

Separating principles below Ramsey's Theorem for Pairs

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February 4, 2013

1 Introduction

In recent years, there has been a substantial amount of work in reverse mathematics concerning natural mathematical principles that are provable from RT_2^2 , Ramsey's Theorem for Pairs. These principles tend to fall outside of the "big five" systems of reverse mathematics and a complicated picture of subsystems below RT_2^2 has emerged. In this paper, we answer two open questions concerning these subsystems, specifically that ADS is not equivalent to CAC and that EM is not equivalent to RT_2^2 .

We begin with a review of the definitions and known results for the relevant systems below RT_2^2 , but will assume a general familiarity with reverse mathematics. We refer the reader to Simpson [6] for background on reverse mathematics and to Hirschfeldt and Shore [4] for background on the general picture of subsystems below RT_2^2 . Unless otherwise specified, we always work in the base theory RCA_0 .

We will have orderings on a variety of structures, but we typically reserve the symbols $<$ and \leq for three contexts: the usual order on \mathbb{N} , extensions of forcing conditions and comparing sets. If F is a finite set and G is a (finite or infinite) set, we write $F < G$ to denote $\max(F) < \min(G)$. Without loss of generality, we assume that the infinite algebraic structures defined below have domain \mathbb{N} .

Definition 1.1. A *2-coloring* of $[\mathbb{N}]^2$ (or simply a *coloring*), where $[\mathbb{N}]^2$ denotes the set of all two element subsets of \mathbb{N} , is a function $c : [\mathbb{N}]^2 \rightarrow \{0, 1\}$. A set $H \subseteq \mathbb{N}$ is *homogeneous* for c if c is constant on $[H]^2$.

(RT_2^2) Ramsey's Theorem for Pairs: Every 2-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.

We refer to an infinite homogeneous set for a coloring c as a *solution* to c . We typically write $c(x, y)$, as opposed to $c(\{x, y\})$, with implicit assumption that $x < y$.

Definition 1.2. Let c be a 2-coloring of $[\mathbb{N}]^2$. Define $A^*(c)$ (respectively $B^*(c)$) to be the set of numbers which are colored 0 (respectively 1) with all but finitely many other numbers.

$$\begin{aligned} A^*(c) &= \{n \mid \exists x \forall y > x (c(n, y) = 0)\} \\ B^*(c) &= \{n \mid \exists x \forall y > x (c(n, y) = 1)\} \end{aligned}$$

The coloring c is *stable* if $A^*(c) \cup B^*(c) = \mathbb{N}$.

(SRT₂²) Stable Ramsey's Theorem for Pairs: Every stable 2-coloring of $[\mathbb{N}]^2$ has an infinite homogeneous set.

Chong, Slaman and Yang [3] have recently shown that SRT₂² is strictly weaker than RT₂².

Definition 1.3. Let $M = (\mathbb{N}, \preceq_M)$ be a poset. For $x, y \in M$, we say that x and y are *comparable* if either $x \preceq_M y$ or $y \preceq_M x$, and we say x and y are *incomparable* (and write $x \not\preceq_M y$) if $x \not\preceq_M y$ and $y \not\preceq_M x$. $S \subseteq \mathbb{N}$ is a *chain* in M if for all $x, y \in S$, x and y are comparable. S is an *antichain* in M if for all $x \neq y \in S$, x and y are incomparable.

(CAC) Chain-AntiChain: Every infinite poset M contains either an infinite chain or an infinite antichain.

A *solution* to an infinite poset M is an infinite set S such that S is either a chain or an antichain. It is straightforward to show that $\text{RT}_2^2 \vdash \text{CAC}$ by transforming instances of CAC into instances of RT₂². Given a partial order $M = (\mathbb{N}, \preceq_M)$, define the coloring c_M by setting $c_M(x, y) = 0$ if x and y are comparable and setting $c_M(x, y) = 1$ otherwise. If H is an infinite homogeneous set for c_M with color 0, then H is an infinite chain in M . If H is an infinite homogeneous set with color 1, then H is an infinite antichain in M .

Hirschfeldt and Shore [4] showed that one cannot give a similar transformation of instances of RT₂² into instances of CAC by showing that $\text{CAC} \not\vdash \text{SRT}_2^2$.

Definition 1.4. Let $M = (\mathbb{N}, \preceq_M)$ be an infinite partial order. Define

$$\begin{aligned} A^*(M) &= \{n \mid \exists x \forall y > x (n \preceq_M y)\} \\ B^*(M) &= \{n \mid \exists x \forall y > x (n \not\preceq_M y)\} \\ C^*(M) &= \{n \mid \exists x \forall y > x (y \preceq_M n)\} \end{aligned}$$

M is *stable* if either $A^*(M) \cup B^*(M) = \mathbb{N}$ or $C^*(M) \cup B^*(M) = \mathbb{N}$.

(SCAC) Stable Chain-Antichain: Every infinite stable poset M contains either an infinite chain or an infinite anti chain.

When we work with SCAC later, we will construct an infinite poset M such that $A^*(M) \cup B^*(M) = \mathbb{N}$. Thus, our notations for $A^*(M)$ and $B^*(M)$ are chosen to parallel the corresponding notations for SRT₂². Although SRT₂² \vdash SCAC by the transformation given above, Hirschfeldt and Shore [4] showed that SCAC $\not\vdash$ CAC.

Definition 1.5. Let $L = (\mathbb{N}, <_L)$ be an infinite linear order. A function $f : \mathbb{N} \rightarrow L$ is an *infinite ascending sequence* in L if for all $n < m$, $f(n) <_L f(m)$ and is an *infinite descending sequence* in L if for all $n < m$, $f(n) >_L f(m)$.

(ADS) Ascending or Descending Sequence: Every infinite linear order L has an infinite ascending sequence or an infinite descending sequence.

Definition 1.6. An infinite linear order L is *stable* if L has order type $\omega + \omega^*$. That is, for every x , there is a y such that either $\forall z > y (x <_L z)$ or $\forall z > y (z <_L x)$.

(SADS) Stable Ascending or Descending Sequence: Every infinite stable linear order has an infinite ascending sequence or an infinite descending sequence.

A *solution* to an infinite linear order L is a function which is either an infinite ascending sequence or an infinite descending sequence.

As above, one can show $\text{CAC} \vdash \text{ADS}$ by transforming instances of ADS into instances of CAC. Given an infinite linear order $(\mathbb{N}, <_L)$, define an infinite partial order $M = (\mathbb{N}, \preceq_M)$ by $x \preceq_M y$ if and only if $x \leq_L y$ and $x \leq y$. Let $S = \{s_0 < s_1 < \dots\}$ be a solution to M and define $f(n) = s_n$. If S is a chain in M , then f is an ascending chain in L . If S is an antichain in M , then f is a descending chain in L .

Hirschfeldt and Shore [4] proved that $\text{SADS} \not\vdash \text{ADS}$, but left open the question of whether ADS implies CAC or SADS implies SCAC. Our first result answers both of these questions in the negative by separating ADS from SCAC in an ω -model.

Theorem 1.7. *There is a Turing ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that the ω -model (ω, \mathcal{I}) satisfies ADS but not SCAC. Therefore, ADS does not imply SCAC.*

This theorem will be proved in Section 2. Our second result concerns infinite tournaments and the Erdős-Moser Theorem.

Definition 1.8. A *tournament* T on a domain $D \subseteq \mathbb{N}$ is an irreflexive binary relation on D such that for all $x \neq y \in D$, exactly one of $T(x, y)$ or $T(y, x)$ holds. T is *transitive* if for all $x, y, z \in D$, if $T(x, y)$ and $T(y, z)$ hold, then $T(x, z)$ holds.

In keeping with our terminology above, an *infinite tournament* refers to a tournament T with domain \mathbb{N} . An *infinite transitive subtournament* of T (or a *solution* to T) is an infinite set $S \subseteq \mathbb{N}$ such that T restricted to domain S is transitive. The Erdős-Moser Principle states that such solutions always exist.

(EM) Erdős-Moser Principle: Every infinite tournament contains an infinite transitive subtournament.

EM follows from RT_2^2 by transforming instances of EM into instances of RT_2^2 . Let T be an infinite tournament and define the coloring c_T for $x < y$ by $c_T(x, y) = 0$ if $T(x, y)$ holds and $c_T(x, y) = 1$ if $T(y, x)$ holds. Suppose H is an infinite homogeneous set for the color 0. Then, H is transitive in T because for all $x \neq y \in H$, $T(x, y)$ holds if and only if $x < y$. Similarly, if H is homogeneous for the color 1, then H is transitive in T because for all $x \neq y \in H$, $T(x, y)$ holds if and only if $x > y$.

Since computable instances of RT_2^2 have Π_2^0 solutions and have low₂ solutions, it follows from this translation that computable instances of EM also have Π_2^0 solutions and have low₂ solutions. In Section 3, we present a proof due to Kach, Lerman, Solomon and Weber that these bounds are best possible.

Theorem 1.9 (Kach, Lerman, Solomon and Weber). *There is a computable instance of EM with no Δ_2^0 solution, and hence no Σ_2^0 solution or low solution.*

Similar techniques were used by Dzhafarov, Kach, Lerman and Solomon to diagonalize against the existence of hyperimmune-free solutions.

Theorem 1.10 (Dzhafarov, Kach, Lerman and Solomon). *There is a computable instance of EM with no hyperimmune-free solution.*

Formalizing Theorem 1.10 in reverse mathematics, which can be done in $\text{RCA}_0 + B\Sigma_2^0$, gives a lower bound on the strength of EM. Hirschfeldt, Shore and Slaman [5] proved that the following version of the Omitting Types Theorem, denoted OPT, is equivalent to the statement that for every X , there is a function not dominated by any X -recursive function (i.e. there is a degree which is hyperimmune relative to X).

(OPT) Omitting Partial Types: Let T be a complete theory and S be a set of partial types of T . There is a model of T that omits all the nonprincipal types in S .

Hence, EM implies OPT over $\text{RCA}_0 + B\Sigma_2^0$. It remains an open question whether EM implies $B\Sigma_2^0$.

Bovykin and Weiermann [1] showed that one can transform an instance c of RT_2^2 into an instance T_c of EM, but that extracting the solution to c from the solution to T_c requires an application of ADS. To see why ADS might be useful, notice that if S is a transitive subtournament of an infinite tournament T , then T defines a linear order on S .

Theorem 1.11 (Bovykin and Weiermann [1]). *EM + ADS implies RT_2^2 .*

Proof. Fix a coloring $c : [\mathbb{N}]^2 \rightarrow \{0, 1\}$. Define an infinite tournament T_c as follows. $T_c(x, y)$ holds if either $x < y$ and $c(x, y) = 1$ or $y < x$ and $c(y, x) = 0$. Let S be an infinite transitive subtournament of T_c and let \leq_S be the linear order on S induced by T_c . By ADS, let f be an infinite ascending sequence or an infinite descending sequence in (S, \leq_S) . By thinning out f , we can assume that $f(0) < f(1) < f(2) < \dots$ and hence the range of f exists in RCA_0 .

Suppose that f is an ascending sequence in \leq_S . Fix $n < m$. Since $f(n) <_S f(m)$, the relation $T_c(f(n), f(m))$ holds. Because $f(n) < f(m)$ and $T_c(f(n), f(m))$ holds, it follows that $c(f(n), f(m)) = 1$. Therefore, the range of f is homogeneous for c with color 1.

Suppose that f is a descending sequence in \leq_S . Fix $n < m$. Since $f(m) <_S f(n)$, the relation $T_c(f(m), f(n))$ holds. Because $f(n) < f(m)$, it follows that $c(f(n), f(m)) = 0$. Therefore, the range of f is homogeneous for c with color 0. \square

Corollary 1.12. *CAC does not prove EM (and hence ADS does not prove EM either).*

Proof. Suppose for a contradiction that CAC implies EM. Since CAC also proves ADS, it follows from Theorem 1.11 that CAC proves RT_2^2 . However, by Hirschfeldt and Shore [4], CAC does not prove RT_2^2 . \square

Corollary 1.13. *EM implies RT_2^2 if and only if EM implies ADS.*

Proof. This follows immediately from Theorem 1.11 and the fact that RT_2^2 implies ADS. \square

An infinite tournament T is *stable* if for all x , there is a y such that either $T(x, z)$ holds for all $z > y$ or $T(z, x)$ holds for all $z > y$.

(SEM) Stable Erdős-Moser Principle: Every infinite stable tournament contains an infinite transitive subtournament.

Corollary 1.14. SEM + SADS implies SRT_2^2 .

Proof. Let c be a stable coloring and define T_c as in Theorem 1.11. We show that T_c is a stable tournament.

Fix x . Let $y > x$ and $i \in \{0, 1\}$ be such that $c(x, z) = i$ for all $z > y$. Suppose that $i = 0$. For every $z > y$, we have $x < z$ and $c(x, z) = 0$, and hence $T_c(z, x)$ holds. On the other hand, suppose $i = 1$. For all $z > y$, we have $x < z$ and $c(x, z) = 1$, we have $T_c(x, z)$ holds. Therefore, T_c is stable.

By SEM, there is an infinite transitive subtournament S of T_c . The corollary follows once we show that the linear order induced by T_c on S is stable. Fix $x \in S$. Since T_c is stable, there is a $y > x$ such that either $T_c(x, z)$ holds for all $z > y$ (and hence $x <_S z$ for all $z > y$ with $z \in S$) or $T_c(z, x)$ holds for all $z > y$ (and hence $z <_S x$ for all $z > y$ with $z \in S$). Therefore, (S, \leq_S) is a stable linear order and SADS suffices to extract an infinite ascending or descending chain in S . \square

Our second result, to be proved in Section 4, is that EM does not imply SRT_2^2 , and hence the inclusion of ADS in Theorem 1.11 cannot be removed.

Theorem 1.15. *There is a Turing ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that the ω -model (ω, \mathcal{I}) satisfies EM but not SRT_2^2 . Therefore, EM does not imply SRT_2^2 .*

Corollary 1.16. EM does not imply SADS (and hence neither EM nor SEM implies either ADS or SADS).

Proof. Suppose for a contradiction that EM implies SADS. Since EM implies SEM, and SEM + SADS implies SRT_2^2 , we have EM implies SRT_2^2 , contradiction Theorem 1.15. \square

2 ADS does not imply SCAC

2.1 Outline

In this section, we prove Theorem 1.7 by constructing a Turing ideal $\mathcal{I} \subseteq \mathcal{P}(\omega)$ such that $(\omega, \mathcal{I}) \models \text{ADS}$ and \mathcal{I} contains a stable partial order $M = (\mathbb{N}, \preceq_M)$ but does not contain a solution to M . The construction proceeds in two steps; we use a ground forcing to build M followed by an iterated forcing to add solutions to infinite linear orders without adding a solution to M .

Recall that for an infinite poset M , $A^*(M)$ is the set of elements which are below almost every element and $B^*(M)$ is the set of elements which are incomparable with almost every element. In the ground forcing, we specify $A^*(M)$ and $B^*(M)$ as we construct M so that $A^*(M) \cup B^*(M) = \mathbb{N}$ and hence M is stable. We satisfy two types of requirements. First, to ensure that M cannot compute a solution to itself it suffices to ensure that if Φ_e^M is infinite, then $\Phi_e^M(a) = \Phi_e^M(b) = 1$ for some $a \in A^*(M)$ and $b \in B^*(M)$. Since we are defining $A^*(M)$ and $B^*(M)$ as we construct M , these are easy to satisfy. Second, we satisfy ground level requirements which guarantee that requirements for the first level of the iteration forcing are appropriately dense (in a sense defined below).

For the first level of the iteration forcing, we begin with M , $A^*(M)$ and $B^*(M)$ already defined. We fix an index e such that Φ_e^M is an infinite linear order and attempt to add a solution f for Φ_e^M to \mathcal{I} so that $M \oplus f$ does not compute a solution to M . As above, the strategy is to show that if $\Phi_e^{M \oplus f}$ is infinite, then there are elements $a \in A^*(M)$ and $b \in B^*(M)$ such that $\Phi_e^{M \oplus f}(a) = \Phi_e^{M \oplus f}(b) = 1$. However, since $A^*(M)$ and $B^*(M)$ are already defined, implementing this strategy requires using the fact that the ground forcing ensured that requirements for the iterated forcing are appropriately dense. This density will mean that as f is defined, if there are lots of options to force large numbers into $\Phi_e^{M \oplus f}$, then there must be numbers from $A^*(M)$ and $B^*(M)$ in $\Phi_e^{M \oplus f}$. In addition to handling these diagonalization strategies, we need to guarantee that the requirements for the next level of the iteration forcing are appropriately dense.

In the construction below, we explain the iteration forcing first (assuming M , $A^*(M)$ and $B^*(M)$ have already been constructed) because it allows us to introduce the density notions that have to be forced at the ground level. After explaining the iteration forcing, we present the ground forcing to construct M , $A^*(M)$ and $B^*(M)$.

Before starting the construction, we restrict the collection of infinite linear orders for which we need to add solutions to \mathcal{I} .

Definition 2.1. A linear ordering (\mathbb{N}, \prec) is *stable-ish* if there is a non-empty initial segment V which has no maximum under \prec , and such that $\mathbb{N} \setminus V$ is non-empty and has no minimum under \prec .

Note that there is no requirement that the set V be computable from \prec .

Lemma 2.2. *If (\mathbb{N}, \prec) is not stable-ish then there is a solution to (\mathbb{N}, \prec) computable from \prec .*

Proof. Assume (\mathbb{N}, \prec) is not stable-ish. Note that if V is a non-empty initial segment with no maximum element, then V can compute an infinite ascending sequence. Let $a_1 \in V$ be arbitrary. Given a_n , there must be infinitely many elements $x \in V$ such that $a_n \prec x$, so simply search (effectively in V) for such an element and set $a_{n+1} = x$.

If there is a non-empty initial segment V with no maximum, observe that since \prec is not stable-ish, either $\mathbb{N} \setminus V = \emptyset$, in which case V is computable, or $\mathbb{N} \setminus V$ has a minimal element b , in which case $V = \{x \mid x \prec b\}$. In either case, V is computable from \prec , and so there is an infinite ascending sequence computable from \prec .

