DEGREES OF ORDERS ON TORSION-FREE ABELIAN GROUPS

ASHER M. KACH, KAREN LANGE, AND REED SOLOMON

ABSTRACT. We show that if $\mathcal{H}$ is an effectively completely decomposable computable torsion-free abelian group, then there is a computable copy $\mathcal{G}$ of $\mathcal{H}$ such that $\mathcal{G}$ has computable orders but not orders of every (Turing) degree.

1. INTRODUCTION

A recurring theme in computable algebra is the study of the complexity of relations on computable structures. For example, fix a natural mathematical relation R on some class of computable algebraic structures such as the successor relation in the class of linear orders or the atom relation in the class of Boolean algebras. One can consider whether each computable structure in the class has a computable copy in which the relation is particularly simple (say computable or low or incomplete) or whether there are structures for which the relation is as complicated as possible in every computable presentation. For the successor relation, Downey and Moses [9] show there is a computable linear order $\mathcal{L}$ such that the successor relation in every computable copy of $\mathcal{L}$ is as complicated as possible, namely complete. On the other hand, Downey [5] shows every computable Boolean algebra has a computable copy in which the set of atoms is incomplete. Alternately, one can explore the connection between definability and the computational properties of the relation R.

More abstractly, one can start with a set S of Turing (or other) degrees and ask whether there is a relation R on a computable structure $\mathcal{A}$ such that the set of degrees of the images of R in the computable copies of $\mathcal{A}$ is exactly S. For example, Hirschfeldt [13] proved that this is possible if S is the set of degrees of a uniformly c.e. collection of sets.

One can also consider relations such as "being a k-coloring" for a computable graph or "being a basis" for a torsion-free abelian group. In these examples, for each fixed computable structure, there are many subsets of the domain (or functions on the domain) satisfying the property. It is natural to ask whether there are computable structures for which all of these instantiations are complicated and whether this complexity depends on the computable presentation. In the case of k-colors of a planar graph, Remmel [25] proves that one can code arbitrary Π^0_1 classes (up to permuting the colors) by the collection of k-colorings. For torsion-free abelian groups, there is a computable group $\mathcal{G}$ such that every basis computes $\mathbf{0}'$. However, for any computable $\mathcal{H}$, one can find a computable copy of the given group in which there is a computable basis (see Dobritsa [4]). Therefore, while every basis can be complicated in one computable presentation, there is always a computable presentation having a computable basis.

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In this paper, we present a result concerning computability-theoretic properties of the spaces of orderings on abelian groups. To motivate these properties, we compare the known results on computational properties of orderings on abelian groups with those for fields. We refer the reader to [11] and [16] for a more complete introduction to ordered abelian groups and to [18] for background on ordered fields.

Definition 1.1. An *ordered abelian group* consists of an abelian group $\mathcal{G} = (G; +, 0)$ and a linear order $\leq_\mathcal{G}$ on G such that $a \leq_\mathcal{G} b$ implies $a + c \leq_\mathcal{G} b + c$ for all $c \in G$. An abelian group $\mathcal{G}$ that admits such an order is *orderable*.

Definition 1.2. The *positive cone* $P(\mathcal{G}; \leq_\mathcal{G})$ of an ordered abelian group $(\mathcal{G}; \leq_\mathcal{G})$ is the set of non-negative elements

$$P(\mathcal{G}; \leq_\mathcal{G}) := \{g \in G \mid 0_\mathcal{G} \leq_\mathcal{G} g\}.$$

Because $a \leq_\mathcal{G} b$ if and only if $b - a \in P(\mathcal{G}; \leq_\mathcal{G})$, there is an effective one-to-one correspondence between positive cones and orderings. Furthermore, an arbitrary subset $X \subseteq G$ is the positive cone of an ordering on $\mathcal{G}$ if and only if X is a semigroup such that $X \cup X^{-1} = G$ and $X \cap X^{-1} = \{0_\mathcal{G}\}$, where $X^{-1} := \{-g \mid g \in X\}$. We let $\mathbb{X}(\mathcal{G})$ denote the space of all positive cones on $\mathcal{G}$. Notice that the conditions for being a positive cone are Π^0_1.

The definitions for ordered fields are much the same, and we let $\mathbb{X}(\mathcal{F})$ denote the space of all positive cones on the field $\mathcal{F}$. We suppress the definitions here as the results for fields are only used as motivation. As in the case of abelian groups, the conditions for a subset of F to be a positive cone are Π^0_1.

Classically, a field $\mathcal{F}$ is orderable if and only if it is formally real, i.e., if $-1_\mathcal{F}$ is not a sum of squares in $\mathcal{F}$; and an abelian group $\mathcal{G}$ is orderable if and only if it is torsion-free, i.e., if $g \in G$ and $g \neq 0_\mathcal{G}$ implies $ng \neq 0_\mathcal{G}$ for all $n \in \mathbb{N}$ with $n > 0$. In both cases, the effective version of the classical result is false: Rabin [24] constructed a computable formally real field that does not admit a computable order, and Downey and Kurtz [6] constructed a computable torsion-free abelian group (in fact, isomorphic to $\mathbb{Z}^\omega$) that does not admit a computable order.

Despite the failure of these classifications in the effective context, we have a good measure of control over the orders on formally real fields and torsion-free abelian groups. Because the conditions specifying the positive cones in both contexts are Π^0_1, the sets $\mathbb{X}(\mathcal{F})$ and $\mathbb{X}(\mathcal{G})$ are closed subsets of 2^F and 2^G respectively, and hence under the subspace topology they form Boolean topological spaces. If $\mathcal{F}$ and $\mathcal{G}$ are computable, then the respective spaces of orders form Π^0_1 classes, and therefore computable formally real fields and computable torsion-free abelian groups admit orders of low Turing degree.

For fields, one can say considerably more. Craven [2] proved that for any Boolean topological space T, there is a formally real field $\mathcal{F}$ such that $\mathbb{X}(\mathcal{F})$ is homeomorphic to T. Translating this result into the effective setting, Metakides and Nerode [23] proved that for any nonempty Π^0_1 class $\mathcal{C}$, there is a computable formally real field $\mathcal{F}$ such that $\mathbb{X}(\mathcal{F})$ is homeomorphic to $\mathcal{C}$ via a Turing degree preserving map. Friedman, Simpson, and Smith [10] proved the corresponding result in reverse mathematics that WKL_0 is equivalent to the statement that every formally real field is orderable.

Most of the corresponding results for abelian groups fail. For example, a countable torsion-free abelian group $\mathcal{G}$ satisfies either $|\mathbb{X}(\mathcal{G})| = 2$ (if the group has rank one) or $|\mathbb{X}(\mathcal{G})| = 2^{\aleph_0}$ and $\mathbb{X}(\mathcal{G})$ is homeomorphic to 2^ω. For a computable torsion-free abelian group $\mathcal{G}$, even if one only considers infinite Π^0_1 classes of separating sets (which are classically homeomorphic to 2^ω) and only requires that the map from $\mathbb{X}(\mathcal{G})$ into the Π^0_1 class be degree preserving, one cannot represent all such classes by spaces of orders on computable torsion-free abelian groups. (See Solomon [28] for a precise statement and proof of this result.) However, the connection to Π^0_1 classes is preserved in the context of reverse mathematics as Hatzikiriakou and Simpson [12] proved that WKL_0 is equivalent to the statement that every torsion-free abelian group is orderable.

Because torsion-free abelian groups are $\mathbb{Z}$-modules, notions such as linear independence play a large role in studying these groups.

Definition 1.3. Let $\mathcal{G}$ be a torsion-free abelian group. Elements $g_0, \dots, g_n$ are *linearly independent* (or just *independent*) if for all $c_0, \dots, c_n \in \mathbb{Z}$,

$$c_0 g_0 + c_1 g_1 + \cdots + c_n g_n = 0_{\mathcal{G}}$$

implies $c_i = 0$ for $0 \leq i \leq n$. An infinite set of elements is *independent* if every finite subset is independent. A maximal independent set is a *basis* and the cardinality of any basis is the *rank* of $\mathcal{G}$.

Solomon [28] and Dabkowska, Dabkowski, Harizanov, and Tonga [3] established that if $\mathcal{G}$ is a computable torsion-free abelian group of rank at least two and B is a basis for $\mathcal{G}$, then $\mathcal{G}$ has orders of every Turing degree greater than or equal to the degree of B. Therefore, the set

$$\deg(\mathbb{X}(\mathcal{G})) := \{\mathbf{d} \mid \mathbf{d} = \deg(P) \text{ for some } P \in \mathbb{X}(\mathcal{G})\}$$

contains all the Turing degrees when the rank of $\mathcal{G}$ is finite (but not one) and contains cones of degrees when the rank is infinite. As mentioned earlier, Dobritsa [4] proved that every computable torsion-free abelian group has a computable copy with a computable basis. Therefore, every computable torsion-free abelian group has a computable copy that has orders of every Turing degree, and hence has a copy in which $\deg(\mathbb{X}(\mathcal{G}))$ is closed upwards.

Our broad goal, which we address one aspect of in this paper, is to better understand which Π^0_1 classes can be realized as $\mathbb{X}(\mathcal{G})$ for a computable torsion-free abelian group $\mathcal{G}$ and how the properties of the space of orders changes as the computable presentation of $\mathcal{G}$ varies. Specifically, is $\deg(\mathbb{X}(\mathcal{G}))$ always upwards closed? If not, does every group $\mathcal{H}$ have a computable copy in which it fails to be upwards closed? We show that if $\mathcal{H}$ is effectively completely decomposable, then there is a computable $\mathcal{G} \cong \mathcal{H}$ such that $\deg(\mathbb{X}(\mathcal{G}))$ contains $\mathbf{0}$ but is not closed upwards. We conjecture that this statement is true for all computable infinite rank torsion-free abelian groups.

Definition 1.4 (Khisamiev and Krykpaeva [14])**.** A computable infinite rank torsion-free abelian group $\mathcal{H}$ is *effectively completely decomposable* if there is a uniformly computable sequence of rank one subgroups $\mathcal{H}_i$ of $\mathcal{H}$, for $i \in \omega$, such that $\mathcal{H}$ is equal to $\oplus_{i \in \omega} \mathcal{H}_i$ (with the standard computable presentation).

There are a number of recent results concerning computability theoretic properties of classically completely decomposable groups in, for example, [7], [8], [15], and [22]. Our main result is the following theorem.

Theorem 1.5. *Let $\mathcal{H}$ be an effectively completely decomposable infinite rank torsion-free abelian group. There is a computable presentation $\mathcal{G}$ of $\mathcal{H}$ and a non-computable, computably enumerable set C such that:*

- *The group $\mathcal{G}$ has exactly two computable orders.*
- *Every C-computable order on $\mathcal{G}$ is computable.*

Thus, the set of degrees of orders on $\mathcal{G}$ is not closed upwards.

If $\mathcal{H}$ is effectively completely decomposable, then $\deg(\mathbb{X}(\mathcal{H}))$ contains all Turing degrees because $\mathcal{H}$ has a computable basis formed by choosing a nonzero element h_i from each $\mathcal{H}_i$. Therefore, although the group $\mathcal{G}$ in Theorem 1.5 is completely decomposable in the classical sense, it cannot be effectively completely decomposable.

