isaac Newton II*, 1960, Cambridge University Press, pp. 126-127.

[7] V. A. Puiseux, "Recherches sur les fonctions algébriques", *J. Math. Pures Appl.*, vol. 15(1850), pp. 365-480.

[8] V. A. Puiseux, "Nouvelles recherches sur les fonctions algébriques", *J. Math. Pures Appl.*, vol. 16(1851), pp. 228-240.

[9] H. Rogers, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, 1967.

[10] Robert J. Walker, *Algebraic Curves*, 1950, Dover.