So suppose there is no such V . Then every non-empty initial segment has a maximum element. Let V be the set of elements with finitely many predecessors. V is either empty or finite, since if V were infinite, it would not have a maximal element. Thus $\mathbb{N} \setminus V$ is computable from \prec , and can have no minimal element. (Any minimal element would have only the finitely many elements of V as predecessors, and would therefore belong to V .) Therefore, by an argument similar to the one above, $\mathbb{N} \setminus V$ contains an infinite descending sequence computable from \prec . \square

We end this subsection by fixing some notation and conventions. If σ and δ are finite strings, then $\sigma \hat{\ } \delta$ denotes the concatenation of σ and δ . We write $\sigma \sqsubseteq \tau$ to denote that σ is an initial segment of τ (i.e. $\tau = \sigma \hat{\ } \delta$ for some string δ). If \prec is a linear order on \mathbb{N} , σ is a

finite sequence which is ascending in \prec and τ is a finite sequence which is descending in \prec , then $\sigma \prec \tau$ means that $\sigma(|\sigma| - 1) \prec \tau(|\tau| - 1)$ (i.e. the last element in σ is strictly below the last element in τ in the \prec order).

For any computation in which part of the oracle is a finite string, for example $\Phi_k^{X \oplus \sigma}$, we follow the standard convention that if $\Phi_k^{X \oplus \sigma}(y)$ converges, then both y and the use of the computation are bounded by $|\sigma|$.

2.2 Iteration Forcing

Assume that we have already used the ground forcing to construct our stable poset (M, \preceq_M) along with $A^*(M)$ and $B^*(M)$. The general context for one step of the iteration forcing will be a fixed set X and an index e meeting the following conditions:

- $M \leq_T X$;
- X does not compute a solution to M ;
- Φ_e^X is the characteristic function for a stable-ish linear order \prec_e^X on \mathbb{N} ; and
- each requirement $\mathcal{K}^{X, A^*(M), B^*(M)}$ is uniformly dense (defined below).

The ground forcing will create this context for $X = M$. Our goal is to find a generic solution G for \prec_e^X (either an infinite ascending or descending sequence) such that $X \oplus G$ does not compute a solution to M and such that for each stable-ish linear order $\prec_{e'}^{X \oplus G}$, the requirements $\mathcal{K}^{X \oplus G, A^*(M), B^*(M)}$ are uniformly dense. We add G to the Turing ideal and note that for any index e' such that $\prec_{e'}^{X \oplus G}$ is a stable-ish linear order, we have created the context for the iteration forcing to continue with $X \oplus G$.

Before giving the specifics of our forcing notion, we describe the basic intuition for constructing a solution G for \prec_e^X while diagonalizing against computing a solution to M from $X \oplus G$. We work with pairs (σ, τ) where σ is a finite ascending sequence in \prec_e^X , τ is a finite descending sequence in \prec_e^X and $\sigma \prec_e^X \tau$. We view this pair as a simultaneous attempt to build an infinite ascending solution and an infinite descending solution to \prec_e^X . The goal is to construct an infinite nested sequence of such pairs (σ_k, τ_k) such that we succeed either with $G = \sigma = \cup \sigma_k$ or with $G = \tau = \cup \tau_k$.

Suppose we have constructed a pair (σ_k, τ_k) . A typical diagonalization requirement is specified by a pair of indices m and n . To meet this requirement, we need to either

- find an ascending sequence σ_{k+1} extending σ_k such that $\sigma_{k+1} \prec_e^X \tau_k$ and there exists a pair of elements $a \in A^*(M)$, $b \in B^*(M)$ such that $\Phi_m^{X \oplus \sigma_{k+1}}(a) = \Phi_m^{X \oplus \sigma_{k+1}}(b) = 1$; or
- find a descending sequence τ_{k+1} extending τ_k such that $\sigma_k \prec_e^X \tau_{k+1}$ and there exists a pair of elements $a \in A^*(M)$, $b \in B^*(M)$ such that $\Phi_n^{X \oplus \tau_{k+1}}(a) = \Phi_n^{X \oplus \tau_{k+1}}(b) = 1$.

That is, we extend our approximation to an ascending solution to \prec_e^X in a manner that diagonalizes or we extend our approximation to a descending solution to \prec_e^X in a manner that diagonalizes. If we can always win on the ascending side, then $G = \cup \sigma_k$ is an infinite

ascending solution to \prec_e^X such that $X \oplus G$ cannot compute a solution to M . Otherwise, there is an index m for which we cannot win on the ascending side. In this case, we must win on the descending side for every index n (when it is paired with m) and hence $G = \cup \tau_k$ is an appropriate infinite descending solution to \prec_e^X .

In general, there is no reason to think we can meet these requirements without some additional information about X . It is the fact that each requirement $\mathcal{K}^{X, A^*(M), B^*(M)}$ is uniformly dense which allows us to meet these requirements. We first focus on formalizing these diagonalization requirements in a general context and then we show why this general context also forces the requirements $\mathcal{K}^{X \oplus G, A^*(M), B^*(M)}$ to be uniformly dense at the next level.

We begin by defining the following sets, each computable from X .

$$\begin{aligned} \mathbb{A}_e^X &= \{\sigma \mid \sigma \text{ is a finite ascending sequence in } \prec_e^X\} \\ \mathbb{D}_e^X &= \{\tau \mid \tau \text{ is a finite descending sequence in } \prec_e^X\} \\ \mathbb{P}_e^X &= \{(\sigma, \tau) \mid \sigma \in \mathbb{A}_e^X \wedge \tau \in \mathbb{D}_e^X \wedge \sigma \prec_e^X \tau\} \end{aligned}$$

\mathbb{P}_e^X is our set of forcing conditions. For $p \in \mathbb{P}_e^X$, we let σ_p and τ_p denote the first and second components of p . For $p, q \in \mathbb{P}_e^X$, we say $q \leq p$ if $\sigma_p \sqsubseteq \sigma_q$ and $\tau_p \sqsubseteq \tau_q$.

To define the generic G , we construct a sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ of conditions $p_n = (\sigma_n, \tau_n) \in \mathbb{P}_e^X$. At the $(n+1)$ st step, we define $p_{n+1} \leq p_n$ to meet the highest priority requirement $\mathcal{K}^{X, A^*(M), B^*(M)}$ which is not yet satisfied. Meeting this requirement will make progress either towards making $\sigma = \cup_n \sigma_n$ our desired infinite ascending solution to \prec_e^X or towards making $\tau = \cup_n \tau_n$ our desired infinite descending solution to \prec_e^X . In the end, we show that one of $G = \sigma$ or $G = \tau$ satisfies all the requirements.

Before defining the requirements, there is one obvious worry we need to address. During this process, we need to avoid taking a step which eliminates either side from being extendible to a solution of \prec_e^X . Because \prec_e^X is stable-ish, we fix a set V for \prec_e^X as in Definition 2.1. We define

$$\mathbb{V}_e^X = \{(\sigma, \tau) \in \mathbb{P}_e^X \mid \sigma \subseteq V \wedge \tau \subseteq \mathbb{N} \setminus V\}.$$

For $(\sigma, \tau) \in \mathbb{V}_e^X$, σ is an initial segment of an increasing solution to \prec_e^X and τ is an initial segment of a decreasing solution to \prec_e^X . Therefore, as long as we choose our generic sequence to lie within \mathbb{V}_e^X , we will never limit either side from being extendible to a solution to \prec_e^X . However, working strictly in \mathbb{V}_e^X has the disadvantage that \mathbb{V}_e^X is not computable from X . We reconcile the advantages of working in \mathbb{P}_e^X (which is computable from X) with working in \mathbb{V}_e^X by using split pairs.

Definition 2.3. A *split pair* below $p = (\sigma_p, \tau_p)$ is a pair of conditions $q_0 = (\sigma_p \widehat{\ } \sigma', \tau_p)$ and $q_1 = (\sigma_p, \tau_p \widehat{\ } \tau')$ such that $\sigma' \prec_e^X \tau'$.

Lemma 2.4. *If $p \in \mathbb{V}_e^X$ and q_0, q_1 is a split pair below p then either $q_0 \in \mathbb{V}_e^X$ or $q_1 \in \mathbb{V}_e^X$.*

Proof. Let $q_0 = (\sigma_p \widehat{\ } \sigma', \tau_p)$ and $q_1 = (\sigma_p, \tau_p \widehat{\ } \tau')$. Suppose $q_0 \notin \mathbb{V}$. Since $\sigma_p \widehat{\ } \sigma' \prec_e^X \tau_p$, it must be that σ' overflows from V into $\mathbb{N} \setminus V$. Therefore, since $\sigma' \prec_e^X \tau'$, $q_1 \in \mathbb{V}$. \square

We will use Lemma 2.4 as follows. Each requirement $\mathcal{K}^{X, A^*(M), B^*(M)}$ will have the property that when we need to meet $\mathcal{K}^{X, A^*(M), B^*(M)}$ below an element p_n in our generic sequence, there

will be a split pair q_0, q_1 (from \mathbb{P}_e^X) below p_n in $\mathcal{K}^{X, A^*(M), B^*(M)}$. Therefore, if $p_n \in \mathbb{V}_e^X$ by induction, then we can meet $\mathcal{K}^{X, A^*(M), B^*(M)}$ within \mathbb{V}_e^X by choosing p_{n+1} to be whichever of q_0 and q_1 is in \mathbb{V}_e^X . Thus, by starting with the empty sequence p_0 (which is in \mathbb{V}_e^X), we can assume that our generic sequence is chosen in \mathbb{V}_e^X .

We have two types of requirements: half requirements and full requirements. For uniformity of presentation, it is easiest to deal with a general definition for the full requirements, although in the end, the only full requirements we need to meet are those made up of a pair of half requirements.

Definition 2.5. We define the following types of requirements and half-requirements.

- A *requirement* is a downward closed set $\mathcal{K}^{X, A^*(M), B^*(M)} \subseteq \mathbb{P}_e^X$ such that

$$\mathcal{K}^{X, A^*(M), B^*(M)} = \{p \in \mathbb{P}_e^X \mid \exists \bar{a} \in A^*(M) \exists \bar{b} \in B^*(M) (K^X(p, \bar{a}, \bar{b}))\}$$

for some relation $K^X(x, \bar{y}, \bar{z})$ computable in X .

- An \mathbb{A} -*side half requirement* is a set $\mathcal{R}^{X, A^*(M), B^*(M)} \subseteq \mathbb{A}_e^X$ which is closed under extensions such that

$$\mathcal{R}^{X, A^*(M), B^*(M)} = \{\sigma \in \mathbb{A}_e^X \mid \exists \bar{a} \in A^*(M) \exists \bar{b} \in B^*(M) (R^X(\sigma, \bar{a}, \bar{b}))\}$$

for some relation $R^X(x, \bar{y}, \bar{z})$ computable in X .

- A \mathbb{D} -*side half requirement* is a set $\mathcal{S}^{X, A^*(M), B^*(M)} \subseteq \mathbb{D}_e^X$ which is closed under extensions such that

$$\mathcal{S}^{X, A^*(M), B^*(M)} = \{\tau \in \mathbb{D}_e^X \mid \exists \bar{a} \in A^*(M) \exists \bar{b} \in B^*(M) (S^X(\tau, \bar{a}, \bar{b}))\}$$

for some relation $S^X(x, \bar{y}, \bar{z})$ computable in X .

- If $\mathcal{R}^{X, A^*(M), B^*(M)}$ is an \mathbb{A} -side half requirement and $\mathcal{S}^{X, A^*(M), B^*(M)}$ is a \mathbb{D} -side half requirement, then $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^{X, A^*(M), B^*(M)}$ is the requirement

$$\mathcal{J}_{\mathcal{R}, \mathcal{S}}^{X, A^*(M), B^*(M)} = \{p \in \mathbb{P}_e^X \mid \sigma_p \in \mathcal{R}^{X, A^*(M), B^*(M)} \vee \tau_p \in \mathcal{S}^{X, A^*(M), B^*(M)}\}.$$

We say $\mathcal{R}^{X, A^*(M), B^*(M)}$ is a half requirement to mean that it is either an \mathbb{A} -side or a \mathbb{D} -side half requirement. Each requirement and half requirement is c.e. in $X \oplus A^*(M) \oplus B^*(M)$ and the dependence on $A^*(M)$ and $B^*(M)$ is positive.

Example 2.6. Fix a pair of indices m and n . The formal version of our basic diagonalization strategy is given by the following half requirements:

$$\mathcal{A}_m^{X, A^*(M), B^*(M)} = \{\sigma \in \mathbb{A}_e^X \mid \exists a \in A^*(M) \exists b \in B^*(M) (\Phi_m^{X \oplus \sigma}(a) = \Phi_m^{X \oplus \sigma}(b) = 1)\},$$

$$\mathcal{D}_n^{X, A^*(M), B^*(M)} = \{\tau \in \mathbb{D}_e^X \mid \exists a \in A^*(M) \exists b \in B^*(M) (\Phi_n^{X \oplus \tau}(a) = \Phi_n^{X \oplus \tau}(b) = 1)\}.$$

These half requirements combine to form the requirement

$$\mathcal{J}_{\mathcal{A}_m, \mathcal{D}_n}^{X, A^*(M), B^*(M)} = \{p \in \mathbb{P}_e^X \mid \sigma_p \in \mathcal{A}_m^{X, A^*(M), B^*(M)} \vee \tau_p \in \mathcal{D}_n^{X, A^*(M), B^*(M)}\}.$$

Notice that if $\sigma \in \mathcal{A}_m^{X, A^*(M), B^*(M)}$ and $\sigma \sqsubseteq G$, then $\Phi_m^{X \oplus G}$ is not a solution to M . Similarly, if $\tau \in \mathcal{D}_n^{X, A^*(M), B^*(M)}$ and $\tau \sqsubseteq G$, then $\Phi_n^{X \oplus G}$ is not a solution to M .

We next describe when an \mathbb{A} -side half requirement $\mathcal{R}^{X,A^*(M),B^*(M)}$ is satisfied by an infinite ascending sequence Λ in \prec_e^X . (With the obvious changes, this description applies to a \mathbb{D} -side half requirement $\mathcal{S}^{X,A^*(M),B^*(M)}$ and an infinite descending sequence Λ .) $\mathcal{R}^{X,A^*(M),B^*(M)}$ is specified by an index i such that

$$\mathcal{R}^{X,A^*(M),B^*(M)} = \{\sigma \in \mathbb{A}_e^X \mid \exists \bar{a} \in A^*(M) \exists \bar{b} \in B^*(M) (\Phi_i^X(\sigma, \bar{a}, \bar{b}) = 1)\}$$

where Φ_i^X is total. For any (typically finite) sets A and B (given by canonical indices), we let

$$\mathcal{R}^{X,A,B} = \{\sigma \in \mathbb{A}_e^X \mid \exists \bar{a} \in A \exists \bar{b} \in B (\Phi_i^X(\sigma, \bar{a}, \bar{b}) = 1)\}.$$

Unlike $\mathcal{R}^{X,A^*(M),B^*(M)}$, the set $\mathcal{R}^{X,A,B}$ is not necessarily closed under extensions. However, for any finite sets A and B , we have $\mathcal{R}^{X,A,B} \leq_T X$.

We write \mathcal{R}^X to indicate the operation mapping A, B to $\mathcal{R}^{X,A,B}$.

Definition 2.7. \mathcal{R}^X is *essential* in Λ if for every n and every x , there is a finite set $A > x$ such that for every y , there is a finite set $B > y$ and an $m > n$ so that $\Lambda \upharpoonright m \in \mathcal{R}^{X,A,B}$. We say the infinite ascending sequence Λ *satisfies* $\mathcal{R}^{X,A^*(M),B^*(M)}$ if either

- \mathcal{R}^X is not essential in Λ , or
- there is an n such that $\Lambda \upharpoonright n \in \mathcal{R}^{X,A^*(M),B^*(M)}$.

Example 2.8. Consider the \mathbb{A} -side diagonalization half requirement

$$\mathcal{A}_m^{X,A^*(M),B^*(M)} = \{\sigma \in \mathbb{A}_e^X \mid \exists a \in A^*(M) \exists b \in B^*(M) (\Phi_m^{X \oplus \sigma}(a) = \Phi_m^{X \oplus \sigma}(b) = 1)\}$$

and an infinite ascending sequence Λ in \prec_e^X . \mathcal{A}_m^X is essential in Λ if and only if $\Phi_m^{X \oplus \Lambda}$ is infinite. Therefore, $\mathcal{A}_m^{X,A^*(M),B^*(M)}$ is satisfied by Λ if and only if either $\Phi_m^{X \oplus \Lambda}$ is finite or there exists $a \in A^*(M)$ and $b \in B^*(M)$ such that $\Phi_m^{X \oplus \Lambda}(a) = \Phi_m^{X \oplus \Lambda}(b) = 1$. In either case, Λ is a solution to \prec_e^X such that $\Phi_m^{X \oplus \Lambda}$ is not a solution to M .

This example does not explain why we need the quantifier alternations in Definition 2.7. This quantifier alternation will be reflected in a similar definition for full requirements and the reason for it will become clear in the ground forcing.

We need similar notions in the context of our (full) requirements. Each requirement $\mathcal{K}^{X,A^*(M),B^*(M)}$ is specified by an index i such that

$$\mathcal{K}^{X,A^*(M),B^*(M)} = \{p \in \mathbb{P}_e^X \mid \exists \bar{a} \in A^*(M) \exists \bar{b} \in B^*(M) (\Phi_i^X(p, \bar{a}, \bar{b}) = 1)\}$$

where Φ_i^X is total. For any (typically finite) sets A and B , we let

$$\mathcal{K}^{X,A,B} = \{p \in \mathbb{P}_e^X \mid \exists \bar{a} \in A \exists \bar{b} \in B (\Phi_i^X(p, \bar{a}, \bar{b}) = 1)\}.$$

As above, the set $\mathcal{K}^{X,A,B}$ need not be downward closed in \mathbb{P}_e^X , but is computable from X when A and B are finite.

Definition 2.9. \mathcal{K}^X is *essential* below $p \in \mathbb{P}_e^X$ if for every x , there is a finite set $A > x$ such that for every y , there is a finite set $B > y$ and a split pair q_0, q_1 below p such that $q_0, q_1 \in \mathcal{K}^{X,A,B}$.