In general, one does not expect the collection of degrees realizing a relation on a fixed computable copy of an algebraic structure to be upwards closed and hence this result is not surprising from that perspective. However, the corresponding result for the basis of a computable torsion-free abelian group fails.

Proposition 1.6. *Let $\mathcal{H}$ be an infinite rank torsion-free abelian group with a computable basis B. For every set D, there is a basis B_D of $\mathcal{H}$ such that $deg(B_D) = deg(D)$.*

Proof. Let $B = \{b_0, b_1, \dots\}$ be effectively listed such that $b_i <_{\mathbb{N}} b_{i+1}$. Fix a set D. Let $B_D = \{n_0 b_0, n_1 b_1, \dots\}$ where the $n_i \in \mathbb{N}$ are chosen so that $n_i b_i <_{\mathbb{N}} n_{i+1} b_{i+1}$ and n_i is even if and only if $i \in D$. It is clear that B_D is a basis for $\mathcal{H}$ and that $B_D \leq_T D$. To compute D from B_D, let $B_D = \{c_0, c_1, \dots\}$ be listed in increasing order. For each i, we can find c_i effectively in B_D, and then we can effectively (with no oracle) find b_i and n_i such that $c_i = n_i b_i$. By testing whether n_i is even or odd, we can determine whether $i \in D$. □

In Section 2, we present background algebraic information. In Section 3, we give the proof of Theorem 1.5. In Section 4, we state some generalizations of our results, present some related open questions, and finish with remarks concerning the following general question.

Question 1.7. Describe the possible degree spectra of orders $\mathbb{X}(\mathcal{G})$ on a computable presentation $\mathcal{G}$ of a computable torsion-free abelian group.

Our notation is mostly standard. In particular we use the following convention from the study of linear orders: If $\leq_{\mathcal{G}}$ is a linear order on $\mathcal{G}$, then $\leq^*_{\mathcal{G}}$ denotes the linear order defined by $x \leq^*_{\mathcal{G}} y$ if and only if $y \leq_{\mathcal{G}} x$. Note that if $(\mathcal{G}; \leq_{\mathcal{G}})$ is an ordered abelian group, then $(\mathcal{G}; \leq^*_{\mathcal{G}})$ is also an ordered group.

2. Algebraic background

In our proof of Theorem 1.5, we will need two facts from abelian group theory. The first fact is that every computable rank one group can be effectively embedded into the rationals. To define this embedding for a rank one $\mathcal{H}$, fix any nonzero element $h \in H$. Every nonzero element $g \in H$ satisfies a unique equation of the form $nh = mg$ where $n \in \mathbb{N}$, $m \in \mathbb{Z}$, $n, m \neq 0$, and $\gcd(n, m) = 1$. Map $\mathcal{H}$ into $\mathbb{Q}$ by sending $0_{\mathcal{H}}$ to $0_{\mathbb{Q}}$, sending h to $1_{\mathbb{Q}}$, and sending g satisfying $nh = mg$ (with constraints as above) to the rational $\frac{n}{m}$. Because this map is effective, the image of $\mathcal{H}$ in $\mathbb{Q}$ is computably enumerable and hence we can view $\mathcal{H}$ as a computably enumerable subgroup of $\mathbb{Q}$. Although the image need not be computable, it does contain $\mathbb{Z}$ and, more generally, is closed under multiplication by any integer.

If $\mathcal{H} = \oplus_{i\in\omega}\mathcal{H}_i$ is effectively completely decomposable, we can effectively map $\mathcal{H}$ into $\mathbb{Q}^\omega = \oplus_{i\in\omega}\mathbb{Q}$ (with its standard computable presentation) by fixing a nonzero element $h_i \in \mathcal{H}_i$ for each i and mapping $\mathcal{H}_i$ into $\mathbb{Q}$ as above. Therefore, we will often treat $\mathcal{H}$ as a computably enumerable subgroup of $\mathbb{Q}^\omega$, and, in particular, treat elements in each $\mathcal{H}_i$ subgroup as rationals.

The second fact we need is Levi's Theorem (see [19] and [1]) giving classical algebraic invariants for rank one groups called Baer sequences. The Baer sequence of a rank one group is a function of the form $f : \omega \to \omega \cup \{\infty\}$ modulo the equivalence relation $\sim$ defined on such functions by $f \sim g$ if and only if $f(n) \neq g(n)$ for at most finitely many n and only when neither $f(n)$ nor $g(n)$ is equal to ∞.

To define the Baer sequence of a rank one group $\mathcal{H}$, fix a nonzero element $h \in H$ and let $\{p_i\}_{i\in\omega}$ denote the prime numbers in increasing order (later, for notational convenience, we alter the indexing to start with one). For a prime p, we say p^k divides h (in $\mathcal{H}$) if $p^k g = h$ for some $g \in H$. We define the *p-height* of an element h by

$$\mathrm{ht}_p(h) := \begin{cases} k & \text{if } k \text{ is greatest such that } p^k \text{ divides } h, \\ \infty & \text{otherwise, i.e., if } p^k \text{ divides } h \text{ for all } k. \end{cases}$$

The Baer sequence of h is the function $B_{\mathcal{H},h}(n) = \mathrm{ht}_{p_n}(h)$. If $h, \hat{h} \in H$ are nonzero elements, then $B_{\mathcal{H},h} \sim B_{\mathcal{H},\hat{h}}$. The Baer sequence $B_\mathcal{H}$ of the group $\mathcal{H}$ is (any representative of) this equivalent class. Levi's Theorem states that for rank one groups, $\mathcal{H}_0 \cong \mathcal{H}_1$ if and only if $B_{\mathcal{H}_0} \sim B_{\mathcal{H}_1}$.

3. Proof of Theorem 1.5

Fix an effectively completely decomposable group $\mathcal{H} = \oplus_{i\in\omega}\mathcal{H}_i$ as in the statement of Theorem 1.5. We divide the proof into three steps. First, we describe our general method of building the computable copy $\mathcal{G} = (G; +_\mathcal{G}, 0_\mathcal{G})$ which is Δ^0_2-isomorphic to $\mathcal{H}$. Second, we describe how the computable ordering $\leq_\mathcal{G}$ on $\mathcal{G}$ is constructed. (The second computable order on $\mathcal{G}$ is $\leq^*_\mathcal{G}$.) Third, we give the construction of C and the diagonalization process to ensure the only C-computable orders on $\mathcal{G}$ are $\leq_\mathcal{G}$ and $\leq^*_\mathcal{G}$.

Part 1. General Construction of $\mathcal{G}$.
The group $\mathcal{G}$ is constructed in stages, with G_s denoting the finite set of elements in G at the end of stage s. We maintain $G_s \subseteq G_{s+1}$ and let $G := \bigcup_s G_s$. We define a partial binary function $+_s$ on G_s giving the addition facts declared by the end of stage s. To make $\mathcal{G}$ a computable group, we do not change any addition fact once it is declared, so we maintain

$$x +_s y = z \quad \Longrightarrow \quad (\forall t \geq s)\,[x +_t y = z]$$

for all $x, y, z \in G_s$. Furthermore, for any pair of elements $x, y \in G_s$, we ensure the existence of a stage t and an element $z \in G_t$ such that we declare $x +_t y = z$.

To define the addition function, we use an approximation $\{b^s_0, b^s_1, \ldots, b^s_s\} \subseteq G_s$ to an initial segment of our eventual basis for G. During the construction, each approximate basis element b^s_i will be redefined at most finitely often, so each will eventually reach a limit. We let $b_i := \lim_s b^s_i$ denote this limit. If k is an even index then the approximate basis element b_k^s will never be redefined, so although we often use the notation b_k^s (for uniformity), we have $b_k = b_k^s$ for all s. Although $\mathcal{G}$ will not be effectively decomposable, the group $\mathcal{G}$ will decompose classically into a countable direct sum using the basis $B = \{b_0, b_1, b_2, \ldots\}$.

At stage 0, we begin with $G_0 := \{0, 1\}$. We let 0 denote the zero element $0_\mathcal{G}$ and we assign 1 the label b_0^0. We declare $0_\mathcal{G} +_0 0_\mathcal{G} = 0_\mathcal{G}$, $0_\mathcal{G} +_0 b_0^0 = b_0^0$, and $b_0^0 +_0 0_\mathcal{G} = b_0^0$.

More generally, at stage s, each element $g \in G_s$ is assigned a $\mathbb{Q}$-linear sum over the stage s approximate basis of the form

$$q_0^s b_0^s + \cdots + q_n^s b_n^s$$

where $n \leq s$, $q_i^s \in \mathbb{Q}$ for $i \leq n$, and $q_n^s \neq 0$. (Later there will be further restrictions on the values of q_i^s to ensure that $\mathcal{G}$ is isomorphic to $\mathcal{H}$.) This assignment is required to be one-to-one, and the zero element $0_\mathcal{G}$ is always assigned the empty sum. It will often be convenient to extend such a sum by adding more approximate basis elements on the end of the sum with coefficients of zero. We define the partial function $+_s$ on G_s by letting $x +_s y = z$ (for $x, y, z \in G_s$) if the assigned sums for x and y add together to form the assigned sum for z.

For each $i \in \omega$, we fix a nonzero element $h_i \in \mathcal{H}_i$ and embed $\mathcal{H}_i$ into $\mathbb{Q}$ by sending h_i to $1_\mathbb{Q}$ as described in Section 2. We equate $\mathcal{H}_i$ with its image in $\mathbb{Q}$ in the sense of treating elements of $\mathcal{H}_i$ as rationals. In particular, since h_i is mapped to $1_\mathbb{Q}$, if $a \in \mathcal{H}_i$ and $a = qh_i$, we view a as being the rational q.

At each stage s, we maintain positive integers N_i^s for $i \leq s$. These integers restrain the (nonzero) coefficients q_i^s of b_i^s allowed in the $\mathbb{Q}$-linear sum for each element $g \in G_s$ by requiring that $q_i^s N_i^s \in \mathcal{H}_i$ and that we have seen this fact by stage s. Using the fact that $N_i := \lim_s N_i^s$ exists and is finite for all i, we will show (using Levi's Theorem) that in the limit, the i-th component of $\mathcal{G}$ is isomorphic to $\mathcal{H}_i$, and hence that $\mathcal{G}$ is a computable copy of $\mathcal{H}$. (Later we will introduce a basis restraint $K \in \omega$ that will prevent us from changing N_i^s too often.)

During stage $s + 1$, we do one of two things – either we leave our approximate basis unchanged or we add a dependency relation for a single b_ℓ^s for some odd index $\ell \leq s$. The diagonalization process dictates which happens.

Case 1. If we leave the basis unchanged, then we define $b_i^{s+1} := b_i^s$ for all $i \leq s$. For each $g \in G_s$ (viewed as an element of G_{s+1}), we define $q_i^{s+1} := q_i^s$ and assign g the same sum with b_i^{s+1} and q_i^{s+1} in place of b_i^s and q_i^s, respectively. It follows that $x +_{s+1} y = z$ (for $x, y, z \in G_s$) if $x +_s y = z$. We set $N_i^{s+1} := N_i^s$ for all $i \leq s$ and $N_{s+1}^{s+1} := 1$.