$\mathcal{K}^{X,A^*(M),B^*(M)}$ is *uniformly dense* if whenever \mathcal{K}^X is essential below p , there is a split pair q_0, q_1 below p belonging to $\mathcal{K}^{X,A^*(M),B^*(M)}$.

Example 2.10. Let $\mathcal{J}_{\mathcal{A}_m, \mathcal{D}_n}^{X,A^*(M),B^*(M)}$ be the requirement from Example 2.6 and fix a condition $p = (\sigma_p, \tau_p)$. Let $q_0 = (\sigma_p \widehat{\sigma}, \tau_p)$ and $q_1 = (\sigma_p, \tau_p \widehat{\tau})$ be a split pair below p . For finite sets A and B , $q_0 \in \mathcal{J}_{\mathcal{A}_m, \mathcal{D}_n}^{X,A,B}$ if

$$\exists a \in A \exists b \in B (\Phi_m^{X \oplus \sigma_p \widehat{\sigma}}(a) = \Phi_m^{X \oplus \sigma_p \widehat{\sigma}}(b) = 1 \vee \Phi_n^{X \oplus \tau_p}(a) = \Phi_n^{X \oplus \tau_p}(b) = 1).$$

For $A > |\tau_p|$, the second disjunct cannot occur by our use convention, and hence

$$q_0 \in \mathcal{J}_{\mathcal{A}_m, \mathcal{D}_n}^{X,A,B} \Leftrightarrow \exists a \in A \exists b \in B (\Phi_m^{X \oplus \sigma_p \widehat{\sigma}}(a) = \Phi_m^{X \oplus \sigma_p \widehat{\sigma}}(b) = 1).$$

Similarly, if $B > |\sigma_p|$, then

$$q_1 \in \mathcal{J}_{\mathcal{A}_m, \mathcal{D}_n}^{X,A,B} \Leftrightarrow \exists a \in A \exists b \in B (\Phi_n^{X \oplus \tau_p \widehat{\tau}}(a) = \Phi_n^{X \oplus \tau_p \widehat{\tau}}(b) = 1).$$

Thus the definition of $\mathcal{J}_{\mathcal{A}_m, \mathcal{D}_n}^X$ being essential below p formalizes a notion of “having lots of options to force large numbers into a potential solution to M ”. Informally, the definition of $\mathcal{J}_{\mathcal{A}_m, \mathcal{D}_n}^{X,A^*(M),B^*(M)}$ being uniformly dense says that whenever there are lots of options to force large numbers into a potential solution to M , then there is an extension which forces numbers from both $A^*(M)$ and $B^*(M)$ into the potential solution.

Definition 2.11. We say an infinite sequence $p_0 > p_1 > \dots$ of conditions *satisfies* $\mathcal{K}^{X,A^*(M),B^*(M)}$ if either

- there are cofinitely many p_i such that \mathcal{K}^X is not essential below p_i , or
- there is some $p_n \in \mathcal{K}^{X,A^*(M),B^*(M)}$.

We have now made all the inductive hypotheses on X precise and can give the formal construction of our generic sequence of conditions. Let $\mathcal{K}_n^{X,A^*(M),B^*(M)}$, for $n \in \omega$, be a list of all requirements. (As we will see below, it suffices for this list to consist of all requirements formed from pairs of half requirements.)

Lemma 2.12. *There is a sequence of conditions $p_0 > p_1 > \dots$ from \mathbb{V}_e^X which satisfies every $\mathcal{K}_n^{X,A^*(M),B^*(M)}$.*

Proof. Let $p_0 = (\sigma_0, \tau_0)$ where both σ_0 and τ_0 are the empty sequence and note that $p_0 \in \mathbb{V}_e^X$. Given p_n , let m be the least index such that \mathcal{K}_m^X is essential below p_n and for all $i \leq n$, $p_i \notin \mathcal{K}_m^{X,A^*(M),B^*(M)}$. By assumption $\mathcal{K}_m^{X,A^*(M),B^*(M)}$ is uniformly dense, so we may apply Lemma 2.4 to obtain $p_{n+1} \leq p_n$ such that $p_{n+1} \in \mathcal{K}_m^{X,A^*(M),B^*(M)}$ and $p_{n+1} \in \mathbb{V}_e^X$. \square

It remains to show that for either $G = \sigma = \cup \sigma_n$ or $G = \tau = \cup \tau_n$, G satisfies the necessary inductive conditions: $X \oplus G$ does not compute a solution to M and all requirements $\mathcal{K}^{X \oplus G, A^*(M), B^*(M)}$ are uniformly dense. We do this in two steps. First we explain the connection between satisfying half requirements and satisfying full requirements. Second, we show that the satisfaction of the appropriate half requirements forces these conditions for $X \oplus G$.

Lemma 2.13. *Let $\mathcal{R}^{X, A^*(M), B^*(M)}$ and $\mathcal{S}^{X, A^*(M), B^*(M)}$ be half-requirements and $p_0 > p_1 > \dots$ be an infinite sequence of conditions with $p_n = (\sigma_n, \tau_n)$. Let $\sigma = \cup_i \sigma_i$, $\tau = \cup_i \tau_i$. If \mathcal{R}^X is essential in σ and \mathcal{S}^X is essential in τ , then $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^X$ is essential below every p_n .*

Proof. Fix p_n . To show $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^X$ is essential below every p_n , fix x . Let $A_0 > x$ witness that \mathcal{R}^X is essential in σ and let $A_1 > x$ witness that \mathcal{S}^X is essential in τ . $A_0 \cup A_1$ will be our witness that $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^X$ is essential below p_n .

Fix y . Let $B_0 > y$ witness that \mathcal{R}^X is essential in σ and let $B_1 > y$ witness that \mathcal{S}^X is essential in τ . $B_0 \cup B_1$ will be our witness that $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^X$ is essential below p_n .

Fix $m_0 > n$ such that $\sigma_{m_0} \in \mathcal{R}^{X, A_0, B_0}$ and fix $m_1 > n$ such that $\tau_{m_1} \in \mathcal{S}^{X, A_1, B_1}$. Because the dependence on A_0, A_1, B_0 and B_1 in these sets is positive, it follows that $\sigma_{m_0} \in \mathcal{R}^{X, A_0 \cup A_1, B_0 \cup B_1}$ and $\tau_{m_1} \in \mathcal{S}^{X, A_0 \cup A_1, B_0 \cup B_1}$. Thus the conditions (σ_{m_0}, τ_n) and (σ_n, τ_{m_1}) are in $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^X$ and form a split pair below p_n . \square

Putting these pieces together, we obtain the following:

Lemma 2.14. *Suppose that for each pair of half-requirements $\mathcal{R}^{X, A^*(M), B^*(M)}$ and $\mathcal{S}^{X, A^*(M), B^*(M)}$, the requirement $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^{X, A^*(M), B^*(M)}$ is uniformly dense. Then there is an infinite sequence $(\sigma_0, \tau_0) > (\sigma_1, \tau_1) > \dots$ of conditions such that, setting $\sigma = \cup_i \sigma_i$ and $\tau = \cup_i \tau_i$, either σ satisfies every \mathbb{A} -side half-requirement or τ satisfies every \mathbb{D} -side half-requirement.*

Proof. Let $p_0 > p_1 > \dots$ be chosen as in Lemma 2.12. Since each $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^{X, A^*(M), B^*(M)}$ is uniformly dense, this sequence satisfies every requirement $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^{X, A^*(M), B^*(M)}$. If σ satisfies every half-requirement, we are done. So suppose there is some $\mathcal{R}^{X, A^*(M), B^*(M)}$ not satisfied by σ , and note that \mathcal{R}^X must be essential in σ . We show that τ satisfies every $\mathcal{S}^{X, A^*(M), B^*(M)}$.

Fix $\mathcal{S}^{X, A^*(M), B^*(M)}$ and assume that \mathcal{S}^X is essential in τ (otherwise this half requirement is trivially satisfied). By Lemma 2.13, $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^X$ is essential for every (σ_n, τ_n) , and since the sequence of conditions satisfies $\mathcal{J}_{\mathcal{R}, \mathcal{S}}^{X, A^*(M), B^*(M)}$, there must be some condition $(\sigma_n, \tau_n) \in \mathcal{J}_{\mathcal{R}, \mathcal{S}}^{X, A^*(M), B^*(M)}$. We cannot have $\sigma_n \in \mathcal{R}^{X, A^*(M), B^*(M)}$, since then σ would satisfy $\mathcal{R}^{X, A^*(M), B^*(M)}$, so $\tau_n \in \mathcal{S}^{X, A^*(M), B^*(M)}$. \square

We set $G = \sigma$ if σ satisfies all the \mathbb{A} -side half requirements and we set $G = \tau$ otherwise. By Lemma 2.14, G satisfies every half requirement (on the appropriate side). It remains to show that $X \oplus G$ does not compute a solution to M and that each requirement $\mathcal{K}^{X \oplus G, A^*(M), B^*(M)}$ is uniformly dense. We work under the hypothesis that $G = \sigma$ and hence restrict our attention to \mathbb{A} -side half requirements. The same arguments, with the obvious changes, give the corresponding results if $G = \tau$ working with \mathbb{D} -side half requirements.

Lemma 2.15. *If G satisfies every $\mathcal{A}_m^{X, A^*(M), B^*(M)}$ half requirement, then $X \oplus G$ does not compute a solution to M .*

Proof. Fix an index m . If $\Phi_m^{X \oplus G}$ is finite, then we are done. So, suppose $\Phi_m^{X \oplus G}$ is infinite. We claim that \mathcal{A}_m^X is essential in G . To prove this claim, fix n and x . Let $a_0 > x$ be such that $\Phi_m^{X \oplus G}(a_0) = 1$ and set $A = \{a_0\}$. Fix y , let $b_0 > y$ be such that $\Phi_m^{X \oplus G}(b_0) = 1$ and set $B = \{b_0\}$. Set $n' > n$ be greater than the use of either of these computations. By definition, $G \upharpoonright n' \in \mathcal{A}_m^{X,A,B}$ and hence \mathcal{A}_m^X is essential in G .

Since G satisfies $\mathcal{A}_m^{X,A^*(M),B^*(M)}$, there must be an n'' such that $G \upharpoonright n'' \in \mathcal{A}_m^{X,A^*(M),B^*(M)}$. Therefore, for some $a \in A^*(M)$ and $b \in B^*(M)$, we have $\Phi_m^{X \oplus G}(a) = \Phi_m^{X \oplus G}(b) = 1$, completing the proof. \square

Finally, we show that for every index e' such that $\prec_{e'}^{X \oplus G}$ is a stable-ish linear order, each requirement $\mathcal{K}^{X \oplus G, A^*(M), B^*(M)} \subseteq \mathbb{P}_{e'}^{X \oplus G}$ is uniformly dense. Recall that $\mathcal{K}^{X \oplus G, A^*(M), B^*(M)}$ is specified by an index i such that

$$\mathcal{K}^{X \oplus G, A^*(M), B^*(M)} = \{p \in \mathbb{P}_{e'}^{X \oplus G} \mid \exists \bar{a} \in A^*(M) \exists \bar{b} \in B^*(M) (\Phi_i^{X \oplus G}(p, \bar{a}, \bar{b}) = 1)\}$$

where $\Phi_i^{X \oplus G}$ is total. As we construct G , we do not know which indices e' will result in $\prec_{e'}^{X \oplus G}$ being a stable-ish linear order and, for each such index e' , which indices i will correspond to requirements $\mathcal{K}^{X \oplus G, A^*(M), B^*(M)} \subseteq \mathbb{P}_{e'}^{X \oplus G}$. Therefore, we define the following \mathbb{A} -side half requirements for every pair of indices e' and i . (Of course, we also define the corresponding \mathbb{D} -side half requirements and all proofs that follow work equally well on the \mathbb{D} -side.)

Definition 2.16. Fix $\sigma \in \mathbb{A}_e^X$ and an index e' . For a pair of finite strings $q = (\sigma_q, \tau_q)$, we say $q \in \mathbb{P}_{e'}^{X \oplus \sigma}$ if for all $i < j < |\sigma_q|$, $\sigma_q(i) \prec_{e'}^{X \oplus \sigma} \sigma_q(j)$, for all $i < j < |\tau_q|$, $\tau_q(j) \prec_{e'}^{X \oplus \sigma} \tau_q(i)$ and $\sigma_p(|\sigma_p| - 1) \prec_{e'}^{X \oplus \sigma} \tau_p(|\tau_p| - 1)$.

We say σ forces $q \notin \mathbb{P}_{e'}^{X \oplus \sigma}$ if either there are $i < j < |\sigma_q|$ such that $\sigma_q(j) \preceq_{e'}^{X \oplus \sigma} \sigma_q(i)$ or there are $i < j < |\tau_q|$ such that $\tau_q(i) \preceq_{e'}^{X \oplus \sigma} \tau_q(j)$ or $\tau_p(|\tau_p| - 1) \preceq_{e'}^{X \oplus \sigma} \sigma_p(|\sigma_p| - 1)$. Note that this definition does not match the usual method for forcing the negation of a statement.

By the use convention, $\mathbb{P}_{e'}^{X \oplus \sigma}$ is finite and we can X -computably quantify over this finite set. Furthermore, we can X -computably determine whether σ forces $q \notin \mathbb{P}_{e'}^{X \oplus \sigma}$.

Definition 2.17. For each pair of indices e' and i and each $q = (\sigma_q, \tau_q)$, we define the \mathbb{A} -side half requirement $\mathcal{T}_{e', i, q}^{X, A^*(M), B^*(M)}$ to be the set of all $\sigma \in \mathbb{A}_e^X$ such that either σ forces $q \notin \mathbb{P}_{e'}^{X \oplus \sigma}$ or there exist strings σ' and τ' such that $q_0 = (\sigma_q \widehat{\ } \sigma', \tau_q)$ and $q_1 = (\sigma_q, \tau_q \widehat{\ } \tau')$ satisfy $q_0, q_1 \in \mathbb{P}_{e'}^{X \oplus \sigma}$ and

$$\exists \bar{a}_0, \bar{a}_1 \in A^*(M) \exists \bar{b}_0, \bar{b}_1 \in B^*(M) (\Phi_i^{X \oplus \sigma}(q_0, \bar{a}_0, \bar{b}_0) = \Phi_i^{X \oplus \sigma}(q_1, \bar{a}_1, \bar{b}_1) = 1)$$

(i.e. σ forces the existence of a split pair below q which lies in $\mathcal{K}^{X, A^*(M), B^*(M)}$).

Let G be the generic constructed by our iterated forcing as in Lemma 2.14 and assume $G = \sigma$. Thus, G satisfies every \mathbb{A} -side half requirement $\mathcal{T}_{e', i, q}^{X, A^*(M), B^*(M)}$.

Fix an index e' such that $\prec_{e'}^{X \oplus G}$ is a stable-ish linear order and fix an index i specifying a requirement

$$\mathcal{K}^{X \oplus G, A^*(M), B^*(M)} = \{q \in \mathbb{P}_{e'}^{X \oplus G} \mid \exists \bar{a} \in A^*(M) \exists \bar{b} \in B^*(M) (\Phi_i^{X \oplus G}(q, \bar{a}, \bar{b}) = 1)\}$$

The following lemma (and its \mathbb{D} -side counterpart) complete our verification of the properties of the iteration forcing.

Lemma 2.18. *If G satisfies $\mathcal{T}_{e',i,q}^{X,A^*(M),B^*(M)}$ for every q , then $\mathcal{K}^{X \oplus G, A^*(M), B^*(M)}$ is uniformly dense in $\mathbb{P}_{e'}^{X \oplus G}$.*

Proof. Fix $q \in \mathbb{P}_{e'}^{X \oplus G}$ and assume that $\mathcal{K}^{X \oplus G}$ is essential below q . We claim that $\mathcal{T}_{e',i,q}^X$ is essential in G . Before proving the claim, notice that this claim suffices to prove the lemma. Since G satisfies $\mathcal{T}_{e',i,q}^{X,A^*(M),B^*(M)}$ and $\mathcal{T}_{e',i,q}^X$ is essential in G , there is an n such that $G \upharpoonright n \in \mathcal{T}_{e',i,q}^{X,A^*(M),B^*(M)}$. By the definition of $\mathcal{T}_{e',i,q}^{X,A^*(M),B^*(M)}$, since $q \in \mathbb{P}_{e'}^{X \oplus G}$, there must be a split pair $q_0, q_1 \in \mathbb{P}_{e'}^{X \oplus G \upharpoonright n}$ below q and $\bar{a}_0, \bar{a}_1 \in A^*(M)$ and $\bar{b}_0, \bar{b}_1 \in B^*(M)$ such that $\Phi_i^{X \oplus G \upharpoonright n}(q_0, \bar{a}_0, \bar{b}_0) = \Phi_i^{X \oplus G \upharpoonright n}(q_1, \bar{a}_1, \bar{b}_1) = 1$. Thus q_0, q_1 give the desired split pair below q in $\mathcal{K}^{X \oplus G, A^*(M), B^*(M)}$.

It remains to prove the claim that $\mathcal{T}_{e',i,q}^X$ is essential in G . Fix n and x . Fix $A > x$ witnessing that $\mathcal{K}^{X \oplus G}$ is essential below q . Fix y and let $B > y$ and the split pair q_0, q_1 below q be such that $q_0, q_1 \in \mathcal{K}^{X \oplus G, A, B}$. Thus,

$$\exists \bar{a}_0, \bar{a}_1 \in A \exists \bar{b}_0, \bar{b}_1 \in B (\Phi_i^{X \oplus G}(q_0, \bar{a}_0, \bar{b}_0) = \Phi_i^{X \oplus G}(q_1, \bar{a}_1, \bar{b}_1) = 1).$$

Let $m > n$ be such that m is greater than the uses of these computations and such that $q, q_0, q_1 \in \mathbb{P}_{e'}^{X \oplus G \upharpoonright m}$. Then we have $G \upharpoonright m \in \mathcal{T}_{e',i,q}^{X,A,B}$ as required. \square

2.3 Ground Forcing

In this section, we define the ground forcing to build $(M, A^*(M), B^*(M))$ such that M does not compute a solution to itself (i.e. it does not compute an infinite subset of $A^*(M)$ or $B^*(M)$) and each requirement $\mathcal{K}^{M, A^*(M), B^*(M)}$ is uniformly dense.