We add two new elements to G_{s+1}, labeling the first by b_{s+1}^{s+1} and labeling the second by $q_0^{s+1} b_0^{s+1} + \cdots + q_n^{s+1} b_n^{s+1}$, where $\langle q_0^{s+1}, \ldots, q_n^{s+1} \rangle$ is the first tuple of rationals (under some fixed computable enumeration of all tuples of rationals) we find such that $n \leq s$, $q_n^{s+1} \neq 0$, $q_i^{s+1} N_i^{s+1} \in \mathcal{H}_i$ at stage s for all $i \leq n$, and this sum is not already assigned to any element of G_{s+1}. (We can effectively search for such a tuple.) This completes the description of G_{s+1} in this case.

Case 2. If we redefine the approximate basis element b_ℓ^s (for the sake of diagonalizing) by adding a new dependency relation, then we proceed as follows. We define $b_i^{s+1} := b_i^s$ for all $i \leq s$ with $i \neq \ell$. The diagonalization process will tell us either to set $b_\ell^s = q b_k^{s+1}$ for some rational q, or to set $b_\ell^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$ for some integers m_1 and m_2. (We will specify properties of these integers below.) In either case, the index k will be even and greater than the basis restraint K and $j, k < \ell$. We assign $g \in G_s$ the same sum except we replace each b_i^s by b_i^{s+1} (for $i \leq s$ and $i \neq \ell$) and we replace b_ℓ^s by either qb_k^{s+1} or $m_1 b_j^{s+1} + m_2 b_k^{s+1}$ (as dictated by the diagonalization process).

For example, if the diagonalization process tells us to make $b_\ell^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$, then the sum for $g \in G_s$ changes from

$$q_0^s b_0^s + \cdots q_j^s b_j^s + \cdots + q_k^s b_k^s + \cdots + q_\ell^s b_\ell^s + \cdots + q_s^s b_s^s$$

at stage s (where we have added zero coefficients if necessary) to

$$\begin{aligned} q_0^s b_0^{s+1} + \cdots + q_j^s b_j^{s+1} + \cdots + q_k^s b_k^{s+1} + \cdots + q_\ell^s(m_1 b_j^{s+1} + m_2 b_k^{s+1}) + \cdots + q_s^s b_s^{s+1} \\ = q_0^s b_0^{s+1} + \cdots + (q_j^s + q_\ell^s m_1) b_j^{s+1} + \cdots + (q_k^s + q_\ell^s m_2) b_k^{s+1} + \cdots + q_s^s b_s^{s+1} \end{aligned}$$

at stage $s+1$. Therefore, we set $q_j^{s+1} := q_j^s + q_\ell^s m_1$, $q_k^{s+1} := q_k^s + q_\ell^s m_2$, and $q_\ell^{s+1} := 0$, while leaving $q_i^{s+1} := q_i^s$ for all $i \notin \{j, k, \ell\}$. Similarly, if the diagonalization process tells us to make $b_\ell^s = qb_k^{s+1}$, then we set $q_k^{s+1} := q_k^s + qq_\ell^s$ and $q_\ell^{s+1} := 0$ while leaving $q_i^{s+1} = q_i^s$ for all $i \notin \{k, \ell\}$.

We define N_i^{s+1}, for $i \leq s$, as follows. If $b_\ell^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$, then $N_i^{s+1} := N_i^s$ for all $i \leq s$. If $b_\ell^s = qb_k^{s+1}$, then $N_i^{s+1} := N_i^s$ for all $i \leq s$ with $i \neq k$ and $N_k^{s+1} := d_q d N_k^s$ where d_q is the denominator of q (when written in lowest terms) and d is the product of all the (finitely many) denominators of coefficients q_ℓ^s for $g \in G_s$. In either case, set $N_{s+1}^{s+1} := 1$.

We add three new elements to G_{s+1}, labeling the first by b_ℓ^{s+1}, labeling the second by b_{s+1}^{s+1}, and labeling the third by $q_0^{s+1} b_0^{s+1} + \cdots + q_n^{s+1} b_n^{s+1}$ where $\langle q_0^{s+1}, \ldots, q_n^{s+1} \rangle$ is the first tuple of rationals we find such that $n \leq s$, $q_n^{s+1} \neq 0$, $q_i^{s+1} N_i^{s+1} \in \mathcal{H}_i$ at stage s for all $i \leq n$, and this sum is not already assigned to any element of G_{s+1}. This completes the description of G_{s+1} in this case.

We note several trivial properties of the transformations of sums in Case 2. First, the approximate basis element b_ℓ^{s+1} does not appear in the new sum for any element of G_s viewed as an element of G_{s+1}. Second, for any element $g \in G_s$, if $q_\ell^s = 0$, then the coefficients q_j^{s+1} and q_k^{s+1} satisfy $q_j^{s+1} = q_j^s$ and $q_k^{s+1} = q_k^s$. Third, by the linearity of the substitutions, if $x +_s y = z$, then $x +_{s+1} y = z$.

We also require two additional properties which place some restrictions on the rational q or the integers m_1 and m_2. The first property is that the assignment of sums to elements of G_s (viewed as elements of G_{s+1}) remains one-to-one. The diagonalization process will place some restrictions on the value of either q or m_1 and m_2, but as long as there are infinitely many possible choices for these values (which we will verify when we describe the diagonalization process), we can assume they are chosen to maintain the one-to-one assignment of sums to elements of G_{s+1}.

The second property is that for each $g \in G_{s+1}$, we need each coefficient q_i^{s+1} to satisfy $q_i^{s+1} N_i^{s+1} \in \mathcal{H}_i$. We will verify this property below under the assumption that when we set $b_\ell^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$, the integers m_1 and m_2 are chosen so that they are divisible by the denominator of each q_ℓ^s coefficient of each $g \in G_s$. (Again, we will verify this property of m_1 and m_2 in the description of the diagonalization process.)

We now check various properties of this construction under these assumptions and the assumption that the limits $b_i := \lim_s b_i^s$ and $N_i := \lim_s N_i^s$ exist for all i (which will be verified in the diagonalization description).

Lemma 3.1. *For $g \in G_s$, the coefficients in the assigned sum $q_0^s b_0^s + \cdots + q_n^s b_n^s$ satisfy $q_i^s N_i^s \in \mathcal{H}_i$.*

Proof. The proof proceeds by induction on s. If g is added at stage s, then the result for g follows trivially. Therefore, fix $g \in G_s$ and assume the condition holds at stage s. Note that if we do not add a dependency relation (i.e., we are in Case 1), then the condition at stage $s+1$ follows immediately. Assume we add a new dependency relation; we split into cases depending on the form of this dependency.

If $b_\ell^s = q b_k^{s+1}$, then for all $i \notin \{k, \ell\}$, the condition holds since $q_i^{s+1} = q_i^s$ and $N_i^{s+1} = N_i^s$. For the index ℓ, we have $q_\ell^{s+1} = 0$ and hence the condition holds trivially. For the index k, we have $q_k^{s+1} = q_k^s + q q_\ell^s$ and $N_k^{s+1} = d_q d N_k^s$. Therefore,

$$q_k^{s+1} N_k^{s+1} = (q_k^s + q q_\ell^s) d_q d N_k^s = q_k^s d_q d N_i^s + q q_\ell^s d_q d N_k^s.$$

Since $q_k^s N_k^s \in \mathcal{H}_k$ and $d_q d \in \mathbb{Z}$, we have $q_k^s d_q d N_k^s \in \mathcal{H}_k$. By definition, $q d_q \in \mathbb{Z}$ and $q_\ell^s d \in \mathbb{Z}$, and hence $q q_\ell^s d_q d N_k^s \in \mathbb{Z} \subseteq \mathcal{H}_k$. Therefore, we have the desired property when $b_\ell^s = q b_k^{s+1}$.

If $b_\ell^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$, then for all $i \notin \{j, k\}$ the condition holds as above. For the index j, we have $q_j^{s+1} = q_j^s + q_\ell^s m_1$ and $N_j^{s+1} = N_j^s$. By assumption, the integer m_1 is divisible by the denominator of q_ℓ^s and hence $q_\ell^s m_1 \in \mathbb{Z}$. Therefore,

$$q_j^{s+1} N_j^{s+1} = (q_j^s + q_\ell^s m_1) N_j^s = q_j^s N_j^s + q_\ell^s m_1 N_j^s \in \mathcal{H}_j$$

since $q_j^s N_j^s \in \mathcal{H}_j$ by the induction hypothesis and $q_\ell^s m_1 N_j^s \in \mathbb{Z}$. The analysis for the index k is identical. □

Let $g \in G$. Suppose there is a stage t such that g is assigned a sum $q_0^t b_0^t + \cdots + q_n^t b_n^t$ that is not later changed in the sense that, for all stages $u \geq t$, the element g is assigned the sum $q_0^u b_0^u + \cdots + q_n^u b_n^u$ with $b_i^u = b_i^t$ and $q_i^u = q_i^t$ for all $i \leq n$. In this case, we refer to this sum as the *limiting sum* for g and denote it by $q_0 b_0 + \cdots + q_n b_n$.

Lemma 3.2 (Basic properties of the construction)**.**

(1) (a) *Each $g \in G$ has a limiting sum with coefficients q_i satisfying $q_i N_i \in \mathcal{H}_i$.*
  (b) *For each rational tuple $\langle q_0, \ldots, q_n \rangle$ such that $q_n \neq 0$ and $q_i N_i \in \mathcal{H}_i$ for all $i \leq n$, there is an element $g \in G$ such that the limiting sum for g is $q_0 b_0 + \cdots + q_n b_n$.*

(2) (a) *If $x +_s y = z$, then $x +_t y = z$ for all $t \geq s$. In particular, if $x +_s y = z$, then the limiting sums for x and y add to form the limiting sum for z.*
  (b) *For each pair $x, y \in G_s$, there is a stage $t \geq s$ and an element $z \in G_t$ such that $x +_t y = z$.*
  (c) *For each $x \in G_s$, there is a stage $t \geq s$ and an element $z \in G_t$ such that $x +_t z = 0_{\mathcal{G}}$.*

Proof. Proof of (1a). When g enters G, it is assigned a sum. The coefficients in this sum only change when a diagonalization occurs. In this case, some approximate basis element b_ℓ^s with nonzero coefficient in the sum for g is made dependent via a relation of the form $b_\ell^s = q b_k^{s+1}$ or $b_\ell^s = m_1 b_j^{s+1} + m_2 b_k^{s+1}$ with $j, k < \ell$. Therefore, each time the sum for g changes, some approximate basis element with nonzero coefficient is replaced by rational multiples of approximate basis elements with lower indices. This process can only occur finitely often before terminating. The last property of the limiting sum follows from Lemma 3.1.

Proof of (1b). For a contradiction, suppose there is a rational tuple violating this lemma. Fix the least such tuple $\langle q_0, \dots, q_n \rangle$ in our fixed computable enumeration of rational tuples. Let $s \geq n$ be a stage such that $b_0^s, \dots, b_n^s$ and $N_0^s, \dots, N_n^s$ have reached their limits, each tuple before $\langle q_0, \dots, q_n \rangle$ which satisfies the conditions in the lemma has appeared as the limiting sum of an element in G_s, and we have seen by stage s that $q_i N_i \in \mathcal{H}_i$ for each $i \leq n$. By our construction, at stage $s+1$, either there is an element that is assigned the sum $q_0 b_0^{s+1} + \cdots + q_n b_n^{s+1}$ or else we add a new element to G_{s+1} and assign it this sum. In either case, this element has the appropriate limiting tuple since $b_0^{s+1}, \dots, b_n^{s+1}$ have reached their limits (and thus we obtain our contradiction).

Proof of (2). Property (2a) follows by induction and the fact that $x +_s y = z$ implies $x +_{s+1} y = z$ at each stage s of the construction. For Property (2b), fixing $x, y \in G_s$, let $u \geq s$ be a stage at which x and y have been assigned their limiting sums

$$x = q_0^u b_0^u + \cdots + q_n^u b_n^u \qquad \text{and} \qquad y = \hat{q}_0^u b_0^u + \cdots + \hat{q}_n^u b_n^u,$$

adding zero coefficients if necessary to make the lengths equal. By Lemma 3.1, for all $t \geq u$ and $i \leq n$, we have that $q_i^t N_i^t \in \mathcal{H}_i$ and $\hat{q}_i^t N_i^t \in \mathcal{H}_i$. Therefore, $(q_i^t + \hat{q}_i^t) N_i^t \in \mathcal{H}_i$. By (1b), there is a stage $t \geq u$ and an element $z \in G_t$ assigned to the sum

$$z = (q_0^t + \hat{q}_0^t) b_0^t + \cdots + (q_n^t + \hat{q}_n^t) b_n^t.$$

Then $x +_t y = z$. The proof of Property (2c) is similar. □

By Properties (1b) and (1a) in Lemma 3.2, the limiting sums of elements of $\mathcal{G}$ are exactly the sums $q_0 b_0 + \cdots + q_n b_n$ with $q_n \neq 0$ and $q_i N_i \in \mathcal{H}_i$ for all $i \leq n$. Using Properties (2a) and (2b) in Lemma 3.2, we define the addition function $+_{\mathcal{G}}$ on $\mathcal{G}$ by putting $x + y = z$ if and only if there is a stage s such that $x +_s y = z$.

Lemma 3.3. *The set $\mathcal{G}$ is a computable copy of $\mathcal{H}$.*

Proof. The domain and addition function on $\mathcal{G}$ are computable. By Property (2c) in Lemma 3.2, every element of $\mathcal{G}$ has an inverse, and it is clear from the construction that the addition operation satisfies the axioms for a torsion-free abelian group.

Let $\mathcal{G}_i$ be the subgroup of $\mathcal{G}$ consisting of all element $g \in G$ with limiting sums of the form $q_i b_i$. Since the limiting sums of elements of $\mathcal{G}$ are exactly the sums of the form $q_0 b_0 + \cdots + q_n b_n$ with $q_n \neq 0$ and $q_i N_i \in \mathcal{H}_i$ for $i \leq n$, it follows that $\mathcal{G} \cong \oplus_{i \in \omega} \mathcal{G}_i$. Therefore, to show that $\mathcal{G} \cong \mathcal{H}$, it suffices to show that $\mathcal{G}_i \cong \mathcal{H}_i$ for every $i \in \omega$.

Fix $i \in \omega$. The group $\mathcal{G}_i$ is a rank one group which is isomorphic to the subgroup of $(\mathbb{Q}, +_{\mathbb{Q}})$ consisting of the rationals q such that $q N_i \in \mathcal{H}_i$. Thus, calculating the Baer sequence for $\mathcal{G}_i$ using the rational $1_{\mathbb{Q}}$, we note that for any prime p_j, $1/p_j^k \in \mathcal{G}_i$ if and only if $N_i / p_j^k \in \mathcal{H}_i$. Therefore, the entries in the Baer sequences for $\mathcal{G}_i$ and $\mathcal{H}_i$ differ only in the values corresponding to the prime divisors of N_i and they differ exactly by the powers of these prime divisors. Therefore, by Levi's Theorem, $\mathcal{G}_i \cong \mathcal{H}_i$. □

Part 2. Defining the Computable Orders on $\mathcal{G}$. We define the computable ordering of $\mathcal{G}$ in stages by specifying a partial binary relation $\leq_s$ on G_s at each stage s. To make the ordering relation computable, we satisfy

$$x \leq_s y \implies (\forall t \geq s)\,[x \leq_t y] \tag{1}$$

for all $x, y \in G_s$. Typically, the relation $\leq_s$ will not describe the ordering between every pair of elements of G_s, but it will have the property that for every pair of elements $x, y \in G_s$, there is a stage $t \geq s$ at which we declare $x \leq_t y$ or $y \leq_t x$, and not both unless $x = y$. Since we will be considering several orderings on $\mathcal{G}$, for an ordering $\preccurlyeq$ on $\mathcal{G}$, we let $(g_1, g_2)_{\preccurlyeq}$ denote the set $\{g \in G \mid g_1 \prec g \prec g_2\}$. Moreover, given $a_1, a_2 \in \mathbb{R}$, we let $(a_1, a_2)_{\leq_{\mathbb{R}}}$ denote the interval $\{a \in \mathbb{R} \mid a_1 <_{\mathbb{R}} a <_{\mathbb{R}} a_2\}$.

To specify the computable order on $\mathcal{G}$, we build a Δ^0_2-map from G into $\mathbb{R}$. (Thus our order will be archimedean.) To describe this order, let $\{p_i\}_{i \geq 1}$ enumerate the prime numbers in increasing order. We map the basis element b_0 to $r_0 = 1_{\mathbb{R}}$. For $i \geq 1$, we will assign (in the limit of our construction) a real number r_i to the basis element b_i such that r_i is a positive rational multiple of $\sqrt{p_i}$. We choose the r_i in this manner so that they are algebraically independent over $\mathbb{Q}$. If the element $g \in G$ is assigned a limiting sum

$$g = q_0 b_0 + \cdots + q_n b_n,$$

then our Δ^0_2-map into $\mathbb{R}$ sends g to the real $q_0 r_0 + \cdots + q_n r_n$. It also sends $0_{\mathcal{G}}$ to 0.

We need to approximate this Δ^0_2-map during the construction. At each stage s, we keep a real number r^s_i as an approximation to r_i, viewing r^s_i as our current target for the image of b_i. The real r^s_0 is always 1 and the real r^s_i is always a positive rational multiple of $\sqrt{p_i}$. Exactly which rational multiple may change during the course of the diagonalization process. However, if k is an even index, then r^s_k will never change.

We could generate a computable order on G_s by mapping G_s into $\mathbb{R}$ using a linear extension of the map sending each b^s_i to r^s_i. However, this would restrict our ability to diagonalize. Therefore, at stage s, we assign each b^s_i (for $i \geq 1$) an interval $(a^s_i, \widehat{a}^s_i)_{\leq_{\mathbb{R}}}$ where a^s_i and $\widehat{a}^s_i$ are positive rationals such that $r^s_i \in (a^s_i, \widehat{a}^s_i)_{\leq_{\mathbb{R}}}$ and $\widehat{a}^s_i - a^s_i \leq 1/2^s$. The image of b^s_i in $\mathbb{R}$ (in the limit) will be contained in this interval.

Because each $x \in G_s$ is assigned a sum describing its relationship to the current approximate basis, we can generate an interval approximating the image of x in $\mathbb{R}$ under the Δ^0_2-map. That is, suppose x is assigned the sum

$$x = q^s_0 b^s_0 + \cdots + q^s_n b^s_n$$

at stage s. The interval constraints on the image of each b^s_i in $\mathbb{R}$ translate into a rational interval constraint on the image of x in $\mathbb{R}$. The endpoints of this constraint can be calculated using the coefficients of the sum for x and the rationals a^s_i and $\widehat{a}^s_i$, with the exact form depending on the signs of the coefficients.

To define $\leq_s$ on G_s at stage s, we look at the interval constraints for each pair of distinct elements $x, y \in G_s$. If the interval constraint for x is disjoint from the interval constraint for y, then we declare $x \leq_s y$ or $y \leq_s x$ depending on which inequality is forced by the constraints. If the interval constraints are not disjoint, then we do not declare any ordering relation between x and y at stage s. Of course, we also declare $x \leq_s x$ for each $x \in G_s$.

To maintain the implication in Equation (1), we will need to check that $x \leq_s y$ implies $x \leq_{s+1} y$. It suffices to ensure that for each $x \in G_s$, the interval constraint for x at stage $s+1$ is contained within the interval constraint for x at stage s.

It will be helpful for us to know that certain approximate basis elements are mapped to elements of $\mathbb{R}$ which are close to $0_{\mathbb{R}}$. Therefore, we will maintain that $0 \le a_k^s \le \widehat{a}_k^s < 1/2^k$ for all stages s and all *even* indices k. (If we worked in a simpler context where each $\mathcal{H}_i = \mathbb{Q}$, or even where each $\mathcal{H}_i \neq \mathbb{Z}$, we could skip this step as any archimedean order on such groups $\mathcal{H}_i$ is dense in $\mathbb{R}$.)

We now describe exactly how r_i^t, a_i^t and $\widehat{a}_i^t$ are defined at each stage t. Recall that at stage $t = 0$, the only elements in G_t are $0_{\mathcal{G}}$ (which is represented by the empty sum and is mapped to $0_{\mathbb{R}}$) and the element represented by b_0^0 (which is mapped to $1_{\mathbb{R}}$). We set $r_0^0 := 1_{\mathbb{R}}$.

At stage $t+1$, the definitions of r_i^{t+1}, a_i^{t+1} and $\widehat{a}_i^{t+1}$ for $i \le t$ depend on whether we add a dependency relation or not. If we do not add a dependency relation, or if i is not an index involved in an added dependency relation, then we define $r_i^{t+1} := r_i^t$ (so we maintain our guess at the target rational multiple of $\sqrt{p_i}$ for b_i) and define a_i^{t+1} and $\widehat{a}_i^{t+1}$ so that

$$(a_i^{t+1}, \widehat{a}_i^{t+1})_{\le_{\mathbb{R}}} \subseteq (a_i^t, \widehat{a}_i^t)_{\le_{\mathbb{R}}}, \quad r_i^{t+1} \in (a_i^{t+1}, \widehat{a}_i^{t+1})_{\le_{\mathbb{R}}}, \quad \text{and} \quad \widehat{a}_i^{t+1} - a_i^{t+1} < 1/2^{t+1}.$$

For the approximate basis element b_{t+1}^{t+1} introduced at this stage, we set r_{t+1}^{t+1} to be a positive rational multiple of $\sqrt{p_{t+1}}$ (requiring $r_{t+1}^{t+1} < 1/2^{t+1}$ if $t+1$ is even) and let a_{t+1}^{t+1} and $\widehat{a}_{t+1}^{t+1}$ be positive rationals so that $r_{t+1}^{t+1} \in (a_{t+1}^{t+1}, \widehat{a}_{t+1}^{t+1})_{\le_{\mathbb{R}}}$ and $\widehat{a}_{t+1}^{t+1} - a_{t+1}^{t+1} < 1/2^{t+1}$ (and also $\widehat{a}_{t+1}^{t+1} < 1/2^{t+1}$ if $t+1$ is even). The diagonalization process may place some requirements on the rational multiple of $\sqrt{p_{t+1}}$ chosen. It remains to handle the indices involved in a dependency relation of the form $b_\ell^t = qb_k^{t+1}$ or $b_\ell^t = m_1 b_j^{t+1} - m_2 b_k^{t+1}$. In either case ℓ will be odd and we define $r_\ell^{t+1} := \sqrt{p_\ell}$ and $a_\ell^{t+1}, \widehat{a}_\ell^{t+1} \in \mathbb{Q}^+$ such that $r_\ell^{t+1} \in (a_\ell^{t+1}, \widehat{a}_\ell^{t+1})_{\le\mathbb{R}}$ and $\widehat{a}_\ell^{t+1} - a_\ell^{t+1} < 1/2^{t+1}$.