Our ground forcing conditions \mathcal{F} are triples (F, A^*, B^*) satisfying

- F is a finite partial order such that $\text{dom}(F)$ is an initial segment of ω and for all $x, y \in \text{dom}(F)$, $x \prec_F y$ implies $x < y$, and
- $A^* \cup B^* \subseteq \text{dom}(F)$, A^* is downwards closed under \prec_F , B^* is upwards closed under \prec_F and $A^* \cap B^* = \emptyset$.

We say $(F, A^*, B^*) \leq (F_0, A_0^*, B_0^*)$ if:

- F extends F_0 as a partial order (i.e. $\text{dom}(F_0) \subseteq \text{dom}(F)$ and for all $x, y \in \text{dom}(F_0)$, $x \preceq_{F_0} y$ if and only if $x \preceq_F y$),
- $A_0^* \subseteq A^*$,
- $B_0^* \subseteq B^*$,
- whenever $a \in A_0^*$ and $x \in \text{dom}(F) \setminus \text{dom}(F_0)$, $a \preceq_F x$,
- whenever $b \in B_0^*$ and $x \in \text{dom}(F) \setminus \text{dom}(F_0)$, $b \not\preceq_F x$ (which implies x is incomparable with b since $b < x$ and hence $x \not\preceq_F b$).

In what follows, we will typically write $x \in M$ rather than $x \in \text{dom}(M)$. We define a generic sequence of conditions $(F_0, A_0^*, B_0^*) > (F_1, A_1^*, B_1^*) > \dots$ and let $M = \cup F_n$. We need to satisfy the following properties:

- (C1) For all i , there is an n such that $i \in A_n^* \cup B_n^*$. (Together with the definitions of our conditions and extensions of conditions, this property guarantees that $A^*(M) = \cup A_n^*$ and $B^*(M) = \cup B_n^*$ and that M is stable.)
- (C2) For all e , if Φ_e^M is infinite, then there are $a \in A^*(M)$ and $b \in B^*(M)$ such that $\Phi_e^M(a) = \Phi_e^M(b) = 1$.
- (C3) If \prec_e^M is a stable-ish linear order and $\mathcal{K}^{M, A^*(M), B^*(M)} \subseteq \mathbb{P}_e^M$ is a requirement (as defined in the previous section), then for all $p \in \mathbb{P}_e^M$, either \mathcal{K}^M is not essential below p or there is a split pair q_0, q_1 below p in $\mathcal{K}^{M, A^*(M), B^*(M)}$.

The next three lemmas show that the appropriate set of conditions forcing these properties are dense. For (C1), we use the following lemma.

Lemma 2.19. *The set of (F, A^*, B^*) such that $i \in A^* \cup B^*$ is dense in \mathcal{F} .*

Proof. Fix (F, A^*, B^*) and $i \in \omega$. Without loss of generality, we assume $i \in F$. If $i \notin A^*$, then $i \not\leq_F a$ for all $a \in A^*$ by the downwards closure of A^* . Let $F_0 = F$, $A_0^* = A^*$ and $B_0^* = B^* \cup \{c \in F \mid i \leq_F c\}$. Then $i \in B_0^*$ and (F_0, A_0^*, B_0^*) extends (F, A^*, B^*) . \square

For (C2), we use the following standard forcing definitions (with G denoting the generic variable). We say $F \Vdash \Phi_e^G$ is finite if

$$\exists k \forall (F_0, A_0^*, B_0^*) \leq (F, A^*, B^*) \forall x (\Phi_e^{F_0}(x) = 1 \rightarrow x \leq k).$$

We say $F \Vdash \Phi_e^G \not\leq A^*(G) \wedge \Phi_e^G \not\leq B^*(G)$ if

$$\exists a \in A^* \exists b \in B^* (\Phi_e^F(a) = 1 \wedge \Phi_e^F(b) = 1).$$

Lemma 2.20. *For each index e , the set of conditions which either force Φ_e^G is finite or force $\Phi_e^G \not\leq A^*(G) \wedge \Phi_e^G \not\leq B^*(G)$ is dense in \mathcal{F} .*

Proof. Fix e and (F, A^*, B^*) and assume that (F, A^*, B^*) has no extension forcing Φ_e^G is finite. Fix $x > F$ and an extension $(F_0, A_0^*, B_0^*) \leq (F, A^*, B^*)$ such that $\Phi_e^{F_0}(x) = 1$. Without loss of generality, we can assume that $A_0^* = A^*$ and $B_0^* = B^*$, so $x \notin A_0^* \cup B_0^*$. By the definition of extensions, we know $b \not\leq_{F_0} x$ for all $b \in B_0^*$. Therefore, the condition (F_1, A_1^*, B_1^*) defined by $F_1 = F_0$, $A_1^* = A_0^* \cup \{c \in F_0 \mid c \leq_{F_0} x\}$ and $B_1^* = B_0^*$ is an extension of (F, A^*, B^*) such that $x \in A_1^*$ and $\Phi_e^{F_1}(a) = 1$.

Since (F_1, A_1^*, B_1^*) does not force Φ_e^G is finite, we can repeat this idea. Fix $y > F_0$ and an extension $(F_2, A_2^*, B_2^*) \leq (F_1, A_1^*, B_1^*)$ such that $\Phi_e^{F_2}(y) = 0$. Again, without loss of generality, we can assume that $A_2^* = A_1^*$ and $B_2^* = B_1^*$, and hence that $y \not\leq_{F_2} a$ for any $a \in A_2^*$. The condition (F_3, A_3^*, B_3^*) defined by $F_3 = F_2$, $A_3^* = A_2^*$ and $B_3^* = B_2^* \cup \{c \in F_2 \mid y \leq_{F_2} c\}$ is an extension of (F, A^*, B^*) forcing $\Phi_e^G \not\leq A^*(G) \wedge \Phi_e^G \not\leq B^*(G)$. \square

We turn to (C3). Fix an index e for a potential stable-ish linear order \prec_e^G . For $p = (\sigma, \tau)$, we say $(F, A^*, B^*) \Vdash p \in \mathbb{P}_e^G$ if σ is a \prec_e^F ascending sequence, τ is a \prec_e^F descending sequence and $\sigma \prec_e^F \tau$. We say $(F, A^*, B^*) \Vdash p \notin \mathbb{P}_e^G$ if no extension of (F, A^*, B^*) forces $p \in \mathbb{P}_e^G$. Obviously, the set of conditions which either force $p \in \mathbb{P}_e^G$ or force $p \notin \mathbb{P}_e^G$ is dense.

Along with the index e , fix an index i for a potential requirement $\mathcal{K}^{G, A^*(G), B^*(G)} \subseteq \mathbb{P}_e^G$. That is, we want to consider the potential requirement

$$\{q \in \mathbb{P}_e^G \mid \exists \bar{a} \in A^*(G) \exists \bar{b} \in B^*(G) (\Phi_i^G(q, \bar{a}, \bar{b}) = 1)\}.$$

Suppose $(F, A^*, B^*) \Vdash p \in \mathbb{P}_e^G$ for $p = (\sigma, \tau)$. We say

$$(F, A^*, B^*) \Vdash \text{there is a split pair } q_0, q_1 \text{ below } p \text{ in } \mathcal{K}^{G, A^*(G), B^*(G)}$$

if there are σ' and τ' such that for $q_0 = (\sigma \frown \sigma', \tau)$ and $q_1 = (\sigma, \tau \frown \tau')$ we have

- $(F, A^*, B^*) \Vdash q_0, q_1 \in \mathbb{P}_e^G$
- $\sigma \frown \sigma' \prec_e^F \tau \frown \tau'$ and
- $\exists \bar{a}_0, \bar{a}_1 \in A^* \exists \bar{b}_0, \bar{b}_1 \in B^* (\Phi_i^F(q_0, \bar{a}_0, \bar{b}_0) = \Phi_i^F(q_1, \bar{a}_1, \bar{b}_1) = 1)$.

Finally, we say that

$$(F, A^*, B^*) \Vdash \mathcal{K}^G \text{ is not essential below } p$$

if for any stable partial order $(\tilde{M}, A^*(\tilde{M}), B^*(\tilde{M}))$ with $\text{dom}(\tilde{M}) = \omega$ extending (F, A^*, B^*) such that $x \prec_{\tilde{M}} y$ implies that $x < y$, $\prec_e^{\tilde{M}}$ is a stable-ish partial order and $\mathcal{K}^{\tilde{M}, A^*(\tilde{M}), B^*(\tilde{M})}$ is a requirement, we have that $\mathcal{K}^{\tilde{M}}$ is not essential below p .

Lemma 2.21. *Fix a pair of indices e and i and let $\mathcal{K}^{G, A^*(G), B^*(G)}$ be the potential requirement specified by these indices. For any p , there is a dense set of (F, A^*, B^*) such that either:*

- $(F, A^*, B^*) \Vdash p \notin \mathbb{P}_e^G$, or
- $(F, A^*, B^*) \Vdash \mathcal{K}^G$ is not essential below p , or
- $(F, A^*, B^*) \Vdash$ there is a split pair below p in $\mathcal{K}^{G, A^*(G), B^*(G)}$.

Proof. Fix (F, A^*, B^*) and $p = (\sigma, \tau)$. If there is any $(F', A', B') \leq (F, A^*, B^*)$ forcing that $p \notin \mathbb{P}_e^G$ then we are done. So assume not, and assume that $(F, A^*, B^*) \Vdash p \in \mathbb{P}_e^G$.

Suppose there is an extension $(F', A^*, B^*) \leq (F, A^*, B^*)$, sets $B_0 > A_0 > A^* \cup B^*$ and a split pair q_0, q_1 below p such that $(F', A^*, B^*) \Vdash q_0, q_1 \in \mathcal{K}^{F', A_0, B_0}$. Let A be the downwards closure of A_0 in F' and B the upwards closure of B_0 in F' .

We claim that A is disjoint from $B \cup B^*$. Fix $x \in A$ and $a \in A_0$ such that $x \preceq_{F'} a$. First, suppose for a contradiction that $x \in B^*$ and hence $x \in F$. If $a \in F$, then $x \preceq_F a$ and hence $a \in B^*$ because B^* is closed upwards in F . But, $a \in A_0$ and $A_0 > B^*$ giving a contradiction. If $a \notin F$, then $a \in F' \setminus F$, so $x \not\preceq_{F'} a$ since $x \in B^*$ and $(F', A^*, B^*) \leq (F, A^*, B^*)$, again giving a contradiction. Therefore, $x \notin B^*$. Second, suppose for a contradiction that $x \in B$. Then

$y \preceq_{F'} x$ for some $y \in B_0$ and hence $y \preceq_{F'} a$. Therefore, $y \leq a$ which contradicts $B_0 > A_0$. Therefore, A is disjoint from $B \cup B^*$.

We also claim that A^* is disjoint from $B \cup B^*$. Fix $x \in A^*$ and note that $x \notin B^*$ since (F, A^*, B^*) is a condition and hence $A^* \cap B^* = \emptyset$. Suppose for a contradiction that $x \in B$. There is a $y \in B_0$ such that $y \preceq_{F'} x$ and hence $y \leq x$, which contradicts $B_0 > A^*$. Therefore, A^* is disjoint from $B \cup B^*$.

Taken together, our claims show that $A \cup A^*$ is disjoint from $B \cup B^*$. Since $A \cup A^*$ is downwards closed and $B \cup B^*$ is upwards closed, $(F', A^* \cup A, B^* \cup B) \leq (F, A^*, B^*)$ is a condition forcing the existence of a split pair below p in $\mathcal{K}^{G, A^*(G), B^*(G)}$.

If there is no such $(F', A^*, B^*) \leq (F, A^*, B^*)$, we claim (F, A^*, B^*) already forces that \mathcal{K}^G is not essential below p : let \tilde{M} be any completion of F to a stable partial ordering satisfying the appropriate conditions from above, and suppose $\mathcal{K}^{\tilde{M}}$ were essential below p . Then in particular, there would be some $A_0 > \max(A^* \cup B^*)$, some $B_0 > \max A_0$, and a split pair q_0, q_1 over p such that $q_0, q_1 \in \mathcal{K}^{\tilde{M}, A_0, B_0}$. But then there would have been some finite restriction $F' = \tilde{M} \upharpoonright [0, m]$ witnessing this, contradicting our assumption. \square

Having verified that any generic for the ground forcing satisfies (C1), (C2) and (C3), we can give the proof of Theorem 1.7.

Proof. We iteratively build a Turing ideal \mathcal{I} containing a partial order M , containing a solution to every infinite linear order in \mathcal{I} , but not containing any solution to M .

Let M be a partial ordering generic for the ground forcing. M is stable by (C1), M does not compute a solution to itself by (C2) and for each stable-ish linear order \prec_e^M , each requirement $\mathcal{K}^{M, A^*(M), B^*(M)} \subseteq \mathbb{P}_e^M$ is uniformly dense by (C3). Thus, we have established the initial conditions for the iterated forcing with $X = M$. For a fixed index e such that \prec_e^M is a stable-ish linear order, let G be a generic solution to \prec_e^M obtained from the iteration forcing. By Lemmas 2.14, 2.15 and 2.18, $M \oplus G$ does not compute a solution to M and for every stable-ish linear order $\prec_{e'}^{M \oplus G}$, each requirement $\mathcal{K}^{M \oplus G, A^*(M), B^*(M)} \subseteq \mathbb{P}_{e'}^{M \oplus G}$ is uniformly dense.

Iterating this process (and choosing stable-ish partial orders systematically to ensure that we eventually consider each one) gives an ideal \mathcal{I} with the property that whenever \prec is a linear order in \mathcal{I} , either \prec is stable-ish, and therefore we added a solution to \mathcal{I} at some stage, or \prec is not stable-ish, and so a solution is computable from \prec , and therefore belongs to \mathcal{I} . We have ensured that $M \in \mathcal{I}$ but that no solution to M belongs to \mathcal{I} .

Therefore (ω, \mathcal{I}) is a model of $\text{RCA}_0 + \text{ADS}$, but is not a model of SCAC . \square

3 EM background

In this section, we present proofs of Theorems 1.9 and 1.10, which are restated below for convenience. We begin with some basic properties of infinite transitive tournaments and their transitive subsets. We regard every tournament T (including finite subtournaments) as containing elements $-\infty$ and ∞ with the property that $T(-\infty, x)$ and $T(x, \infty)$ hold for every $x \in T$. If T is a transitive tournament, then the T relation defines a linear order on the domain of T with $-\infty$ as the least element and ∞ as the greatest element. We will denote

this order by \leq_T , or by \leq_F if F is a finite transitive subset of some ambient (nontransitive) tournament T .

Definition 3.1. Let T be an infinite tournament and let $a, b \in T$ be such that $T(a, b)$ holds. The *interval* (a, b) is the set of all $x \in T$ such that both $T(a, x)$ and $T(x, b)$ hold. That is, (a, b) is the set of points “between” a and b in T .

Definition 3.2. Let $F \subseteq T$ be a finite transitive subset of an infinite tournament T . For $a, b \in F$ such that $T(a, b)$ holds (i.e. $a \leq_F b$), we say (a, b) is a *minimal interval of F* if there is no $c \in F$ such that $T(a, c)$ and $T(c, b)$ both hold (i.e. b is the successor of a in \leq_F).

In the context of Definition 3.2, (a, b) is an interval in T well as in F . However, the fact that (a, b) is a minimal interval of F is a property of this interval in F .

Definition 3.3. Let T be an infinite tournament and $F \subseteq T$ be a finite transitive set. F is *extendable* if F is a subset of some solution to T . A *one point transitive extension of F* is a transitive set $F \cup \{a\}$ such that $a \notin F$.

Lemma 3.4. Let T be an infinite transitive tournament and $F \subseteq T$ be a finite transitive set. F is extendable if and only if F has infinitely many one point transitive extensions.

Proof. If F is extendable, then it clearly has infinitely many one point extensions. Conversely, suppose F has infinitely many one point extensions. Let T' be the set of all $a \in T \setminus F$ such that $F \cup \{a\}$ is transitive. Since F is transitive, we can list F in \leq_F order as

$$-\infty <_F x_0 <_F x_1 <_F \cdots <_F x_k <_F \infty$$

Because T' is infinite and there are finitely many minimal intervals in F , there must be a minimal interval (a, b) of F such that $(a, b) \cap T'$ is infinite. (Note that a could be $-\infty$, if there are infinitely many elements $a \in T'$ such that $T(a, x_0)$ holds. Similarly, b could be ∞ .) Fix such a minimal interval (a, b) in F and let $T'' = T' \cap (a, b)$. T'' is an infinite subtournament of T and hence (viewing T'' as an infinite tournament), T'' contains an infinite transitive tournament T''' . Since T''' is contained in a minimal interval of F , $T''' \cup F$ is transitive, and hence is a solution to T containing F . \square

Lemma 3.5. Let T be an infinite tournament. For any $a \in T$, there is a solution to T containing a .

Proof. Fix $a \in T$ and let $F = \{a\}$. For all $b \in T$, $\{a, b\}$ is a transitive, so F has infinitely many one point transitive extensions. By Lemma 3.4, F is extendable. \square

Lemma 3.6. Let T be an infinite transitive tournament and let $F \subseteq T$ be a finite transitive extendible set. Cofinitely many of the one point transitive extensions of F are extendable.

Proof. Suppose for a contradiction that there are infinitely many $x \in T \setminus F$ such that $F \cup \{x\}$ is transitive but not extendable. Let T' be the set of all such x . As in the proof of Lemma 3.4, there must be a minimal interval (a, b) of F such that $T' \cap (a, b)$ is infinite. Fix such an interval (a, b) and let $T'' = T' \cap (a, b)$. T'' is an infinite subtournament of T , so there is an infinite transitive set $T''' \subseteq T''$. $F \cup T'''$ is a solution to T containing F as well as infinitely many point from T' giving the desired contradiction. \square

Theorem 3.7 (Kach, Lerman, Solomon and Weber). *There is a computable infinite tournament T with no infinite Σ_2^0 transitive subtournaments.*

Proof. Since every infinite Σ_2^0 set contains an infinite Δ_2^0 subset, it suffices to construct an infinite computable tournament T with no infinite Δ_2^0 transitive subtournaments. We build T in stages to meet the following requirements.