For the other indices involved in an added dependency relation, we split into cases depending on the type of relation added.

(1) If we add a dependency of the form $b_\ell^t = qb_k^{t+1}$, then we set $r_k^{t+1} := r_k^t$. The action of the diagonalization strategy will ensure that we can choose $a_k^{t+1}, \widehat{a}_k^{t+1} \in \mathbb{Q}^+$ such that $(a_k^{t+1}, \widehat{a}_k^{t+1})_{\le_{\mathbb{R}}} \subseteq (a_k^t, \widehat{a}_k^t)_{\le_{\mathbb{R}}}$, $\widehat{a}_k^{t+1} - a_k^{t+1} < 1/2^{t+1}$ and

$$(qa_k^{t+1}, q\widehat{a}_k^{t+1})_{\le_{\mathbb{R}}} \subseteq (a_\ell^t, \widehat{a}_\ell^t)_{\le_{\mathbb{R}}} \tag{2}$$

(2) If we add a dependency of the form $b_\ell^t = m_1 b_j^{t+1} - m_2 b_k^{t+1}$, then we set $r_j^{t+1} := r_j^t$ and $r_k^{t+1} := r_k^t$. We will be in one of two contexts.

2(a). If we are in a context in which (in $\mathbb{R}$)

$$0 < na_k^t < n\widehat{a}_k^t < a_\ell^t < \widehat{a}_\ell^t < a_j^t < \widehat{a}_j^t < (n+1)a_k^t < (n+1)\widehat{a}_k^t, \tag{3}$$

then we will choose $m_1, m_2 \in \mathbb{N}$ such that $m_1 \le m_2/n$ and

$$(m_1 a_j^{t+1} - m_2 \widehat{a}_k^{t+1}, m_1 \widehat{a}_j^{t+1} - m_2 a_k^{t+1})_{\le_{\mathbb{R}}} \subseteq (a_\ell^t, \widehat{a_\ell}^t)_{\le_{\mathbb{R}}}. \tag{4}$$

2(b). If we are in a context in which (in $\mathbb{R}$)

$$0 < na_k^t < n\widehat{a}_k^t < a_j^t < \widehat{a}_j^t < a_\ell^t < \widehat{a}_\ell^t < (n+1)a_k^t < (n+1)\widehat{a}_k^t, \tag{5}$$

then we will choose $m_1, m_2 \in \mathbb{N}$ such that $m_1 \le m_2(n+1)$ and

$$(m_1 a_k^{t+1} - m_2 \widehat{a}_j^{t+1}, m_1 \widehat{a}_k^{t+1} - m_2 a_j^{t+1})_{\le_{\mathbb{R}}} \subseteq (a_\ell^t, \widehat{a}_\ell^t)_{\le_{\mathbb{R}}}. \tag{6}$$

By Lemma 3.5 (given below), in each of these contexts, there are infinitely many such choices for m_1 and m_2 satisfying the given conditions. Moreover, we can assume that m_1 and m_2 satisfy the divisibility conditions required by the general group construction.

To explain why appropriate $m_1, m_2 \in \mathbb{N}$ exist for the two contexts above, we rely on the following fact about the reals.

Lemma 3.4. *Let r_1 and r_2 be positive reals that are linearly independent over $\mathbb{Q}$. For any rational numbers $q_1 < q_2$ and any integer $d \geq 1$, there are infinitely many $m_1, m_2 \in \mathbb{N}$ such that $m_1 r_1 - m_2 r_2 \in (q_1, q_2)_{\leq_\mathbb{R}}$ and both m_1 and m_2 are divisible by d.*

Lemma 3.5. *If we are in the context of (3) (respectively (5)), then there are infinitely many choices for m_1 and m_2 that are divisible by any fixed integer $d \geq 1$ and satisfy (4) (respectively (6)).*

Proof. First, suppose we are in the context of (3). We have that b_j^t and b_k^t are currently identified with the rational multiples r_j^t and r_k^t of $\sqrt{p_j}$ and $\sqrt{p_k}$ respectively, so r_j^t and r_k^t are linearly independent over $\mathbb{Q}$. Hence, by Lemma 3.4 (requiring m_1 and m_2 to be divisible by nd where n comes from the context (3) and d comes from the statement of this lemma), there are infinitely many choices of $m_1, m_2 \in \mathbb{N}$ such that $m_1 r_j^t - m_2 r_k^t \in (a_\ell^t, \widehat{a}_\ell^t)_{\leq_\mathbb{R}}$. We let $\tilde{m}_2 := \frac{m_2}{n}$. We can choose $a_j^{t+1}, \widehat{a}_j^{t+1}, a_k^{t+1}, \widehat{a}_k^{t+1} \in \mathbb{Q}$ with $a_j^{t+1} < r_j^t < \widehat{a}_j^{t+1}$ and $a_k^{t+1} < r_k^t < \widehat{a}_k^{t+1}$ satisfying (4) by shrinking the intervals $(a_j^t, \widehat{a}_j^t)_{\leq_\mathbb{R}}$ and $(a_k^t, \widehat{a}_k^t)_{\leq_\mathbb{R}}$ appropriately.

It remains to see why we must have $m_1 \leq \frac{m_2}{n} = \tilde{m}_2$. Suppose $m_1 > \frac{m_2}{n} = \tilde{m}_2$, so $m_1 - 1 \geq \tilde{m}_2$. Then

$$\begin{aligned} m_1 r_j^t - \tilde{m}_2 n r_k^t &= r_j^t + (m_1 - 1) r_j^t - \tilde{m}_2 n r_k^t \\ &\geq r_j^t + \tilde{m}_2 r_j^t - \tilde{m}_2 n r_k^t \\ &= r_j^t + \tilde{m}_2 (r_j^t - n r_k^t) \\ &> r_j^t \end{aligned}$$

because $r_j^t - n r_k^t > 0$ by (3). We have reached a contradiction since $m_1 r_j^t - \tilde{m}_2 n r_k^t \in (a_\ell^t, \widehat{a}_\ell^t)_{\leq_\mathbb{R}}$ and $r_j^t \in (a_j^t, \widehat{a}_j^t)_{\leq_\mathbb{R}}$ but $\widehat{a}_\ell^t < a_j^t$. So, $m_1 \leq \frac{m_2}{n} = \tilde{m}_2$ as desired.

Now suppose we are in the context of (5). Since r_j^t and r_k^t are linearly independent over $\mathbb{Q}$, by Lemma 3.4 (requiring m_1 and m_2 to be divisible by $(n+1)d$) there are infinitely many choices of $m_1, m_2 \in \mathbb{N}$ such that $m_1 r_k^t - m_2 r_j^t \in (a_\ell^t, \widehat{a}_\ell^t)_{\leq_\mathbb{R}}$. We let $\tilde{m}_1 := \frac{m_1}{(n+1)}$. As before, we can choose $a_j^{t+1}, \widehat{a}_j^{t+1}, a_k^{t+1}, \widehat{a}_k^{t+1} \in \mathbb{Q}$ satisfying (6).

It remains to see why $m_1 = \tilde{m}_1(n+1) \leq m_2(n+1)$. Suppose $m_1 = \tilde{m}_1(n+1) > m_2(n+1)$, so $\tilde{m}_1 - 1 \geq m_2$. Then

$$\begin{aligned} m_1 r_k^t - m_2 r_j^t &= \tilde{m}_1 (n+1) r_k^t - m_2 r_j^t \\ &\geq \tilde{m}_1 (n+1) r_k^t - (\tilde{m}_1 - 1) r_j^t \\ &> \tilde{m}_1 (n+1) r_k^t - (\tilde{m}_1 - 1)(n+1) r_k^t \\ &= (n+1) r_k^t. \end{aligned}$$

The first inequality follows because $\tilde{m}_1 - 1 \geq m_2$ and r_j^t is positive, and the second inequality follows because $r_j^t < (n+1) r_k^t$ by (5). We have reached a contradiction since $m_1 r_k^t - m_2 r_j^t \in (a_\ell^t, \widehat{a}_\ell^t)_{\leq_\mathbb{R}}$ but $\widehat{a}_\ell^t < (n+1) a_k^t$. □

We define $\leq_{\mathcal{G}}$ on $\mathcal{G}$ by $x \leq_{\mathcal{G}} y$ if and only if $x \leq_s y$ for some s. We verify that $\leq_{\mathcal{G}}$ is a computable order under the assumptions that each approximate basis element b_i^s eventually reaches a limit and that we choose our intervals and associated rationals in the manner described above.

Lemma 3.6. *The relation $\leq_{\mathcal{G}}$ is a computable order on $\mathcal{G}$. Furthermore, $\mathcal{G}$ is classically isomorphic to an ordered subgroup of $(\mathbb{R}; +, 0_{\mathbb{R}})$ under the standard ordering.*

Proof. We begin by verifying the following properties of the construction.

(1) For every pair of elements $x, y \in G_s$, if $x \leq_s y$, then $x \leq_{s+1} y$.
(2) For each i, the limit $r_i := \lim_s r_i^s$ exists and is a rational multiple of $\sqrt{p_i}$. Furthermore, once r_i^s reaches its limit, the rational intervals $(a_i^t, \widehat{a}_i^t)_{\leq_{\mathbb{R}}}$ for $t \geq s$ form a nested sequence converging to r_i.
(3) For each pair $x, y \in G_s$, there is a stage $t \geq s$ for which either $x \leq_t y$ or $y \leq_t x$.

Proof of (1). It suffices to show that for each $g \in G_s$, the interval constraint for g at stage $s+1$ is contained in the interval constraint for g at stage s. This fact follows from three observations. Fix $g \in G_s$. First, if $q_i^s b_i^s$ occurs in the sum for g at stage s and the index i is not involved in an added dependency relation, then $q_i^{s+1} = q_i^s$ and $(a_i^{s+1}, \widehat{a}_i^{s+1})_{\leq_{\mathbb{R}}} \subseteq (a_i^s, \widehat{a}_i^s)_{\leq_{\mathbb{R}}}$. Therefore, the constraint imposed on g by these terms at stage $s+1$ is contained in the constraint imposed at stage s.

Second, suppose we add a dependency relation of the form $b_\ell^s = q b_k^{s+1}$ and $q_\ell^s \neq 0$. From stage s to stage $s+1$, the $q_k^s b_k^s + q_\ell^s b_\ell^s$ part of the sum for g turns into $(q_k^s + q q_\ell^s) b_k^{s+1} + 0 b_\ell^{s+1}$ where $b_k^{s+1} = b_k^s$. Since the constraint on r_ℓ^{s+1} plays no role in the constraint on g at stage $s+1$ and since we have, by (2), that $(q a_k^{s+1}, q \widehat{a}_k^{s+1})_{\leq_{\mathbb{R}}} \subseteq (a_\ell^s, \widehat{a}_\ell^s)_{\leq_{\mathbb{R}}}$, it follows that the constraint imposed by the indices k and ℓ at stage $s+1$ is contained in the constraint imposed at stage s.