R_e : If $D_e(x) = \lim_s \varphi_e(x, s)$ exists for every x , then D_e is finite or D_e is not transitive.

As stage s , we determine whether $T(x, s)$ or $T(s, x)$ holds for each $x < s$ by acting in substages $e < s$. At substage e , R_e chooses the $2e + 2$ least elements $x_0 < x_1 < \dots < x_{2e+1}$ (less than s) that φ_e currently claims are in D_e . (If there are not $2e + 2$ many such elements, then we proceed to the next substage.) Let x_i and x_j be the least from this set which have not been chosen as witnesses by a higher priority requirement at this stage and assume that $T(x_i, x_j)$ holds. Declare that $T(s, x_i)$ and $T(x_j, s)$ hold so that $\{x_i, x_j, s\}$ is not transitive. Proceed to the next substage. When all substages are complete, declare $T(x, s)$ for any $x < s$ for which we have not declared either $T(x, s)$ or $T(s, x)$. This ends stage s .

It is clear that this process defines a computable infinite tournament T . To see that R_e is met, assume that $D_e(x)$ is defined for all x . Let $x_0 < x_1 < \dots < x_{2e+1}$ be least such that $D_e(x_i) = 1$ and let s be such that φ_e claims that each x_i is in D_e for all $t \geq s$. For every $t \geq s$, R_e chooses a pair of element from $\{x_0, \dots, x_{2e+1}\}$ to make a cycle with t . Therefore, $\{x_0, \dots, x_{2e+1}\}$ has only finitely many one point transitive extensions and hence is not a subset of any infinite transitive subtournament. \square

Theorem 3.8 (Dzhafarov, Kach, Lerman and Solomon). *There is a computable infinite tournament T with no infinite hyperimmune-free transitive subtournaments.*

Proof. We build T in stages to meet, for each e , the requirement R_e that if $\{D_{\varphi_e(x)} \mid x \in \mathbb{N}\}$ is a strong array, then there are $x_0 < x_1$ such that for all $y_0 \in D_{\varphi_e(x_0)}$ and all $y_1 \in D_{\varphi_e(x_1)}$, the set $\{y_0, y_1\}$ is not extendible.

The strategy to meet an individual requirement R_e in isolation is straightforward. We wait for $\varphi_e(x_0)$ to converge for some x_0 , and start defining $T(y, s)$ for all $y \in D_{\varphi_e(x_0)}$ and all s . If $\{D_{\varphi_e} : e \in \omega\}$ is a strong array, we must eventually find an x_1 such that $\varphi_e(x_1)$ converges with $T(y_0, y_1)$ for all $y_0 \in D_{\varphi_e(x_0)}$ and all $y_1 \in D_{\varphi_e(x_1)}$. We then start defining $T(s, y)$ for all $y \in D_{\varphi_e(x_0)}$ and all s , and $T(y, s)$ for all $y \in D_{\varphi_e(x_1)}$ and all s . Thus ensures that R_e is met. Sorting out competing requirements can be handled via a standard finite injury priority argument, as we now show.

At stage s , we define $T(x, s)$ or $T(s, x)$ for all $x < s$. We proceed by substages $e \leq s$. At substage e , we act as follows, breaking into three cases.

Case 1: R_e has no witnesses. Let x_0 be the least $x < s$, if it exists, such that

- (1) $\varphi_{e,s}(x) \downarrow$;
- (2) $D_{\varphi_e(x)} \neq \emptyset$ and each element of $D_{\varphi_e(x)}$ is $< s$;
- (3) for all $i < e$ and any witness y of R_i , $x > y$ and $D_{\varphi_e(x)}$ is disjoint from $D_{\varphi_i(y)}$.

If there is no such x_0 , then do nothing and proceed to the next substage. If there is such an x_0 , then call x_0 the *first witness* of R_e , define $T(y, s)$ for all $y \in D_{\varphi_e(x_0)}$, cancel the witnesses of each R_i with $i > e$ and proceed to the next substage.

Case 2: R_e has exactly one witness. Call this first witness x_0 . Let x_1 be the least $x < s$, if it exists, that satisfies conditions (1)–(3) above, as well as

$$(4) \ T(y_0, y_1) \text{ for all } y_0 \in D_{\varphi_e(x_0)} \text{ and all } y_1 \in D_{\varphi_e(x)}.$$

If there is no such x_1 , then define $T(y, s)$ for all $y \in D_{\varphi_e(x_0)}$ and proceed to the next substage. If there is such a witness x_1 , then call x_1 the *second witness* of R_e , define $T(s, y)$ for all $y \in D_{\varphi_e(x_0)}$ and $T(y, s)$ for all $y \in D_{\varphi_e(x_1)}$, cancel the witnesses of each R_i with $i > e$ and proceed to the next substage.

Case 3: R_e has two witnesses. Let x_0 be the first witness and x_1 be the second witness. Define $T(s, y)$ for all $y \in D_{\varphi_e(x_0)}$ and $T(y, s)$ for all $y \in D_{\varphi_e(x_1)}$. Proceed to the next substage.

When all substages $e < s$ are complete, define $T(x, s)$ for any $x < s$ for which neither $T(x, s)$ nor $T(s, x)$ has been defined. This completes the description for the construction.

It is clear that T is a computable tournament on \mathbb{N} . To verify that requirement R_e is met, suppose $\{D_{\varphi_e(x)} : x \in \mathbb{N}\}$ is a strong array. By induction, support further that each R_i , $i < e$, is satisfied. Since each requirement R_i has at most two witnesses at any stage, and since it can lose these witnesses only for the sake of some $R_{i'}$, $i' < i$, being assigned a witness, we let s' be the least stage such that no R_i , $i < e$, is assigned a witness at any stage $s \geq s'$. By minimality of s' , it must be that R_e has no witnesses at stage s' . Since $\{D_{\varphi_e(x)} : x \in \mathbb{N}\}$ is a strong array, we let $s_0 \geq s'$ be the least stage such that some $x < s_0$ satisfies conditions (1)–(3) of the construction. Then the least such x is assigned as a first witness x_0 of R_e , and this witness is never cancelled.

If, at any later stage $s_1 > s_0$, we assign a second witness x_1 for R_e , then R_e will be satisfied. (Because x_1 will never be canceled, we have $T(y_0, y_1)$, $T(s, y_0)$ and $T(y_1, s)$ for all $s > s_1$, all $y_0 \in D_{\varphi_e(x_0)}$ and all $y_1 \in D_{\varphi_e(x_1)}$. Therefore, $\{y_0, y_1\}$ is not extendible.) So suppose we never find a second witness x_1 . Then by construction, we define $T(y, s)$ for all $s \geq s_0$ and all $y \in D_{\varphi_e(x_0)}$. But if s is large enough that for some $x < s$, $\varphi_{e,s}(x) \downarrow$ and all elements of $D_{\varphi_e(x)}$ lie between s_0 and s , then x will satisfy conditions (1)–(4) of the construction. The least such x is assigned as a second witness x_1 of R_e for the desired contradiction. \square

4 EM does not imply SRT_2^2

Before giving the proof of Theorem 1.15 in a style similar to the proof of Theorem 1.7, we present some motivating ideas for the forcing construction. Fix an index e . We sketch a strategy to meet a single diagonalization requirement towards constructing a stable coloring c_∞ such that if $\Phi_e^{c_\infty}$ is the characteristic function for an infinite tournament $T_e^{c_\infty}$ given by $\Phi_e^{c_\infty}$, then there is a solution S_∞ to $T_e^{c_\infty}$ such that $c_\infty \oplus S_\infty$ does not compute a solution to c_∞ . A single diagonalization requirements has the form

$$R_i : \Phi_i^{c_\infty \oplus S_\infty} \text{ is not a solution to } c_\infty.$$

To approximate c_∞ we use a triples (c, A^*, B^*) (called partial stable colorings) such that c is a 2-coloring of the two element subsets of a finite domain $[0, |c|]$, and A^* and B^* are disjoint subsets of this domain. We say $(c_\alpha, A_\alpha^*, B_\alpha^*)$ *extends* $(c_\beta, A_\beta^*, B_\beta^*)$ if

- $c_\beta \subseteq c_\alpha$, $A_\beta^* \subseteq A_\alpha^*$, $B_\beta^* \subseteq B_\alpha^*$,
- if $a \in A_\beta^*$ and $|c_\beta| < x \leq |c_\alpha|$, then $c_\alpha(a, x) = 0$, and
- if $b \in B_\beta^*$ and $|c_\beta| < x \leq |c_\alpha|$, then $c_\alpha(b, x) = 1$.

In the full construction, these partial stable colorings will be our ground forcing conditions, and we can force statements such as “ F is a finite transitive subtournament of $T_e^{c_\infty}$ ” or “ I is a minimal interval in F which is infinite in $T_e^{c_\infty}$ ” in a standard manner. For example, the set of (c, A^*, B^*) such that $i \in A^* \cup B^*$ is obviously dense, so a generic coloring c_∞ will be stable. Given $\alpha = (c_\alpha, A_\alpha^*, B_\alpha^*)$, we let \mathbb{C}_α denote the set of suitably generic infinite stable colorings extending α .

To approximate a solution S_∞ to $T_e^{c_\infty}$, we augment a partial stable coloring α by adding a finite transitive subtournament F_α of $T_e^{c_\alpha}$ and a minimal interval I_α of F_α such that I_α is infinite in every tournament $T_e^{c_\infty}$ for $c_\infty \in \mathbb{C}_\alpha$. F_α denotes the part of S_∞ specified so far and I_α witnesses the fact that no matter how c_α is (generically) extended to c_∞ , F_α is extendible in $T_e^{c_\infty}$. Thus, a condition for the purposes of this sketch has the form $\alpha = (c_\alpha, A_\alpha^*, B_\alpha^*, F_\alpha, I_\alpha)$. We say α extends β if the partial colorings extend as above, $F_\beta \subseteq F_\alpha$, I_α is a subinterval of I_β and for each $x \in F_\alpha \setminus F_\beta$, $x > \max(F_\beta)$ and $x \in I_\beta$.

Given a condition α , we would like to meet R_i by extending c_α to c_β and F_α to F_β so that $\Phi_i^{c_\beta \oplus F_\beta}(y) = 1$ for some large $y \notin A_\alpha^* \cup B_\alpha^*$. Assuming we can do this without expanding $A_\alpha^* \cup B_\alpha^*$, we are free to add y to either A_α^* or B_α^* . Therefore, if we can perform such an expansion twice, we will arrive at a condition γ such that

$$\exists a \in A_\gamma^* \exists b \in B_\gamma^* (\Phi_i^{c_\gamma \oplus F_\gamma}(a) = \Phi_i^{c_\gamma \oplus F_\gamma}(b) = 1)$$

and hence will have successfully diagonalized. The obvious difficulty is that we have to maintain that F_γ is extendible in $T_e^{c_\infty}$ for all $c_\infty \in \mathbb{C}_\gamma$. We use following partition theorem to help address this problem.

Lemma 4.1. *Let T be an infinite tournament, F be a finite transitive set and (a, b) be a minimal interval of F which is infinite in T . For any finite set $J \subseteq (a, b)$ such that $F \cup J$ is transitive, there is a partition $P \cup Q = J$ such that both $F \cup P$ and $F \cup Q$ are extendible and contain minimal intervals in (a, b) which are infinite in T .*

Given a condition α , we ask our main question: is there a coloring $c_\infty \in \mathbb{C}_\alpha$ extending c_α , an infinite transitive set S in $T_e^{c_\infty}$ contained in I_α with $F_\alpha < S$, and a finite initial segment J of S such that for all partitions $P \cup Q = J$, there is a transitive $F \subseteq P$ or $F \subseteq Q$ for which $\Phi_i^{c_\infty \oplus (F_\alpha \cup F)}(y) = 1$ for some $y \notin A_\alpha^* \cup B_\alpha^*$?

Suppose the answer to this question is yes. We collect a finite set Y disjoint from $A_\alpha^* \cup B_\alpha^*$ such that for each partition $P \cup Q = J$, there is some $F_{P,Q} \subseteq P$ or $F_{P,Q} \subseteq Q$ and some $y \in Y$

so that $\Phi_i^{c_\infty \oplus (F_\alpha \cup F_{P,Q})}(y) = 1$. Set c_β to be an initial segment of c_∞ extending c_α which is long enough to force all these computations, and set $A_\beta^* = A_\alpha^* \cup Y$ and $B_\beta^* = B_\alpha^*$ (or vice versa).

We have defined our extension β of α with the exception of F_β and I_β . To define F_β , we need to find a partition $P \cup Q = J$ such that both $F_\alpha \cup P$ and $F_\alpha \cup Q$ are extendible in every $c_\infty \in \mathbb{C}_\beta$. There are finitely many partitions and we consider each such partition in turn. For a given partition $P \cup Q$, we ask whether there is an extension of $(c_\beta, A_\beta^*, B_\beta^*)$ which forces one of $F_\alpha \cup P$ or $F_\alpha \cup Q$ to be non-extendible. (Forcing such a property depends only on the part of the condition dealing with the approximation of the coloring.) If there is no such an extension then we take $F = F_{P,Q}$ and $F_\beta = F_\alpha \cup F_{P,Q}$. If there is such an extension, replace $(c_\beta, A_\beta^*, B_\beta^*)$ by this extension and consider the next partition. We must eventually find a partition which cannot be forced to be finite, since otherwise we end with a condition $(c_\beta, A_\beta^*, B_\beta^*)$ extending $(c_\alpha, A_\alpha^*, B_\alpha^*)$ which forces either $F_\alpha \cup P$ or $F_\alpha \cup Q$ to be non-extendible for any $P \cup Q = J$, and hence they must be non-extendible in any $c_\infty \in \mathbb{C}_\beta$, contradicting Lemma 4.1.

To define I_β , we need to find a minimal subinterval of I_α which is infinite in every $c_\infty \in \mathbb{C}_\beta$. F subdivides I_α into finitely many minimal intervals and we follow the same procedure as before, considering each such minimal interval in turn. For a given minimal interval of I_α (as subdivided by F), ask whether there is an extension of $(c_\beta, A_\beta^*, B_\beta^*)$ which forces this interval to be finite. (Forcing such a property depends only on the part of the condition dealing with the approximation of the coloring.) If there is no such extension, then set I_β equal to this minimal interval and we have completed the description of β . If there is such an extension, replace $(c_\beta, A_\beta^*, B_\beta^*)$ by this extension and consider the next minimal interval. We must eventually find a minimal interval which cannot be forced to be finite because otherwise, we end with a condition $(c_\beta, A_\beta^*, B_\beta^*)$ extending $(c_\alpha, A_\alpha^*, B_\alpha^*)$ which forced all the minimal subintervals of I_α to be finite and hence forced I_α to be finite. This contradicts the assumption that I_α was infinite in all $c_\infty \in \mathbb{C}_\alpha$.

Thus, when the answer to the main question is yes, we are able to extend α to β forcing $\Phi_i^{c_\beta \oplus F_\beta}(y) = 1$ for a large element y which we could put into either A_β^* or B_β^* . So, suppose the answer to the main question is no. To deal with this case, we introduce a Matthias component to our forcing conditions.

Fix $c_\infty \in \mathbb{C}_\alpha$. A solution S to $T_e^{c_\infty}$ is consistent with the restrictions placed by α if S is contained in I_α and $F_\alpha < S$. We narrow this collection of solutions to consider only those that satisfy

$$\forall \text{ finite } B \subseteq S \forall y (\Phi_i^{c_\infty \oplus (F_\alpha \cup B)}(y) = 1 \rightarrow y \in A_\alpha^* \cup B_\alpha^*).$$

Let $\mathbb{S}_{\alpha, c_\infty}$ denote this collection of solutions to $T_e^{c_\infty}$. (We show this set is nonempty below.) If our eventual generic coloring is c_∞ and we restrict our choices for extensions of F_α to those which are subsets of elements of $\mathbb{S}_{\alpha, c_\infty}$, then our generic solution S_∞ will meet R_i because $\Phi^{c_\infty \oplus S_\infty}$ will be finite. Therefore, we want to add a Matthias component to our conditions requiring that when we extend c_α and F_α to c_β and F_β in the future, we must have that F_β is a subset of some element of $\mathbb{S}_{\beta, c_\infty}$ for each c_∞ extending c_β .

Lemma 4.2. $\mathbb{S}_{\alpha, c_\infty}$ is nonempty.

Proof. Fix a solution $S = \{s_1 < s_2 < \dots\}$ of $T_e^{c_\infty}$ consistent with α . For $n \geq 1$, let

$S_n = \{s_1, \dots, s_n\}$. Let $R(P, Q, n)$ be a predicate that holds if and only if $P \cup Q = S_n$ is a partition such that for all F contained in P or Q and all y , $\Phi_i^{c_\infty \oplus (F_\alpha \cup F)}(y) = 1$ implies $y \in A_\alpha^* \cup B_\alpha^*$. Because the answer to the main question was no, for every n , there is a partition $P \cup Q = S_n$ such that $R(P, Q, n)$ holds. We define a finitely branching tree T such that the nodes at level n of T are labelled by the ‘‘halves’’ of such partitions so that if σ is an extension of τ on T , then the label of τ is a subset of the label of σ .

For level $n = 0$, let T have a root λ with label $\ell(\lambda) = \emptyset$. For level $n = 1$, there is only one partition $\{s_1\} \cup \emptyset$ of S_1 and therefore, this partition must satisfy R . Let T have two nodes σ_0 and σ_1 at level 1 with labels $\ell(\sigma_0) = \{s_1\}$ and $\ell(\sigma_1) = \emptyset$.

For level $n + 1$, consider each partition $P \cup Q = S_{n+1}$ satisfying $R(P, Q, n + 1)$. Let $P' \cup Q' = S_n$ be the restriction of this partition to S_n and note that $R(P', Q', n)$ holds. Therefore, by induction, there are nodes σ' and τ' at level n of T such that $\ell(\sigma') = P'$ and $\ell(\tau') = Q'$. Add nodes σ and τ to T at level $n + 1$ with σ a successor of σ' and label $\ell(\sigma) = P$ and with τ a successor of τ' with label $\ell(\tau) = Q$.

This completes the description of T . Notice that T is infinite, finitely branching and has the property that if σ is an extension of τ , then $\ell(\tau) \subseteq \ell(\sigma)$. To extract an element of $\mathbb{A}_{\alpha, c_\infty}$, we break into two cases.

First, suppose there is a level n and a node σ at level n such that for every $m \geq n$, there is an extension τ_m of σ at level m with $\ell(\tau_m) = \ell(\sigma)$. In this case, for every $m \geq n$, $\ell(\sigma) \cup (S_m \setminus \ell(\sigma)) = S_m$ satisfies R . Therefore, $S' = \cup_{m \geq n} (S_m \setminus \ell(\sigma))$ is an infinite subset of S and hence is a solution to $T_e^{c_\infty}$ consistent with α . Furthermore, if $B \subseteq S'$ is finite, then $B \subseteq (S_m \setminus \ell(\sigma))$ for some m and hence if $\Phi_i^{c_\infty \oplus (F_\alpha \cup B)}(y) = 1$, then $y \in A_\alpha^* \cup B_\alpha^*$. Therefore, $S' \in \mathbb{S}_{\alpha, c_\infty}$.

Second, suppose there is no such σ . In this case, for every σ in T , either the tree above σ is finite or there is a level m such that for all τ extending σ at level m , $\ell(\sigma) \subsetneq \ell(\tau)$. Because T is infinite and finitely branching, it has an infinite path $\sigma_0 \subseteq \sigma_1 \subseteq \dots$. Let $S' = \cup \ell(\sigma_k)$. By our case assumption, S' is an infinite subset of S and as above, $S' \in \mathbb{S}_{\alpha, c_\infty}$. \square

To fully implement the strategy sketched in this section, we need a more uniform method to represent these collections of ‘‘allowable’’ solutions to $T_e^{c_\infty}$. We wait for the full construction to spell out the details of representing the solutions in an appropriate matter.

4.1 Iteration forcing

As with the proof of Theorem 1.7, we begin the formal proof of Theorem 1.15 with the iteration forcing. Assume we have already used the ground forcing to build a stable 2-coloring c of $[\mathbb{N}]^2$ along with $A^*(c)$ and $B^*(c)$. The general context for one step of the iteration forcing is a fixed set X and an index e such that

- $c \leq_T X$;
- X does not compute a solution to c ;
- Φ_e^X is the characteristic function for an infinite tournament T_e^X on \mathbb{N} ; and

- each requirement $\mathcal{K}^{X,A^*(c),B^*(c)}$ is uniformly dense (in a sense defined below).

Our goal is to define a generic solution G to T_e^X such that $X \oplus G$ does not compute a solution to c and the requirements $\mathcal{K}^{X \oplus G, A^*(c), B^*(c)}$ are uniformly dense. We let \mathbb{E}_e^X denote the set of finite subtournaments of T_e^X and let variations on E (such as E' , \tilde{E} , E_0 , etc.) denote finite subtournaments of T_e^X . We let \mathbb{F}_e^X denote the set of finite transitive subtournament of T_e^X and let variations on F denote finite transitive subtournaments of T_e^X .

Definition 4.3. A *family of subtournaments* of T_e^X (or simply a *family*) is a function $S^X(n)$ computable from X such that

- (A1) for each n , $S^X(n)$ is a finite collection of finite subtournaments of T ; and
- (A2) whenever $E \in S^X(n+1)$, there is an $E' \in S^X(n)$ such that $E' \subsetneq E$ and for all $x \in E \setminus E'$, $x > \max(S^X(n)) = \max(\cup\{\tilde{E} \mid \tilde{E} \in S^X(n)\})$.

We say S is *infinite* if for every n , $S(n)$ is non-empty.

To be more precise, the value of $S^X(n)$ is the canonical index for the finite collection of finite subtournaments so we can calculate $\max(S^X(n))$ effectively in X . A family of subtournaments is represented by its index, so later when such families appear as part of our forcing conditions, we mean that the index is listed a part of the condition. Note that there is a trivial infinite family of subtournaments given by $S(n) = \{[0, n]\}$.

For the first part of this construction, we will typically drop the superscript X and denote a family by S . Later when we consider forcing uniform density of the requirements $\mathcal{K}^{X \oplus G, A^*(c), B^*(c)}$ at the next level, we will be more careful about the oracle as it will have the form $X \oplus F$ for some finite transitive subtournament F of T_e^X .

We view a family of subtournaments S as a finitely branching tree whose nodes at level n are the finite subtournaments in $S(n)$. (We artificially add a root node at the bottom of the tree.) The node $E \in S(n+1)$ is a successor of $E' \in S(n)$ if and only if $E' \subseteq E$ and for all $x \in E \setminus E'$, $x > \max(S(n))$. Since there is a unique such node E' , the family S forms a finitely branching tree under this successor relation, and X can compute the number of branches at each level.

If $E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \dots$ is an infinite path in a family S (viewed as a tree), then $\cup E_n$ is an infinite subtournament of T_e^X . We say that an infinite subtournament $U \subseteq T_e^X$ is *coded by* S if $U = \cup E_n$ for some infinite path through S . We will use families in our forcing conditions in the style of Matthias forcing to restrict our attention to solutions of T_e^X which are contained in an infinite subtournament coded by S .

Definition 4.4. Let E be a finite subtournament of T_e^X and let S be a family of subtournaments such that $E < E'$ for all $E' \in S(0)$. We write $E + S$ for the family given by

$$(E + S)(n) = \{E \cup E' \mid E' \in S(n)\}.$$

It is straightforward to check that under the conditions of this definition, $E + S$ is a family of subtournaments and that the subtournaments coded by $E + S$ are exactly the subtournaments coded by S unioned with E .

Definition 4.5. Let S' and S be families of subtournaments. We say $S' \leq S$ if for every n , there is an m such that whenever $E' \in S'(n)$, there is an $E \in S(m)$ with $E' \subseteq E$.

If $S' \leq S$, then S' is similar to a subtree of S , but is more flexible. For example, let $S(n) = \{[0, n]\}$ and let $S'(n) = \{\text{Even}(n), \text{Odd}(n)\}$ where $\text{Even}(n)$ is the set of even numbers less than n and $\text{Odd}(n)$ is the set of odd numbers less than n . Then $S' \leq S$ despite the fact that S contains a single branch and S' contains two branches.

Lemma 4.6. *Let S and S' be infinite families of subtournaments such that $S' \leq S$. Every infinite subtournament coded by S' is contained in an infinite subtournament coded by S .*

Proof. Let $E'_0 \subsetneq E'_1 \subsetneq \dots$ be an infinite path in S' . Let $m(k)$ be a function such that for each k , $E'_k \subseteq E$ for some $E \in S(m(k))$. Consider the subtree of S formed by taking the downward closure of all $E \in S(m(k))$ such that $E'_k \subseteq E$. This subtree is infinite because $S' \leq S$ and therefore has an infinite path $E_0 \subsetneq E_1 \subsetneq \dots$ because it is finitely branching. By the definition of the subtree, $\cup E'_k \subseteq \cup E_k$ and hence the subtournament coded by S' is contained in a subtournament coded by S . \square

Definition 4.7. Our set of forcing conditions, \mathbb{Q}_e^X , is the set of triples (F, I, S) such that

- F is a finite, transitive subtournament of T_e^X ,
- I is a minimal interval of F , and
- S is an infinite family of subtournaments such that $F < E$ for all $E \in S(0)$ and such that for each n , $E \subseteq I$ for all $E \in S(n)$.

We say $(F', I', S') \leq (F, I, S)$ if

- $F \subseteq F'$,
- I' is a sub-interval of I , and
- $(F' \setminus F) + S' \leq S$.

The last condition implies that there is an n and an $E \in S(n)$ such that $F' \setminus F \subseteq E$.

There are several points worth noting about this definition. First, for each $(F, I, S) \in \mathbb{Q}_e^X$, F is extendible in T_e^X . In particular, if U is a subtournament coded by S and $S \subseteq U$ is an infinite transitive set, then $F \cup S$ is a solution of T_e^X because S is contained in the minimal interval I of F .

Second, if $(F', I', S') \leq (F, I, S)$, then $F < x$ for all $x \in F' \setminus F$. To see why, by the third condition in the definition of an extension, we can fix n such that $F' \setminus F \subseteq \widehat{E}$ for some $\widehat{E} \in S(n)$. Since (F, I, S) is a condition, $F < E$ for all $E \in S(0)$. Applying Condition (A2) in Definition 4.3, $F < \widehat{E}$ and hence $F < (F' \setminus F)$, completing the explanation. If $F, F' \in \mathbb{F}_e^X$, then we say F' extends F if $F \subseteq F'$ and $F < (F' \setminus F)$. Thus, if $(F', I', S') \leq (F, I, S)$, then F' extends F .

Third, let $(F', I', S') \leq (F, I, S)$ and let U be a subtournament coded by S' . Because $(F' \setminus F) + S' \leq S$, there is a subtournament V coded by S such that $(F' \setminus F) \cup U \subseteq V$. Therefore,

when passing from (F, I, S) to (F', I', S') , we extend the fixed part of our transitive solution from F to F' while narrowing the interval in T_e^X where we can look for future transitive extensions from I to I' and further restricting where in I' we can find these extensions in a way that is consistent with the restriction to the previous family S .

Our construction of the generic solution G to T_e^X will be spelled out in detail later, but it proceeds in the usual way. We specify a sequence of conditions $(F_0, I_0, S_0) \geq (F_1, I_1, S_1) \geq \dots$ meeting the appropriate requirements and set $G = \cup_n F_n$.

Having defined our forcing conditions, we turn to the requirements and the notion of uniform density.

Definition 4.8. A *requirement* is a set $\mathcal{K}^{X, A^*(c), B^*(c)}$ together with a partial function $a_{\mathcal{K}}^X$ meeting the following conditions.

- $\mathcal{K}^{X, A^*(c), B^*(c)}$ is a set of finite transitive subtournaments of T_e^X which is closed under extensions and is defined by

$$\mathcal{K}^{X, A^*(c), B^*(c)} = \{F \in \mathbb{F}_e^X \mid \exists a \in A^*(c) \exists b \in B^*(c) (R_{\mathcal{K}}^X(F, a, b))\}$$

for an X -computable relation $R_{\mathcal{K}}^X(x, y, z)$ which is symmetric in the y and z variables.

- $a_{\mathcal{K}}^X$ is a partial X -computable function with $\text{domain}(a_{\mathcal{K}}^X) \subseteq \mathbb{F}_e^X$ such that if F' extends F and $a_{\mathcal{K}}^X(F)$ converges, then $a_{\mathcal{K}}^X(F') = a_{\mathcal{K}}^X(F)$.

Note that $\mathcal{K}^{X, A^*(c), B^*(c)}$ is specified by a pair of indices, one for the relation $R_{\mathcal{K}}^X$ and one for the partial function $a_{\mathcal{K}}^X$.

Example 4.9. For each m , we define the requirement

$$\mathcal{W}_m^{X, A^*(c), B^*(c)} = \{F \in \mathbb{F}_e^X \mid \exists a \in A^*(c) \exists b \in B^*(c) (\Phi_m^{X \oplus F}(a) = \Phi_m^{X \oplus F}(b) = 1)\}$$

together with the function $a_{\mathcal{W}_m}^X(F) = u$ where u is the first number (if any) on which $\Phi_m^{X \oplus F}(u) = 1$ in appropriately dovetailed computations.

Suppose a condition (F, I, S) used to construct our generic G satisfies $F \in \mathcal{W}_m^{X, A^*(c), B^*(c)}$. Because F is an initial segment of G , we have successfully diagonalized against $\Phi_m^{X \oplus G}$ computing a solution to c . On the other hand, if $a_{\mathcal{W}_m}^X(F)$ diverges for every condition (F, I, S) in the sequence defining G , then $\Phi_m^{X \oplus F}$ is empty for every initial segment F of G and hence $\Phi_m^{X \oplus G}$ does not compute a solution to c .

As in the previous construction, we can replace the sets $A^*(c)$ and $B^*(c)$ in a requirement $\mathcal{K}^{X, A^*(c), B^*(c)}$ by finite sets A and B and consider the set

$$\mathcal{K}^{X, A, B} = \{F \in \mathbb{F}_e^X \mid \exists a \in A \exists b \in B (R_{\mathcal{K}}^X(F, a, b))\}.$$

Typically, we will work in this context with $A = \{a_{\mathcal{K}}^X(F)\}$ for some fixed F for which $a_{\mathcal{K}}^X(F)$ converges. We abuse notation and write $\mathcal{K}^{X, a_{\mathcal{K}}^X(F), B}$ in this situation.

Definition 4.10. We say \mathcal{K}^X is *essential* below (F, I, S) if $a_{\mathcal{K}}^X(F)$ converges and for every x there is a finite set $B > x$ and a level n such that whenever $E \in S(n)$ and $E = E_0 \cup E_1$ is a partition, there is an $i \in \{0, 1\}$ and a transitive $F' \subseteq E_i$ such that $F \cup F' \in \mathcal{K}^{X, a_{\mathcal{K}}^X(F), B}$.

Definition 4.11. We say $\mathcal{K}^{X,A^*(c),B^*(c)}$ is *uniformly dense* if whenever \mathcal{K}^X is essential below (F, I, S) , there is some level n such that whenever $E \in S(n)$ and $E = E_0 \cup E_1$ is a partition, there is an $i \in \{0, 1\}$ and a transitive $F' \subseteq E_i$ such that $F \cup F' \in \mathcal{K}^{X,A^*(c),B^*(c)}$.

Definition 4.12. We say (F, I, S) *settles* $\mathcal{K}^{X,A^*(c),B^*(c)}$ if $a_{\mathcal{K}}^X(F)$ converges and either $F \in \mathcal{K}^{X,A^*(c),B^*(c)}$ or there is an x such that whenever $E \in S(n)$ is on an infinite path through S and $F' \subseteq E$ is transitive, $F \cup F' \notin \mathcal{K}^{X,a_{\mathcal{K}}^X(F),(x,\infty)}$.

We give one example to illustrate settling and prove one essential property of this notion.

Example 4.13. Suppose (F, I, S) settles $\mathcal{W}_m^{X,A^*(c),B^*(c)}$. We claim that if (F, I, S) appears in a sequence defining a generic G , then $\Phi_m^{X \oplus G}$ is not a solution to c . If $(F, I, S) \in \mathcal{W}_m^{X,A^*(c),B^*(c)}$, then this claim was verified in Example 4.9. So, assume that (F, I, S) settles $\mathcal{W}_m^{X,A^*(c),B^*(c)}$ via the second condition in this definition and fix the witness x . We claim that for all $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$ and all $b > x$, $\Phi_m^{X \oplus \tilde{F}}(b) \neq 1$. It follows immediately from this claim that $\Phi_m^{X \oplus G}$ is finite and hence is not a solution to c .

To prove the claim, fix $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$. By the definition of $a_{\mathcal{W}_m}^X(F)$, we know that $\Phi_m^{X \oplus F}(a_{\mathcal{W}_m}^X(F)) = 1$. Since \tilde{F} extends F , we have $\Phi_m^{X \oplus \tilde{F}}(a_{\mathcal{W}_m}^X(F)) = 1$. Suppose for a contradiction that there is a $b > x$ such that $\Phi_m^{X \oplus \tilde{F}}(b) = 1$. Then

$$\exists b > x \left(\Phi_m^{X \oplus \tilde{F}}(a_{\mathcal{W}_m}^X(F)) = \Phi_m^{X \oplus \tilde{F}}(b) = 1 \right)$$

and hence $\tilde{F} \in \mathcal{W}_m^{X,a_{\mathcal{W}_m}^X(F),(x,\infty)}$.

Let $F' = \tilde{F} \setminus F$, so $F \cup F' \in \mathcal{W}_m^{X,a_{\mathcal{W}_m}^X(F),(x,\infty)}$. Because $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$, we have $(\tilde{F} \setminus F) + S' = F' + S' \leq S$ and hence there is a level n and an $E \subseteq S(n)$ such that $F' \subseteq E$. Therefore, F' shows that our fixed x does not witness the second condition for (F, I, S) to settle $\mathcal{W}_m^{X,A^*(c),B^*(c)}$ giving the desired contradiction.

Lemma 4.14. *If (F, I, S) settles $\mathcal{K}^{X,A^*(c),B^*(c)}$ and $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$, then $(\tilde{F}, \tilde{I}, \tilde{S})$ settles $\mathcal{K}^{X,A^*(c),B^*(c)}$.*

Proof. Assume (F, I, S) settles $\mathcal{K}^{X,A^*(c),B^*(c)}$ and fix $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$. Since $a_{\mathcal{K}}^X(F)$ converges and \tilde{F} is an extension of F , we know that $a_{\mathcal{K}}^X(\tilde{F})$ converges.

If $F \in \mathcal{K}^{X,A^*(c),B^*(c)}$, then since $\mathcal{K}^{X,A^*(c),B^*(c)}$ is closed under extensions, $\tilde{F} \in \mathcal{K}^{X,A^*(c),B^*(c)}$ and $(\tilde{F}, \tilde{I}, \tilde{S})$ settles $\mathcal{K}^{X,A^*(c),B^*(c)}$.

On the other hand, suppose (F, I, S) settles $\mathcal{K}^{X,A^*(c),B^*(c)}$ via the second condition in Definition 4.12 with the witness x . We claim that the same witness x works to show that $(\tilde{F}, \tilde{I}, \tilde{S})$ settles $\mathcal{K}^{X,A^*(c),B^*(c)}$. Suppose for a contradiction that there is an $\tilde{E} \in \tilde{S}(\tilde{n})$ on an infinite path through \tilde{S} and a transitive $\tilde{F}' \subseteq \tilde{E}$ such that $\tilde{F} \cup \tilde{F}' \in \mathcal{K}^{X,a_{\mathcal{K}}^X(\tilde{F}),(x,\infty)}$. Note that $\tilde{F} \cap \tilde{F}' = \emptyset$ because $\tilde{F}' \subseteq \tilde{E} \in \tilde{S}(\tilde{n})$.