Third, if we add a dependency relation of the form $b_\ell^s = m_1 b_j^{s+1} - m_2 b_k^{s+1}$, then a similar analysis using (4) and (6) yields that the constraint imposed by the indices j, k and ℓ at stage $s+1$ is contained in the constraint imposed at stage s.

Proof of (2). We have $r_i^{s+1} \neq r_i^s$ only when $b_i^{s+1} \neq b_i^s$. Since the latter happens only finitely often, each r_i^s reaches a limit. The remainder of the statement is immediate from the construction.

Proof of (3). Since $x \leq_s x$ for all $x \in G_s$, we consider distinct elements $x, y \in G_s$. Let $t \geq s$ be a stage such that x and y have reached their limiting sums and such that for each b_i^t occurring in these sums, the real r_i^t has reached its limit r_i. Because the reals r_i are algebraically independent over $\mathbb{Q}$ and the nested approximations $(a_i^u, \widehat{a}_i^u)_{\leq_{\mathbb{R}}}$ (for $u \geq t$) converge to r_i, there is a stage at which the interval constraints for x and y are disjoint. At the first such stage, we declare an ordering relation between x and y.

Proof of Lemma. By Statements (1) and (3), $\leq_{\mathcal{G}}$ is computable and every pair of elements is ordered. By construction, the Δ_2^0-map from G to $\mathbb{R}$ that sends

$$q_0 b_0 + q_1 b_1 \cdots + q_n b_n \mapsto q_0 + q_1 r_1 + \cdots + q_n r_n$$

is order preserving. □

Part 3. Building C and Diagonalizing. It remains to show how to use this general construction method to build the ordered group $(\mathcal{G}; \leq_{\mathcal{G}})$ together with a noncomputable c.e. set C such that the only C-computable orders on $\mathcal{G}$ are $\leq_{\mathcal{G}}$ and $\leq_{\mathcal{G}}^*$.

The requirements

$$\mathcal{S}_e : \Phi_e \text{ total} \implies C \neq \Phi_e$$

to make C noncomputable are met in the standard finitary manner. The strategy for $\mathcal{S}_e$ chooses a large witness x, keeps x out of C, and waits for $\Phi_e(x)$ to converge to 0. If this convergence never occurs, the requirement is met because $x \notin C$. If the convergence does occur, then $\mathcal{S}_e$ is met by enumerating x into C and restraining C.

The remaining requirements are

$$\mathcal{R}_e : \text{If } \Phi_e^C(x,y) \text{ is an ordering on } \mathcal{G}, \text{ then } \Phi_e^C \text{ is either } \leq_{\mathcal{G}} \text{ or } \leq_{\mathcal{G}}^*.$$

We explain how to meet a single $\mathcal{R}_e$ in a finitary manner, leaving it to the reader to assemble the complete finite injury construction in the usual manner. After explaining one requirement in isolation, we examine the interaction between $\mathcal{R}_e$ strategies in detail to clarify the finitely nature of the construction.

To simplify the notation, we let $\leq_e^C$ be the binary relation on $\mathcal{G}$ computed by Φ_e^C. We will assume throughout that $\leq_e^C$ never directly violates any of the Π_1^0 conditions in the definition of a group order. For example, if we see at some stage s that $\leq_e^C$ has violated transitivity, then we can place a finite restraint on C to preserve these computations and win $\mathcal{R}_e$ trivially.

The strategy to satisfy $\mathcal{R}_e$ is as follows. For $\mathcal{R}_e$, we set the basis restraint $K := e$. (This restraint is used in the verification that each N_i^s reaches a limit.) If $\leq_{\mathcal{G}} \neq \leq_e^C$ and $\leq_{\mathcal{G}}^* \neq \leq_e^C$, then there must eventually be a stage s, an approximate basis element b_j^s, a nonnegative integer n, and an even index $k > K$ such that:

- we have declared $0 <_s nb_k^s <_s b_j^s <_s (n+1)b_k^s$ in G_s, and
- the order $\leq_e^C$ has declared either (a) $b_k^s >_e^C 0_{\mathcal{G}}$ and either $b_j^s <_e^C nb_k^s$ or $b_j^s >_e^C (n+1)b_k^s$, or (b) $b_k^s <_e^C 0_{\mathcal{G}}$ and either $b_j^s >_e^C nb_k^s$ or $b_j^s <_e^C (n+1)b_k^s$.

We verify such objects exist in Lemma 3.9. In the latter case, we work with the ordering $\leq_e^{C^*}$, transforming the latter case into the former case. We therefore assume that we are in the former case.

While waiting for these witnesses, the construction of $\mathcal{G}$ proceeds as in the general description with no dependencies added. When such s, b_j^s, n, and k are found, we say $\mathcal{R}_e$ is *activated*, and we restrain C to preserve the computations ordering $0_{\mathcal{G}}$, b_j^s, nb_k^s, and $(n+1)b_k^s$.

At stage $s+1$ (without loss of generality, we assume $s+1$ is odd), we order the new approximate basis element b_{s+1}^{s+1} depending on whether $b_j^s <_e^C nb_k^s$ or $b_j^s >_e^C (n+1)b_k^s$. We say that $\mathcal{R}_e$ is *set up to diagonalize* with diagonalization witness b_{s+1}^{s+1}.

(D1) If $b_j^s <_e^C nb_k^s$, we order b_{s+1}^{s+1} so that $nb_k^s <_{s+1} b_{s+1}^{s+1} <_{s+1} b_j^s$, that is, we choose r_{s+1}^{s+1} to be a rational multiple of $\sqrt{p_{s+1}}$ and rationals a_{s+1}^{s+1} and $\widehat{a}_{s+1}^{s+1}$ so that $n\widehat{a}_k^s < a_{s+1}^{s+1} < r_{s+1}^{s+1} < \widehat{a}_{s+1}^{s+1} < a_j^s$ and $\widehat{a}_{s+1}^{s+1} - a_{s+1}^{s+1} < 1/2^{s+1}$.

(D2) If $b_j^s >_e^C (n+1)b_k^s$, we order b_{s+1}^{s+1} so that $b_j^s <_{s+1} b_{s+1}^{s+1} <_{s+1} (n+1)b_k^s$, that is, we choose r_{s+1}^{s+1} to be a rational multiple of $\sqrt{p_{s+1}}$ and rationals a_{s+1}^{s+1} and $\widehat{a}_{s+1}^{s+1}$ so that $\widehat{a}_j^s < a_{s+1}^{s+1} < r_{s+1}^{s+1} < \widehat{a}_{s+1}^{s+1} < (n+1)a_k^s$ and $\widehat{a}_{s+1}^{s+1} - a_{s+1}^{s+1} < 1/2^{s+1}$.

We then wait for a stage $t+1$ so that $\leq_e^C$ declares $b_{s+1}^t <_e^C nb_k^s$ or $nb_k^s <_e^C b_{s+1}^t <_e^C (n+1)b_k^s$ or $b_{s+1}^t >_e^C (n+1)b_k^s$. While waiting, we assume that no higher priority $\mathcal{S}_i$

strategy enumerates a number into C below the restraint and that $b^u_j = b^s_j$, $b^u_k = b^s_k$, and $b^u_{s+1} = b^{s+1}_{s+1}$ at all stages $u \geq s+1$ until $\mathcal{R}_e$ finds such a stage $t+1$ or for all $u \geq s+1$ if $\mathcal{R}_e$ never sees such a stage. (We discuss how to handle $\mathcal{R}_e$ if either of these conditions is violated below when we examine the interaction between strategies.) If these conditions hold, then we say $\mathcal{R}_e$ has been *activated with potentially permanent witnesses.*

We assume that such a stage $t+1$ is found, else $\mathcal{R}_e$ is trivially satisfied. At stage $t+1$, $\mathcal{R}_e$ acts to diagonalize by restraining C to preserve the computations ordering b^t_{s+1}, nb^t_k, and $(n+1)b^t_k$ under $\leq^C_e$ and adding a dependency relation as follows.

Case 1. If $\leq^C_e$ declares $b^t_{s+1} <^C_e nb^t_k$ or $b^t_{s+1} >^C_e (n+1)b^t_k$, then we will add a relation of the form $b^t_{s+1} = qb^{t+1}_k$. Since $nb^t_k <_t b^t_{s+1} <_t (n+1)b^t_k$, we know that

$$nr^t_k <_{\mathbb{R}} a^t_{s+1} <_{\mathbb{R}} \widehat{a}^t_{s+1} <_{\mathbb{R}} (n+1)r^t_k.$$

There are infinitely many rationals $q \in (n, n+1)_{\leq_{\mathbb{R}}}$ such that $qr^t_k \in (a_{s+1}, \widehat{a}^t_{s+1})_{\leq_{\mathbb{R}}}$. For each such q, there are rationals a^{t+1}_k and $\widehat{a}^{t+1}_k$ such that

$$r^{t+1}_k = r^t_k \in (a^{t+1}_k, \widehat{a}^{t+1}_k)_{\leq_{\mathbb{R}}} \subseteq (a^t_k, \widehat{a}^t_k)_{\leq_{\mathbb{R}}},$$

$\widehat{a}^{t+1}_k - a^{t+1}_k \leq_{\mathbb{R}} 1/2^{t+1}$, and $(qa^{t+1}_k, q\widehat{a}^{t+1}_k)_{\leq_{\mathbb{R}}} \subseteq (a^t_{s+1}, \widehat{a}^t_{s+1})_{\leq_{\mathbb{R}}}$. Choose q, a^{t+1}_k, and $\widehat{a}^{t+1}_k$ to be the first rationals meeting these conditions such that the assignment of sums to elements of G_t remains one-to-one.

These choices satisfy the necessary requirements for both the group construction and the ordering construction. Furthermore, we have successfully diagonalized against $\leq^C_e$ being an ordering of $\mathcal{G}$ since any order under which $b^{t+1}_k = b^t_k$ is positive must place b^t_{s+1} between nb^{t+1}_k and $(n+1)b^{t+1}_k$. However, $0_{\mathcal{G}} <^C_e b^{t+1}_k$ and either $b^t_{s+1} <^C_e nb^t_k$ or $b^t_{s+1} >^C_e (n+1)b^t_k$.

Case 2. If $\leq^C_e$ declares $nb^t_k <^C_e b^t_{s+1} <^C_e (n+1)b^t_k$, then we know $0_{\mathcal{G}} <^C_e b^t_{s+1}$ since $0_{\mathcal{G}} <^C_e b^t_k$. We act depending on whether $b^t_{s+1} <_t b^t_j$ or $b^t_{s+1} >_t b^t_j$.

Case 2(a): If $b^t_{s+1} <_t b^t_j$, then it is because we acted in (D1) and hence we know that $b^{t+1}_j <^C_e nb^{t+1}_k$ and we are in the context of Equation (3) with $\ell = s+1$. Let d be the product of all denominators of coefficients q^t_{s+1} for all $g \in G_t$. We declare $b^t_{s+1} = m_1 b^{t+1}_j - m_2 b^{t+1}_k$ for positive integers m_1 and m_2 both divisible by d that satisfy $m_1 \leq_{\mathbb{N}} m_2/n$ and the ordering constraints in Equation (4) and maintain the one-to-one assignment of sums to elements of G_{t+1}. (This choice is possible by Lemma 3.5.)