Let $F' = (\tilde{F} \cup \tilde{F}') \setminus F$, so $F \cup F' \in \mathcal{K}^{X,a_{\mathcal{K}}^X(\tilde{F}),(x,\infty)} = \mathcal{K}^{X,a_{\mathcal{K}}^X(F),(x,\infty)}$. Because $\tilde{F} \cap \tilde{F}' = \emptyset$, we have $F' = (\tilde{F} \setminus F) \cup \tilde{F}'$. It follows from $(\tilde{F} \setminus F) + \tilde{S} \leq S$ that there is a level n and an $E \in S(n)$ on an infinite path in S such that $(\tilde{F} \setminus F) \cup \tilde{E} \subseteq E$. Therefore, $F' \subseteq E \in S(n)$ and $F \cup F' \in \mathcal{K}^{X,a_{\mathcal{K}}^X(F),(x,\infty)}$ contradicting the fact that x was a witness for (F, I, S) settling $\mathcal{K}^{X,A^*(c),B^*(c)}$ via the second condition in Definition 4.12. \square

The heart of this construction is the following theorem.

Theorem 4.15. *Let $\mathcal{K}^{X,A^*(c),B^*(c)}$ be a uniformly dense requirement and let (F, I, S) be a condition for which $a_{\mathcal{K}}^X(F)$ converges. There is an extension $(F', I', S') \leq (F, I, S)$ settling $\mathcal{K}^{X,A^*(c),B^*(c)}$.*

We prove Theorem 4.15 at the end of this subsection after showing how it is used to construct our generic G and verifying that $X \oplus G$ does not compute a solution to c and that for any index e' such that $\Phi_{e'}^{X \oplus G}$ defines an infinite tournament, the associated requirements $\mathcal{K}^{X \oplus G, A^*(c), B^*(c)}$ are uniformly dense.

To define G , let $\mathcal{K}_n^{X,A^*(c),B^*(c)}$, for $n \in \omega$, be a list of all the requirements. We define a sequence of conditions

$$(F_0, I_0, S_0) \geq (F_1, I_1, S_1) \geq \dots$$

by induction. Let $F_0 = \emptyset$, $I_0 = (-\infty, \infty)$ and $S_0(n) = \{[0, n]\}$. Assume (F_k, I_k, S_k) has been defined. Let n be the least index such that $a_{\mathcal{K}_n}^X(F_k)$ converges and $\mathcal{K}_n^{X,A^*(c),B^*(c)}$ is not settled by (F_k, I_k, S_k) . Applying Theorem 4.15, we choose $(F_{k+1}, I_{k+1}, S_{k+1})$ so that it settles $\mathcal{K}_n^{X,A^*(c),B^*(c)}$. We define our generic by $G = \cup F_n$.

The next lemma shows that we eventually settle each condition that is not trivially satisfied.

Lemma 4.16. *Let $\mathcal{K}_n^{X,A^*(c),B^*(c)}$ be a requirement and let (F_j, I_j, S_j) be the sequence of conditions defining G . If there is an i such that $a_{\mathcal{K}_n}^X(F_i)$ converges, then there is an index k such that (F_k, I_k, S_k) settles $\mathcal{K}_n^{X,A^*(c),B^*(c)}$.*

Proof. Fix the least i such that $a_{\mathcal{K}_n}^X(F_i)$ converges. For all $j \geq i$, $a_{\mathcal{K}_n}^X(F_j)$ converges because F_j extends F_i . By Lemma 4.14, once a requirement $\mathcal{K}_n^{X,A^*(c),B^*(c)}$ has been settled by a condition in the sequence defining G , it remains settled by all future conditions. Therefore there must be a stage $k \geq i$ at which for all $n' < n$ such that $a_{\mathcal{K}_{n'}}^X(F_k)$ converges, $\mathcal{K}_{n'}^{X,A^*(c),B^*(c)}$ is settled by (F_k, I_k, S_k) . Either $\mathcal{K}_n^{X,A^*(c),B^*(c)}$ is already settled by (F_k, I_k, S_k) , or, by construction, $(F_{k+1}, I_{k+1}, S_{k+1})$ settles $\mathcal{K}_n^{X,A^*(c),B^*(c)}$. \square

We can now verify the properties of G starting with the fact that $X \oplus G$ does not compute a solution to c .

Lemma 4.17. *$X \oplus G$ does not compute a solution to c .*

Proof. Fix an index m and we show that $\Phi_m^{X \oplus G}$ is not a solution to c using the requirement $\mathcal{W}_m^{X,A^*(c),B^*(c)}$. If $\Phi_m^{X \oplus G}(u)$ is never equal to 1, then $\Phi_m^{X \oplus G}$ does not compute an infinite set and we are done. Therefore, assume that $\Phi_m^{X \oplus G}(u) = 1$ for some u . In this case, $a_{\mathcal{W}_m}^X(F_i)$ converges for some i and hence $\mathcal{W}_m^{X,A^*(c),B^*(c)}$ is settled by some condition (F_k, I_k, S_k) in the sequence defining G . In Example 4.13 we verified that if $\mathcal{W}_m^{X,A^*(c),B^*(c)}$ is settled by a condition in a sequence defining a generic G , then $\Phi_m^{X \oplus G}$ does not compute a solution to c . \square

Next, we describe the requirements forcing uniform density at the next level. To specify a potential requirement at the next level, we need to fix three indices: an index e' for a potential infinite transitive tournament $T_{e'}^{X \oplus G}$ and indices for the relation $R_{\mathcal{K}}^{X \oplus G}$ (defining $\mathcal{K}^{X \oplus G, A^*(c), B^*(c)}$) and the function $a_{\mathcal{K}}^{X \oplus G}$. We regard the indices for $R_{\mathcal{K}}^{X \oplus G}$ and $a_{\mathcal{K}}^{X \oplus G}$ as a pair \mathcal{K} and will represent this choice of indices by indicating e' and \mathcal{K} . For each choice of these indices and each $q = (F_q, I_q, S_q)$, representing a potential condition in $\mathbb{Q}_{e'}^{X \oplus G}$, we will have a requirement $\mathcal{T}_{e', \mathcal{K}, q}^X$.

To describe these requirements, we fix some forcing definitions for a fixed index e' . These definitions are exactly what one expects and in each case reflect that F is a big enough to guarantee the convergence of relevant computations for any generic G extending F . In these definitions, F is a finite transitive subtournament of T_e^X . E , F' and similar variations are finite sets which are potentially subtournaments or transitive subtournaments of $T_{e'}^{X \oplus G}$. For $q = (F_q, I_q, S_q)$, we assume F_q is a finite set, I_q is a pair of distinct elements of F_q and S_q is an index for a potential infinite family of subtournaments in $T_{e'}^{X \oplus G}$.

- $F \Vdash (E \text{ is a subtournament})$ if for every $u \in E$, $\Phi_{e'}^{X \oplus F}(u, u) = 0$ and for each distinct $u, v \in E$, there is an $i \in \{0, 1\}$ such that $\Phi_{e'}^{X \oplus F}(u, v) = i$ and $\Phi_{e'}^{X \oplus F}(v, u) = 1 - i$.
- $F \Vdash (E \text{ is not a subtournament})$ if either there is a $u \in E$ such that $\Phi_{e'}^{X \oplus F}(u, u) = 1$ or there are distinct $u, v \in E$ such that $\Phi_{e'}^{X \oplus F}(u, v) = \Phi_{e'}^{X \oplus F}(v, u)$.
- $F \Vdash (E \text{ is transitive})$ if $F \Vdash (E \text{ is a subtournament})$ and the relation on E given by $\Phi_{e'}^{X \oplus F}$ is transitive.
- $F \Vdash (E \text{ is not transitive})$ if $F \Vdash (E \text{ is a subtournament})$ and the relation on E given by $\Phi_{e'}^{X \oplus F}$ is not transitive.
- Fix an index S for a potential family of subtournaments of $T_{e'}^{X \oplus G}$.
 - $F \Vdash (S \text{ is a potential family at level } m)$ if $S^{X \oplus F}(m)$ converges and outputs a finite set $\{E_0, \dots, E_k\}$ such that $F \Vdash (E_j \text{ is a subtournament})$ for each j .
 - $F \Vdash (S \text{ is a family up to level } n)$ if for all $m \leq n$, $F \Vdash (S(m) \text{ is a potential family at level } m)$, and if $m < n$, then the families of finite sets given by $S^{X \oplus F}(m)$ and $S^{X \oplus F}(m + 1)$ satisfy Condition (A2) in Definition 4.3.
 - $F \Vdash (S \text{ is not a family})$ if either there is an $m \leq |F|$ such that $S^{X \oplus F}(m) = \{E_0, \dots, E_k\}$ and for some j , $F \Vdash (E_j \text{ is not a subtournament})$, or there is an $m < |F|$ such that $F \Vdash (S \text{ is a family up to level } m)$, $F \Vdash (S \text{ is a potential family at level } m + 1)$, but the families of finite sets given by $S^{X \oplus F}(m)$ and $S^{X \oplus F}(m + 1)$ do not meet Condition (A2) in Definition 4.3.
- $F \Vdash (q \notin \mathbb{Q}_{e'}^{X \oplus G})$ if either $F \Vdash (F_q \text{ is not transitive})$; or $F \Vdash (F_q \text{ is transitive})$ but I_q is not a minimal interval in F_q ; or $F \Vdash (S_q \text{ is not a family})$; or there is an $E \in S_q^{X \oplus F}(0)$ such that $F_q \not\prec E$; or there is a $E \in S_q^{X \oplus F}(n)$ for some $n \leq |F|$ such that E is not contained in I_q .

- $F \Vdash (q \text{ is a condition to level } n)$ if $F \Vdash (F_q \text{ is transitive; } I_q \text{ is a minimal interval in } F_q;$
 $F \Vdash (S \text{ is a family up to level } n);$ for all $E \in S_q^{X \oplus F}(0)$, $F_q < E$; and for all $m \leq n$ and
all $E \in S_q^{X \oplus F}(n)$, $E \subseteq I_q$.

Note that all of these forcing statements are X -computable and are closed under extensions. If $q \in \mathbb{Q}_{e'}^{X \oplus G}$ for our generic G , then for each n , there is an index k_n such that $F_{k_n} \Vdash (q \text{ is a condition up to level } n)$ where $(F_{k_n}, I_{k_n}, S_{k_n})$ appears in the sequence defining G . However, if $q \notin \mathbb{Q}_e^{X \oplus G}$, then this statement is not necessarily forced because, in order to keep our statements X -computable, we have not included conditions to handle divergence computations.

The requirement $\mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$ consists of all finite transitive subtournaments F of T_e^X such that either

(C1) $F \Vdash q \notin \mathbb{Q}_{e'}^{X \oplus G}$; or

(C2) there is an $n \leq |F|$ such that $F \Vdash (q \text{ is a condition up to level } n)$ and for all $E \in S_q^{X \oplus F}(n)$ and all partitions $E = E_0 \cup E_1$, there is an $i \in \{0, 1\}$ and a transitive $F' \subseteq E_i$ such that $\exists a \in A^*(c) \exists b \in B^*(c) (R_{\mathcal{K}}^{X \oplus F}(F_q \cup F', a, b))$.

The function $a_{\mathcal{T}_{e', \mathcal{K}, q}^X}^X(F)$ is defined by $a_{\mathcal{T}_{e', \mathcal{K}, q}^X}^X(F) = a_{\mathcal{K}}^{X \oplus F}(F_q)$.

Lemma 4.18. *Let $G = \cup F_k$ be a generic defined by a sequence of conditions (F_k, I_k, S_k) and let e' be an index such that $T_{e'}^{X \oplus G}$ is an infinite tournament. Each requirement $\mathcal{K}^{X \oplus G, A^*(c), B^*(c)}$ is uniformly dense in $\mathbb{Q}_{e'}^{X \oplus G}$.*

Proof. Fix a requirement $\mathcal{K}^{X \oplus G, A^*(c), B^*(c)}$ and a condition $q = (F_q, I_q, S_q) \in \mathbb{Q}_{e'}^{X \oplus G}$ such that $\mathcal{K}^{X \oplus G}$ is essential below q . Being essential below q implies that $a_{\mathcal{K}}^{X \oplus G}(F_q)$ converges, and so there is an i such that $a_{\mathcal{K}}^{X \oplus F_i}(F_q) = a_{\mathcal{K}}^{X \oplus G}(F_q)$ converges. By definition, $a_{\mathcal{T}_{e', \mathcal{K}, q}^X}^X(F_i) = a_{\mathcal{K}}^{X \oplus F_i}(F_q)$, so it also converges. By Lemma 4.16, there is a condition (F_k, I_k, S_k) settling $\mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$. For notational simplicity, let $a = a_{\mathcal{T}_{e', \mathcal{K}, q}^X}^X(F_k) = a_{\mathcal{K}}^{X \oplus G}(F_q)$.

By Definition 4.12, there are two ways in which (F_k, I_k, S_k) could settle $\mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$. We consider these options separately. First, we show that if (F_k, I_k, S_k) settles $\mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$ because $F_k \in \mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$, then $\mathcal{K}^{X \oplus G, A^*(c), B^*(c)}$ satisfies the required condition for uniform density in Definition 4.11 with respect to the condition q . Second, we show that (F_k, I_k, S_k) cannot settle $\mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$ in the case when $F_k \notin \mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$, completing the proof.

First, suppose $F_k \in \mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$. Because $q \in \mathbb{Q}_{e'}^{X \oplus G}$, F_k must be in $\mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$ because of Condition (C2). However, replacing the oracle in (C2) by $X \oplus G$ and comparing the result with the definition of uniform density for $\mathcal{K}^{X \oplus G, A^*(c), B^*(c)}$ with respect to the condition q shows that we have obtained exactly what we need.

Second, suppose (F_k, I_k, S_k) settles $\mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$ but $F_k \notin \mathcal{T}_{e', \mathcal{K}, q}^{X, A^*(c), B^*(c)}$. By the definition of settling, we fix x such that whenever $E \in S_k^X(n)$ is on an infinite path in S_k and $F' \subseteq E$ is transitive, $F_k \cup F' \notin \mathcal{T}_{e', \mathcal{K}, q}^{X, a, (x, \infty)}$. We derive a contradiction by constructing such an F' (called H' below) for which $F_k \cup F' \in \mathcal{T}_{e', \mathcal{K}, q}^{X, a, (x, \infty)}$.

Because $\mathcal{K}^{X \oplus G}$ is essential below q , there is a level n (for this fixed x) and a finite set $B > x$ such that for all $E \in S_q^{X \oplus G}(n)$ and all partitions $E = E_0 \cup E_1$, there is an $i \in \{0, 1\}$ and a transitive $F' \subseteq E_i$ such that $F_q \cup F' \in \mathcal{K}^{X \oplus G, a, B}$, and hence $F_q \cup F' \in \mathcal{K}^{X \oplus G, a, (x, \infty)}$. That is, for each such F'

$$\exists b > x (R_{\mathcal{K}}^{X \oplus G}(F_q \cup F', a, b)).$$

Let H be a finite initial segment of G which is long enough to force these statements for each of the finitely many F' sets. We also assume that H is long enough that $S_q^{X \oplus H}(m)$ converges for all $m \leq n$ and F_q is a transitive subset of $T_{e'}^{X \oplus H}$.

Let $H' = H \setminus F_k$. Because G is a subset of some set coded by S_k , there is a level m and an $E \subseteq S_k(m)$ which is on an infinite path through S_k (namely the one containing G) such that $H' \subseteq E$. Furthermore, because G is transitive in $T_{e'}^X$, so is H' . Finally, since $F_k \cup H' = H$, we have for each F'

$$\exists b > x (R_{\mathcal{K}}^{X \oplus (F_k \cup H')}(F_q \cup F', a, b))$$

which means $F_k \cup H' \in \mathcal{T}_{e', \mathcal{K}, q}^{X, a, (x, \infty)}$ giving the desired contradiction. \square

We have now completed the description of the iteration forcing except for the proof of Theorem 4.15. Note that we have met the required conditions: G is an infinite transitive subtournament of $T_{e'}^X$ such that $X \oplus G$ does not compute a solution to c (by Lemma 4.17) and such that for each index e' for which $T_{e'}^{X \oplus G}$ is an infinite tournament, each requirement $\mathcal{K}^{X \oplus G, A^*(c), B^*(c)}$ is uniformly dense (by Lemma 4.18).

We end this subsection by proving Theorem 4.15. Fix a requirement $\mathcal{K}^{X, A^*(c), B^*(c)}$ which is uniformly dense and a condition (F, I, S) . We need to find a condition $(F', I', S') \leq (F, I, S)$ which settles $\mathcal{K}^{X, A^*(c), B^*(c)}$. Finding this condition breaks into two cases: when \mathcal{K}^X is essential below (F, I, S) and when \mathcal{K}^X is not essential below (F, I, S) . In each case, we will need a partition lemma for S .

We begin with the case when \mathcal{K}^X is essential below (F, I, S) . We state the required partition lemma, show that it suffices and then return to the proof of the partition lemma. For finite subtournaments E and E' , we write $E \rightarrow E'$ if $T_e^X(x, y)$ holds for all $x \in E$ and $y \in E'$.

Lemma 4.19. *Let S be an infinite family of subtournaments. Fix a level n and an $E \in S(n)$ on an infinite path through S . There is a partition $E = E_0 \cup E_1$ and an infinite family S' such that $E_0 + S' \leq S$, $E_1 + S' \leq S$, and for all m and all $E' \in S'(m)$, $E_0 \rightarrow E'$ and $E_1 \rightarrow E'$.*

Lemma 4.20. *Let \mathcal{K}^X be essential below (F, I, S) and let $\mathcal{K}^{X, A^*(c), B^*(c)}$ be uniformly dense. There is an $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$ settling $\mathcal{K}^{X, A^*(c), B^*(c)}$.*

Proof. If $F \in \mathcal{K}^{X, A^*(c), B^*(c)}$, then there is nothing to show, so assume $F \notin \mathcal{K}^{X, A^*(c), B^*(c)}$. By uniform density, we can fix a level n such that whenever $E \in S(n)$ and $E = E_0 \cup E_1$ is a partition, there is an $i \in \{0, 1\}$ and a transitive $F' \subseteq E_i$ such that $F \cup F' \in \mathcal{K}^{X, A^*(c), B^*(c)}$. Fix $E \in S(n)$ such that E is on an infinite path through S . Let $E = E_0 \cup E_1$ and S' be the partition and infinite family guaranteed by Lemma 4.19. Fix an $i \in \{0, 1\}$ and a transitive $F' \subseteq E_i$ such that $F \cup F' \in \mathcal{K}^{X, A^*(c), B^*(c)}$. Since $F \notin \mathcal{K}^{X, A^*(c), B^*(c)}$, we have $F' \neq \emptyset$.