To see that we have successfully diagonalized, we show that $\leq^C_e$ must violate the order axioms. Since $b^t_{s+1} = m_1 b^{t+1}_j - m_2 b^{t+1}_k$ and $0_{\mathcal{G}} <^C_e b^t_{s+1}, b^{t+1}_k$, we know $0_{\mathcal{G}} <^C_e b^{t+1}_j$. Because $m_1 \leq_{\mathbb{N}} m_2/n$ and $0_{\mathcal{G}} <^C_e b^{t+1}_j$, we have

$$b^t_{s+1} = m_1 b^{t+1}_j - m_2 b^{t+1}_k \leq^C_e (m_2/n) b^{t+1}_j - m_2 b^{t+1}_k.$$

By our case assumption that $b^{t+1}_j <^C_e nb^{t+1}_k$, we get

$$b^t_{s+1} \leq^C_e (m_2/n) b^{t+1}_j - m_2 b^{t+1}_k <^C_e (m_2/n) n b^{t+1}_k - m_2 b^{t+1}_k = 0_{\mathcal{G}}.$$

We have arrived at a contradiction since we have both $0_{\mathcal{G}} <^C_e b^t_{s+1}$ (since we are in Case 2) and $b^t_{s+1} <^C_e 0_{\mathcal{G}}$ by this calculation.

Case 2(b): If $b^t_{s+1} >_t b^t_j$, then it is because we acted in (D2) and hence we know $(n+1)b^{t+1}_k <^C_e b^{t+1}_j$ and we are in the context of Equation (5) with $\ell = s+1$. Let d be as in Case 2(a) and declare $b^t_{s+1} = m_1 b^{t+1}_k - m_2 b^{t+1}_j$ for positive integers m_1 and m_2 both divisible by d that satisfy $m_1 \leq_{\mathbb{N}} m_2(n+1)$ and the ordering constraints in Equation (6) and maintain the one-to-one assignment of sums to elements of G_{t+1} (again by Lemma 3.5.)

We show that $<^C_e$ must violate the order axioms. Since $0_{\mathcal{G}} <^C_e b^{t+1}_k$ and $m_1 \leq_{\mathbb{N}} m_2(n+1)$, we have

$$b^t_{s+1} = m_1 b^{t+1}_k - m_2 b^{t+1}_j \leq^C_e m_2(n+1)b^{t+1}_k - m_2 b^{t+1}_j.$$

By our case assumption that $(n+1)b^{t+1}_k <^C_e b^{t+1}_j$, we have

$$b^t_{s+1} \leq^C_e m_2(n+1)b^{t+1}_k - m_2 b^{t+1}_j <^C_e m_2 b^{t+1}_j - m_2 b^{t+1}_j = 0_{\mathcal{G}}.$$

Again, we have arrived at a contradiction since $0_{\mathcal{G}} <^C_e b^t_{s+1}$ (since we are in Case 2) and $b^t_{s+1} <^C_e 0_{\mathcal{G}}$ (by this calculation).

This completes our description of the action of a single requirement $\mathcal{R}_e$.

In the full construction, we set up priorities between $\mathcal{S}_i$ requirements and $\mathcal{R}_e$ requirements in the usual way. If $i < e$, then $\mathcal{S}_i$ is allowed to enumerate its diagonalizing witness even if it destroys a restraint imposed by $\mathcal{R}_e$, but if $e \leq i$, then $\mathcal{S}_i$ must pick a new large witness when $\mathcal{R}_e$ imposes a restraint.

There is also a potential conflict between different $\mathcal{R}_e$ requirements. Consider requirements $\mathcal{R}_e$ and $\mathcal{R}_i$ involved in the following scenario. Assume that at stage s_0, $\mathcal{R}_i$ is the highest priority activated requirement with witnesses $b^{s_0}_{j_0}$, $b^{s_0}_{k_0}$, and n_0. At stage s_0+1, $\mathcal{R}_i$ sets up to diagonalize with witness $b^{s_0+1}_{s_0+1}$ (via either (D1) or (D2)). At stage $s_1 > s_0$, while $\mathcal{R}_i$ is still waiting to diagonalize, $\mathcal{R}_e$ is activated with witnesses $b^{s_1}_{j_1}$, $b^{s_1}_{k_1}$, and n_1 with $j_1 = s_0 + 1$. Then $\mathcal{R}_e$ sets up to diagonalize with $b^{s_1+1}_{s_1+1}$ at stage $s_1 + 1$.

At stages after s_1+1, $\mathcal{R}_e$ is waiting for $\leq^C_e$ to declare an ordering relation between certain elements (which may never appear) and it needs to maintain $b^u_{j_1} = b^{s_1}_{j_1}$ (which means $b^u_{s_0+1} = b^{s_1}_{s_0+1}$) to remain in a position to diagonalize. On the other hand, when $\mathcal{R}_i$ sees $\leq^C_i$ declare the appropriate order relations, it wants to add a dependency of the form $b^t_{s_0+1} = q b^{t+1}_{k_0}$ or $b^t_{s_0+1} = m_1 b^{t+1}_{j_0} + m_2 b^{t+1}_{k_0}$ which would cause $b^{t+1}_{s_0+1}$ (and hence $b^{t+1}_{j_1}$) to be redefined.

In this scenario, if $e < i$, then when $\mathcal{R}_e$ sets up to diagonalize at stage $s_1 + 1$, it cancels $\mathcal{R}_i$'s claim on the diagonalizing witness $b^{s_0+1}_{s_0+1}$, thus removing the potential conflict. The requirement $\mathcal{R}_i$ remains activated (since the appropriate $\leq^C_i$ computations have been preserved) and at the next odd stage $s_2 + 1$ at which $\mathcal{R}_i$ is the highest priority activated requirement, it will set up to diagonalize with a new witness $b^{s_2+1}_{s_2+1}$.

If $i < e$, then no cancelation of setup witnesses takes place when $\mathcal{R}_e$ sets up to diagonalize. If $\mathcal{R}_e$ acts to diagonalize first, there is no conflict because $\mathcal{R}_e$ adds a dependency relation which causes $b^{t+1}_{s_1+1}$ to be redefined, but leaves $b^{t+1}_{j_1} = b^t_{j_1}$ (and hence $b^{t+1}_{s_0+1} = b^t_{s_0+1}$). If $\mathcal{R}_i$ acts first, then it does cause $b^{t+1}_{s_0+1}$ (and hence $b^{t+1}_{j_1}$) to be redefined, injuring $\mathcal{R}_e$. In this case, the witnesses in the activation for $\mathcal{R}_e$ were not potentially permanent and $\mathcal{R}_e$ is deactivated and has to look for new activating witnesses.

Thus, in the full construction, an $\mathcal{R}_e$ requirement can be injured by a higher priority $\mathcal{S}_i$ requirement (which becomes permanently satisfied) or by a higher priority $\mathcal{R}_i$ requirement (either because $\mathcal{R}_i$ diagonalizes and is permanently satisfied or because $\mathcal{R}_i$ cancels $\mathcal{R}_e$'s diagonalizing witness and $\mathcal{R}_e$ can pick a new diagonalizing witness with the same activation witnesses). Thus, the full construction is finite injury.

To verify the construction succeeds, we show that the limits $\lim_s b_i^s$ and $\lim_s N_i^s$ exist and that if $\leq_e^C$ is an order but is not equal to $\leq_{\mathcal{G}}$ or $\leq_{\mathcal{G}}^*$, then $\mathcal{R}_e$ is eventually activated with potentially permanent witnesses.

Lemma 3.7. *The limit $b_i := \lim_s b_i^s$ exists for all i.*

Proof. The only approximate basis elements which are redefined are those chosen as diagonalizing witnesses by some $\mathcal{R}_e$ requirement. Therefore, at stage $s+1$, if b_{s+1}^{s+1} is not chosen as a diagonalizing witness, then it is never redefined. If b_{s+1}^{s+1} is chosen as a diagonalizing witness by $\mathcal{R}_e$, then it can be redefined at most once when $\mathcal{R}_e$ acts to diagonalize. □

Lemma 3.8. *The limit $N_i := \lim_s N_i^s$ exists for all i.*

Proof. The only time $N_i^{s+1} \neq N_i^s$ is when we add a dependency relation of the form $b_\ell^s = q b_k^{s+1}$ causing $N_k^{s+1} = d_q d N_k^s$. However, in this case, the index k is even and a requirement $\mathcal{R}_e$ can only add such a dependency if $k > K = e$. Therefore, only $\mathcal{R}_e$ with $e < k$ can cause N_k^s to changes value. Since these requirements only act finitely often, the value of N_k^s changes only finitely often. □

Lemma 3.9. *If we fail to find a stage s where $\mathcal{R}_e$ is activated with potentially permanent witnesses, then either $\leq_e^C$ is not an order or $\leq_{\mathcal{G}} = \leq_e^C$ or $\leq_{\mathcal{G}}^* = \leq_e^C$.*

Proof. Assume that $\leq_e^C$ is an order on $\mathcal{G}$. Let s' be a stage such that all higher priority requirements have finished acting by s'. It suffices to show that if we fail to find a stage $s \geq s'$ at which $\mathcal{R}_e$ is activated with some witnesses b_j^s, n, and k, then $\leq_e^C$ is equal to $\leq_{\mathcal{G}}$ or $\leq_{\mathcal{G}}^*$.

First, we claim that if we fail to find a stage $s' \geq s$ at which $\mathcal{R}_e$ is activated, then either $0_{\mathcal{G}} <_e^C b_j$ for all j or $b_j <_e^C 0_{\mathcal{G}}$ for all j.

To prove this claim, suppose that $\mathcal{R}_e$ is never activated after s' and that j_0 and j_1 are indices with $b_{j_1} <_e^C 0_{\mathcal{G}} <_e^C b_{j_0}$. Fix a stage $s \geq s'$ such that $b_{j_1}^s = b_{j_1}$, $b_{j_0}^s = b_{j_0}$ and $b_{j_1} <_e^C 0_{\mathcal{G}} <_e^C b_{j_0}$ is permanently fixed by stage s. Consider a stage $t \geq s$ and an even index k greater than the basis restraint for $\mathcal{R}_e$ such that $b_k^t = b_k$ has reached its limit and there are $n_0, n_1 \in \omega$ for which

$$0_{\mathcal{G}} <_t n_0 b_k^t <_t b_{j_0}^t <_t (n_0+1) b_k^t \qquad \text{and} \qquad 0_{\mathcal{G}} <_t n_1 b_k^t <_t b_{j_1} <_t (n_1+1) b_k^t.$$

Since $\leq_e^C$ is an order, there must be a stage $u \geq t$ at which it declares either $0_{\mathcal{G}} <_e^C b_k^u$ or $b_k^u <_e^C 0_{\mathcal{G}}$ permanently.

If $0_{\mathcal{G}} <_e^C b_k^u$, then we must eventually see $b_{j_1}^v <_e^C 0_{\mathcal{G}} <_e^C n_1 b_k^v$ for some $v \geq u$. Therefore, $\mathcal{R}_e$ is activated at stage v (with $j = j_1$, $k = k$, and $n = n_1$) for the desired contradiction. Alternately, if $b_k^u <_e^C 0_{\mathcal{G}}$, then we must eventually see $n_0 b_k^v <_e^C 0_{\mathcal{G}} <_e^C b_{j_0}^v$ for some $v \geq u$. Again, $\mathcal{R}_e$ is activated at stage v (with $j = j_0$, $k = k$, and $n = n_0$) for the desired contradiction. This completes the proof of the claim.

To complete the proof of this lemma, assume that $\mathcal{R}_e$ is never activated after s' and $0_{\mathcal{G}} <_e^C b_j$ for all j. We show that $\leq_e^C = \leq_{\mathcal{G}}$. It follows by a similar argument that if $b_j <_e^C 0_{\mathcal{G}}$ for all j, then $\leq_e^C = \leq_{\mathcal{G}}^*$.

By construction, $(\mathcal{G}; +_\mathcal{G}, 0_\mathcal{G}, \leq_\mathcal{G})$ can be embedded (as an ordered group) into $(\mathbb{R}; +_\mathbb{R}, 0_\mathbb{R}, \leq_\mathbb{R})$ by sending each basis element $b_i \in \mathcal{G}$ to $r_i \in \mathbb{R}$. To show that $\leq_\mathcal{G} = \leq_e^C$, it suffices to show that the same map is an ordered group embedding of $(\mathcal{G}; +_\mathcal{G}, 0_\mathcal{G}, \leq_e^C)$ into $(\mathbb{R}; +_\mathbb{R}, 0_\mathbb{R}, \leq_\mathbb{R})$.

For each even index k, we fix $n_{0,k} \in \omega$ such that

$$n_{0,k} b_k \leq_\mathcal{G} b_0 \leq_\mathcal{G} (n_{0,k}+1) b_k.$$

By the construction, this condition is equivalent to $n_{0,k} r_k \leq_\mathbb{R} r_0 \leq_\mathbb{R} (n_{0,k}+1) r_k$. Since k is even, we have $(n_{0,k}+1) r_k - n_{0,k} r_k = r_k \leq 1/2^k$ and hence

$$\lim_{k\to\infty} n_{0,k} r_k = \lim_{k\to\infty} (n_{0,k}+1) r_k = r_0 = 1$$

where the limits (and all limits throughout this lemma) are taken over even indices k. More generally, for each index $i \in \omega$ and each even index k, we fix $n_{i,k} \in \omega$ such that

$$n_{i,k} b_k \leq_\mathcal{G} b_i \leq_\mathcal{G} (n_{i,k}+1) b_k.$$

As above, this condition is equivalent to $n_{i,k} r_k \leq_\mathbb{R} r_i \leq_\mathbb{R} (n_{i,k}+1) r_k$ and we have

$$\lim_{k\to\infty} n_{i,k} r_k = \lim_{k\to\infty} (n_{i,k}+1) r_k = r_i.$$

Combining these limits, we have

$$\lim_{k\to\infty} \frac{n_{i,k}}{n_{0,k}+1} = \lim_{k\to\infty} \frac{n_{i,k} r_k}{(n_{0,k}+1) r_k} = \frac{r_i}{1} = r_i$$

and

$$\lim_{k\to\infty} \frac{n_{i,k}+1}{n_{0,k}} = \lim_{k\to\infty} \frac{(n_{i,k}+1) r_k}{n_{0,k} r_k} = \frac{r_i}{1} = r_i.$$

We now translate these results to $(\mathcal{G}, \leq_e^C)$. Because $\mathcal{R}_e$ is never activated after s' and $0_\mathcal{G} <_e^C b_k$ for all even k, the inequalities $n_{i,k} b_k \leq_e^C b_i \leq_e^C (n_{i,k}+1) b_k$ hold for all i and all even k such that k is greater than the basis restraint for $\mathcal{R}_e$. In particular, combining the inequalities $n_{0,k} b_k \leq_e^C b_0 \leq_e^C (n_{0,k}+1) b_k$ and $n_{i,k} b_k \leq_e^C b_i \leq_e^C (n_{i,k}+1) b_k$, we have

$$\frac{n_{i,k}}{n_{0,k}+1} b_0 \leq_e^C b_i \leq_e^C \frac{n_{i,k}+1}{n_{0,k}} b_0$$

where this inequality is interpreted as representing the corresponding inequality after multiplying through by the denominators so all the coefficients are integers. (Alternately, this inequality can be viewed in the divisible closure of $\mathcal{G}$ using the fact that an order on an abelian group has a unique extension to an order on its divisible closure.) The limits above show that the map sending b_i to r_i defines an embedding of $(\mathcal{G}; +_\mathcal{G}, 0_\mathcal{G}, \leq_e^C)$ into $(\mathbb{R}; +_\mathbb{R}, 0_\mathbb{R}, \leq_\mathbb{R})$ as required. □

4. Remarks and Open Questions

Since the construction of the presentation $\mathcal{G}$ and the set C is a typical finite injury construction, certain modifications to the constructions are straightforward.

Remark 4.1. Rather than building $\mathcal{G}$ so that there are exactly two computable orders, it is an easy modification to build exactly any even number or an infinite number of computable orders (with no other C-computable orders).

For example, to build $\mathcal{G}$ with four computable orders, we double the number of $\mathcal{R}_e$ requirements. We build a computable order $\leq_\mathcal{G}^0$ in which $0 <_\mathcal{G}^0 b_0 <_\mathcal{G}^0 b_1$ and a computable order $\leq^1_{\mathcal{G}}$ in which $0 <^1_{\mathcal{G}} b_1 <^1_{\mathcal{G}} b_0$. For each of these orders, we meet a slightly modified requirement for $i \in \{0, 1\}$:

$$\mathcal{R}^i_e\text{: If } \Phi^C_e \text{ is an ordering of } \mathcal{G}\text{, then } 0_{\mathcal{G}} \leq^C_e b_i \leq^C_e b_{1-i} \text{ implies } \leq^C_e = \leq^i_{\mathcal{G}}$$
$$\text{and } b_{1-i} \leq^C_e b_i \leq^C_e 0_{\mathcal{G}} \text{ implies } \leq^C_e = {\leq^i_{\mathcal{G}}}^*.$$

Note that this requirement suffices because (as shown in Lemma 3.9) if b_0 and b_1 lie on opposite sides of $0_{\mathcal{G}}$ under $\leq^C_e$, then $\mathcal{R}^i_e$ will be activated and the diagonalization will guarantee that Φ^C_e is not an order of $\mathcal{G}$. Since these requirements are still finitary (both restraint and injury) in nature, these combine easily to yield the desired result.

The result in Remark 4.1 contrasts with the classical situation. As mentioned in Section 1, a countable torsion free abelian group admits either two or continuum many orders. More generally, it is possible for a countable (nonabelian) group to admit either a finite number of orders greater than 2 or countably many orders. In the finite case, the number of orders must be even and the best known results are that is possible to have exactly $4n$ or $2(4n + 3)$ many orders (see [17] and [21]). It is an open question whether it is possible to get exactly $2n$ number of orders for each n.

Remark 4.2. We note that the computably enumerable set C cannot be complete. The reason is that $\mathbf{0}'$ can compute a basis for any computable torsion-free abelian group $\mathcal{G}$, and hence $\mathcal{G}$ has orders of degree $\mathbf{0}'$.

We also note that, as long as the construction remains finitary (both restraint and injury), additional requirements on C can be added. For example, lowness requirements could be added, though this would be counter-productive (the weaker C is computationally, the weaker the result).

Though making C computationally weak is counter-productive, we ask if it is possible to make C computationally strong.

Question 4.3. Can the set C in Theorem 1.5 have high degree?

Question 4.4. Does Theorem 1.5 remain true when $\mathcal{G}$ is allowed to be an arbitrary computable torsion-free abelian group?

We end with a result concerning the general project of understanding the possible degree spectra of orders on computable torsion-free abelian groups.

Proposition 4.5 (With Daniel Turetsky)**.** *If $\mathcal{G}$ is a computable presentation of a torsion-free abelian group with infinite rank, then deg($\mathbb{X}(\mathcal{G})$) contains infinitely many low degrees.*

Proof. We inductively show deg($\mathbb{X}(\mathcal{G})$) must contain at least n-many low degrees for all n. Fix two linearly independent elements $g, h \in G$ and let T_0 be a computable tree such that $[T_0]$ (the set of infinite paths through T_0) contains exactly the orders $\leq_{\mathcal{G}}$ on $\mathcal{G}$ satisfying

$$0_{\mathcal{G}} <_{\mathcal{G}} g <_{\mathcal{G}} h <_{\mathcal{G}} 4g.$$

Note that the set of orders on $\mathcal{G}$ satisfying this constraint is a Π^0_1 class and hence can be represented in this manner. The Low Basis Theorem applied to T_0 yields a low order of some degree $\mathbf{d}_0$. To get a second order of low degree $\mathbf{d}_1 \neq \mathbf{d}_0$, it suffices (as low over low is low) to build a nonempty $\mathbf{d}_0$-computable subtree T_1 of T_0

having no $\mathbf{d}_0$-computable paths. From this, we obtain a low (low over $\mathbf{d}_0$) order of some degree $\mathbf{d}_1$ not computable from $\mathbf{d}_0$.

The subset T_1 of T_0 is constructed (using an oracle of degree $\mathbf{d}_0$) by killing paths that agree with the e^{th} (candidate) $\mathbf{d}_0$-computable order $\leq_e$ on the relative ordering of g and h for a sufficiently large amount of precision. In particular, to diagonalize against $\leq_e$, we attempt to find positive rationals $q_0 <_{\mathbb{Q}} q_1$ such that $q_1 - q_0 < 2^{-e}$ and $q_0 g <_e h <_e q_1 g$. If and when such rationals are found, we kill initial segments of T_0 that specify $q_0 g <_{\mathcal{G}} h <_{\mathcal{G}} q_1 g$ (if any exist). Notice that $[T_1] \neq \emptyset$ as $\sum_{e=0}^{\infty} 2^{-e} = 2 < 4$ and as for every $q \in (1,4)_{\leq_{\mathbb{R}}}$ and rational $\varepsilon > 0$, there is an order on $\mathcal{G}$ with $(q-\varepsilon)g < h < (q+\varepsilon)g$.

To get a third order of low degree $\mathbf{d}_2 \notin \{\mathbf{d}_0, \mathbf{d}_1\}$, we repeat this process to construct a $(\mathbf{d}_0 \oplus \mathbf{d}_1)$-computable subtree T_2 of T_1 such that T_2 has no $\mathbf{d}_1$-computable paths. We note that T_2 cannot have any $\mathbf{d}_0$-computable paths as it is a subtree of T_1. The only change we need to make is to require the rationals q_0 and q_1 (being used to diagonalize against the e^{th} (candidate) $\mathbf{d}_1$-computable order $\leq_e$) to satisfy $q_1 - q_0 < 2^{-(e+1)}$. Since $\sum_{e=0}^{\infty} 2^{-(e+1)} = 1 < 2$, we guarantee that $[T_2] \neq \emptyset$.

Continuing to repeat this process in the obvious way yields the proposition. □

Note that this proposition also holds for other classes of degrees which form a basis for Π^0_1 classes and relativize in the appropriate manner. For example, $\deg(\mathbb{X}(\mathcal{G}))$ must contain infinitely many hyperimmune-free degrees.

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Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
E-mail address: kach@math.uchicago.edu
URL: http://www.math.uchicago.edu/~kach/

Department of Mathematics, Wellesley College, Wellesley, MA 02481, USA
E-mail address: klange2@wellesley.edu
URL: http://palmer.wellesley.edu/~klange

Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA
E-mail address: david.solomon@uconn.edu
URL: http://www.math.uconn.edu/~solomon