Let $\tilde{F} = F \cup F'$ and $\tilde{S} = S'$. To define \tilde{I} , let c and d be such that I is the F -minimal interval (c, d) . Since F' is transitive and hence is a finite linear order contained in I , let a denote the minimal element of F' and b denote the maximal element of F' . (If $|F'| = 1$, then $a = b$.) If $i = 0$, then for all m and all $E' \in S'(m)$, we have $F' \rightarrow E'$ and hence E' is contained in the minimal interval (b, d) of \tilde{F} . In this case, set $\tilde{I} = (b, d)$. If $i = 1$, then for all m and all $E' \in S'(m)$, $E' \rightarrow F'$ and hence E' is contained in the minimal interval (c, a) of \tilde{F} . In this case, set $\tilde{I} = (c, a)$.

It is clear that $(\tilde{F}, \tilde{I}, \tilde{S})$ is a condition and it settles $\mathcal{K}^{X, A, *(c), B^*(c)}$ because $\tilde{F} = F \cup F' \in \mathcal{K}^{X, A^*(c), B^*(c)}$. To see that $(\tilde{F}, \tilde{I}, \tilde{S}) \leq (F, I, S)$, note that $\tilde{F} \setminus F = F' \subseteq E \in S(n)$, and $(\tilde{F} \setminus F) + S' = F' + S' \leq S$. \square

We next turn to the proof of Lemma 4.19. For a family of subtournaments S , let $\bigcup S$ be the set of all x such that $x \in E$ for some $E \in S(n)$.

Definition 4.21. Let S be a family of subtournaments. A *pointwise partition* of S is a function $g : \bigcup S \rightarrow \{0, 1\}$. Given a partition g of S , we say g *generates the functions* S_0 and S_1 defined by

$$\begin{aligned} S_0(n) &= \{E \mid \exists E' \in S(n) (x \in E \leftrightarrow (x \in E' \wedge g(x) = 0))\} \\ S_1(n) &= \{E \mid \exists E' \in S(n) (x \in E \leftrightarrow (x \in E' \wedge g(x) = 1))\} \end{aligned}$$

where in both sets, E ranges over the finite subtournaments of T_e^X .

The functions S_0 and S_1 need not be families of potential subtournaments. Condition (A2) in Definition 4.3 can fail because a subtournament E appears in both $S_0(n)$ and $S_0(n+1)$. In particular, if S is infinite, then S_0 and S_1 are also infinite in the sense that for $i \in \{0, 1\}$ and for all levels n , $S_i(n) \neq \emptyset$. This fact follows because each $E' \in S(n)$ must be partitioned into an S_0 -half and an S_1 -half. Note that one of these halves could be \emptyset , i.e. we can have $\emptyset \in S_i(n)$.

S_0 and S_1 do satisfy a condition similar to (A2): For every $E' \in S_i(n+1)$, either $E' \in S_i(n)$ or there is an $E'' \in S_i(n)$ such that $E'' \subsetneq E'$ and for all $x \in E' \setminus E''$, $x > \max(S(n))$. Therefore, S_0 and S_1 are finitely branching trees and we can treat S_0 and S_1 like infinite families with the exception that an infinite path in S_i may code a finite set. In particular, we write $S' \leq S_i$, for a family S' , to denote that for every level n in S' , there is a level $m \geq n$ in S_i such that for all $E' \in S'(n)$, there is an $E \in S_i(m)$ such that $E' \subseteq E$. Note that if $S' \leq S_i$, then $S' \leq S$.

In the next lemma, we show that we can always refine at least one of S_0 or S_1 to obtain a family of subtournaments contained in S .

Lemma 4.22. *Let S be an infinite family of subtournaments and let S_0 and S_1 be generated by a pointwise partition g of S . Then there is an infinite family $S' \leq S_i$ for some $i \in \{0, 1\}$.*

Proof. The proof splits into two cases. For the first case, suppose that there is an n and an $E_0 \in S_0(n)$ such that $E_0 \in S_0(m)$ for all $m \geq n$. Then, for every $m \geq n$, there is an $E_1^m \in S_1(m)$ such that $E_0 \cup E_1^m \in S(m)$. In particular, we have $g(x) = 0$ for all $x \in E_0$ and $g(x) = 1$ for all $x \in E_1^m$. Define $S'(k)$ by

$$S'(k) = \{E_1 \in S_1(n+k) \mid E_0 \cup E_1 \in S(n+k)\}.$$

We show that S' is an infinite family of potential subtournaments from which it immediately follows that $S' \leq S_1$.

Condition (A1) in Definition 4.3 for S' is clear, so consider (A2). Fix $E_1 \in S'(\ell + 1)$. We need to find $E'_1 \in S'(\ell)$ such that $E'_1 \subsetneq E_1$ and for all $x \in E_1 \setminus E'_1$, $x > \max(S'(\ell))$.

Since $E_1 \in S'(\ell + 1)$, we have $E_0 \cup E_1 \in S(n + \ell + 1)$ and we can fix $E \in S(n + \ell)$ such that $E \subsetneq E_0 \cup E_1$ and for all $x \in (E_0 \cup E_1) \setminus E$, $x > \max(S(n + \ell))$. Since $E_0 \in S_0(n + \ell)$, and hence is contained in some element of $S(n + \ell)$, every element $x \in E_0$ satisfies $x \leq \max(S(n + \ell))$. Therefore, $E_0 \subseteq E$. Set $E'_1 = E \setminus E_0$ and we check that E'_1 has the desired properties.

To see that $E'_1 \subseteq E_1$, notice that $E'_1 \subseteq E \subseteq E_0 \cup E_1$ and $E'_1 \cap E_0 = \emptyset$. Therefore, $E'_1 \subseteq E_1$. Furthermore, since $E_0 \subseteq E \subsetneq E_0 \cup E_1$, it follows that $E'_1 \subsetneq E_1$. If $x \in E_1 \setminus E'_1$, then $x \in (E_0 \cup E_1) \setminus E$ and hence $x > \max(S(n + \ell)) \geq \max(S'(\ell))$. To see that $E'_1 \in S'(\ell)$, note that $E'_1 \subseteq E_1$ implies that $g(x) = 1$ for all $x \in E'_1$. Since $E \in S(n + \ell)$ and $E = E_0 \cup E'_1$, it follows that $E'_1 \in S_1(n + \ell)$ and $E'_1 \in S'(\ell)$.

To show that S' is infinite, it suffices to show that for every k , there is an $E \in S(n + k)$ such that $E_0 \subseteq E$ and $E \setminus E_0 \in S_1(n + k)$. If this property failed for some k , then for all $E'_0 \in S_0(n + k)$ with $E_0 \subseteq E'_0$, we would have $E_0 \subsetneq E'_0$ and hence $E_0 \notin S_0(n + k)$, contrary to our case assumption.

Now, assume that the first case fails. Then, for each level n and each $E_0 \in S_0(n)$, there is a level $m > n$ such that $E_0 \notin S_0(m)$. (This can happen either because E_0 has been properly extended or because the corresponding branch in S has been eliminated.) Because the levels of S_0 are finite, it follows that for each level n , there is a level m such that for all $E_0 \in S_0(n)$, $E_0 \notin S_0(m)$. (Recall that S_0 is infinite because S is infinite.) We define $S' \leq S_0$ inductively. Let $S'(0) = S_0(0)$. Given $S'(n) = S_0(n_0)$, let $S'(n + 1) = S_0(m)$ for the least $m > n_0$ such that for every $E_0 \in S_0(n_0)$, $E_0 \notin S_0(m)$. The fact that S' is an infinite family of subtournaments follows almost immediately from this definition. \square

Definition 4.21 and Lemma 4.22 can be extended to pointwise partitions of S into any fixed finite number of pieces.

Definition 4.23. Let S be a family of subtournaments and let $E \in S(n)$. $S \upharpoonright E$ is the family of subtournaments defined by

$$(S \upharpoonright E)(m) = \{E' \mid E' \cap E = \emptyset \wedge E' \cup E \in S(n + m + 1) \wedge \forall x \in E' (x > \max(S(n)))\}.$$

As a tree, $S \upharpoonright E$ is formed by taking the subtree of S above E and removing the set E from each node.

If $E \in S(n)$ is on an infinite path through S , then $S \upharpoonright E$ is an infinite family of subtournaments. Note that $E < E'$ for all $E' \in (S \upharpoonright E)(0)$ and that $E + (S \upharpoonright E)$ corresponds to the subtree of S above E . We can now give the proof of Lemma 4.19.

Proof. Fix an infinite family S and a node $E \in S(n)$ which is on an infinite path through S . Let P_E be the (finite) set of (ordered) partitions of E defined by

$$P_E = \{\langle E_0, E_1 \rangle \mid E_0 \cup E_1 = E \text{ and } E_0 \cap E_1 = \emptyset\}.$$

We define a pointwise partition g of the infinite family $S \upharpoonright E$. Let g be the function from $\bigcup(S \upharpoonright E)$ into P_E defined by $g(x) = \langle E_0, E_1 \rangle$ where

$$E_0 = \{a \in E \mid T_e^X(a, x) \text{ holds}\} \text{ and } E_1 = \{b \in E \mid T_e^X(x, b) \text{ holds}\}.$$

By the extended version of Lemma 4.22, there is an infinite family S' such that $S' \leq S_{\langle E_0, E_1 \rangle}$ for a fixed partition $\langle E_0, E_1 \rangle$. Since $E < E'$ for all $E' \in S'(0)$, we have that $E_0 + S' \leq S$ and $E_1 + S' \leq S$ are infinite families.

Fix m and $E' \in S'(m)$. Since $S' \leq S_{\langle E_0, E_1 \rangle}$, there is an $k \geq m$ and an $\tilde{E} \in S_{\langle E_0, E_1 \rangle}(k)$ such that $E' \subseteq \tilde{E}$. It follows that for all $x \in E'$, $a \in E_0$ and $b \in E_1$, we have $T_e^X(a, x)$ and $T_e^X(x, b)$. Therefore, $E_0 \rightarrow E'$ and $E' \rightarrow E_1$ as required. \square

Lastly, we turn to the remaining case in the proof of Theorem 4.15, when \mathcal{K}^X is not essential below (F, I, S) and $a_{\mathcal{K}}^X(F)$ converges.

Lemma 4.24. *If \mathcal{K}^X is not essential below (F, I, S) and $a_{\mathcal{K}}^X(F)$ converges, then there is an infinite family $S'' \leq S$ such that $(F, I, S'') \leq (F, I, S)$ settles $\mathcal{K}^{X, A^*(c), B^*(c)}$.*

Proof. Assume \mathcal{K}^X is not essential below (F, I, S) and $a_{\mathcal{K}}^X(F)$ converges. Fix x such that for all levels n and all $B > x$, there is an $E \in S(n)$ and a partition $E_0 \cup E_1 = E$ such that for all $i \in \{0, 1\}$ and all transitive $F' \subseteq E_i$, $F \cup F' \notin \mathcal{K}^{X, a_{\mathcal{K}}^X(F), B}$. Because the dependence on B in $\mathcal{K}^{X, a_{\mathcal{K}}^X(F), B}$ is positive, we can restrict our attention to sets B of the form $(x, x + v + 2)$ for all v . Before proceeding, we examine this hypothesis in more detail.

Let $Q(u, v)$ be the predicate such that for any finite tournament E and any v , $Q(E, v)$ holds if and only if

$$\exists \text{ partition } E_0 \cup E_1 = E \forall i \in \{0, 1\} \forall \text{ transitive } F' \subseteq E_i (F \cup F' \notin \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, x+v+2)}).$$

Because

$$F \cup F' \notin \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, x+v+2)} \Leftrightarrow \forall b (x < b < x + v + 2 \rightarrow \neg R_{\mathcal{K}}^X(F \cup F', a_{\mathcal{K}}^X(F), b))$$

the predicate $Q(E, v)$ is X -computable. We write $Q(E, \infty)$ for the same predicate with the inequality $x < b < x + v + 2$ replaced by $x < b$ and note that $Q(E, \infty)$ is a $\Pi_1^{0, X}$ predicate.

We can restate our hypothesis in terms of Q . For every level n and for every v , there exists $E \in S(n)$ such that $Q(E, v)$. Since each level $S(n)$ is finite, we have that for every level n there exists $E \in S(n)$ such that $Q(E, \infty)$.

Suppose $E \in S(n+1)$ is a successor of $\tilde{E} \in S(n)$. If $Q(E, v)$ holds (allowing the possibility that $v = \infty$), then $Q(\tilde{E}, v)$ holds as well because the witnessing partition $E_0 \cup E_1 = E$ restricts to a witnessing partition $\tilde{E}_0 \cup \tilde{E}_1 = \tilde{E}$ for $Q(\tilde{E}, v)$. Therefore, we can define two subtrees, S' and \tilde{S} , of S as follows:

$$\begin{aligned} S'(n) &= \{E \in S(n) \mid Q(E, n) \text{ holds}\} \\ \tilde{S}(n) &= \{E \in S(n) \mid Q(E, \infty) \text{ holds}\}. \end{aligned}$$

Because the relation $Q(E, n)$ is X -computable, $S' \leq S$ is an infinite family of subtournaments. \tilde{S} satisfies all the requirements for being an infinite subtournament except it is not X -computable. However, if $E \in S'(n)$ is on an infinite path through S' then $E \in \tilde{S}(n)$.

To define S'' , we will partition S' . We consider each $E \in S'(n)$ and look at all partitions $E_0 \cup E_1 = E$ such that for all $i \in \{0, 1\}$ and all transitive $F' \subseteq E_i$, $F' \cup F' \notin \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, x+n+2)}$. We will form a tree T_0 of all “left halves” of such splittings (and hence implicitly also a tree of all “right halves” of such splittings) where we choose the “left halves” in a coherent manner. Then, we show how to define an appropriate infinite family $S'' \leq S$ from T_0 which settles $\mathcal{K}^{X, A^*(c), B^*(c)}$.

Formally, we proceed as follows. Let $R(u, v, w, z)$ be the X -computable predicate such that $R(E_0, E_1, E, n)$ holds if and only if $E \in S'(n)$ and $E_0 \cup E_1 = E$ is a partition such that for all $i \in \{0, 1\}$ and all transitive $F' \subseteq E_i$, $F' \cup F' \notin \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, x+n+2)}$. Notice that R is symmetric in the E_0 and E_1 variables and that for all $E \in S(n)$, $Q(E, n)$ holds if and only if there are E_0 and E_1 such that $R(E_0, E_1, E, n)$ holds. We define the tree T_0 inductively, starting with placing a root in T_0 .

To define the nodes at level 1 in T_0 , consider each $E \in S'(0)$ in turn. Find the set of all (unordered) partitions $\{E_0, E_1\}$ such that $R(E_0, E_1, E, 0)$ holds (and hence also $R(E_1, E_0, E, 0)$ holds). For each such $\{E_0, E_1\}$, add a node σ at level 1 to T_0 and label this node by an arbitrarily chosen element of $\{E_0, E_1\}$, suppose it is E_0 . We indicate this labeling by $E_\sigma = E_0$. Also, include a second label for σ indicating the set $E \in S'(0)$ which has been split. We write $S_\sigma = E$ to indicate this information.

To define the nodes at level $n + 1$ (for $n > 0$), consider each $E \in S'(n)$ in turn. Fix $E \in S'(n)$ and let $E' \in S'(n - 1)$ be the predecessor of E in the tree S' . Find the set of all (unordered) partitions $\{E_0, E_1\}$ such that $R(E_0, E_1, E, n)$ holds. Consider each of these sets in turn.

The partition $E_0 \cup E_1 = E$ restricts to a partition $E'_0 \cup E'_1 = E'$ such that $R(E'_0, E'_1, E', n - 1)$ holds. By induction, there is a node δ at level n of T_0 such that $S_\delta = E'$ and either $E_\delta = E'_0$ or $E_\delta = E'_1$. Add a node σ to T_0 as a successor of δ . Set $S_\sigma = E$ and $E_\sigma = E_i$ where i is chosen such that $E_\delta = E'_i$.

This completes the description of T_0 . Notice that T_0 is an infinite finitely branching tree. For any nodes δ and σ such that σ is a successor of δ , we have $E_\delta \subseteq E_\sigma$ and $S_\delta \subseteq S_\sigma$ is the predecessor of S_σ in S' . It follows easily by induction that if τ is an extension of σ on T_0 with $E_\sigma = E_0$, $E_0 \cup E_1 = S_\sigma$, $E_\tau = E'_0$ and $E'_0 \cup E'_1 = S_\tau$, then $E_0 \subseteq E'_0$ and $E_1 \subseteq E'_1$.

Furthermore, we claim that if $E_0 = E_\sigma$, $E_1 \cup E_0 = S_\sigma$ and σ is on an infinite path through T_0 , then for each $i \in \{0, 1\}$ and all transitive $F' \subseteq E_i$, $F' \cup F' \notin \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, \infty)}$. (Below, we refer to this claim as the *main claim*.) To prove this claim, suppose for a contradiction that there is a transitive $\tilde{F} \subseteq E_i$ such that $F \cup \tilde{F} \in \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, \infty)}$. Fix k such that $F \cup \tilde{F} \in \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, x+k+2)}$ and let $\tau \in T_0$ be an extension of σ at level k . Let $E'_0 = E_\tau$ and let $E'_1 \cup E'_0 = S_\tau$. Because τ is on T_0 , for every transitive $F' \subseteq E'_i$, $F' \cup F' \notin \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, x+k+2)}$. However, because τ extends σ , we have $E_0 \subseteq E'_0$ and $E_1 \subseteq E'_1$. Therefore, $\tilde{F} \subseteq E_i \subseteq E'_i$ is transitive and $F \cup \tilde{F} \in \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, x+k+2)}$, giving the desired contradiction.

It remains to extract the infinite family $S'' \leq S$ which settles $\mathcal{K}^{X, A^*(c), B^*(c)}$. This extraction breaks into two cases.

For the first case, assume that there is a level n and an E_0 such that for every level $m \geq n$, there is a node $\sigma \in T_0$ at level m such that $E_\sigma = E_0$. Fix n and E_0 . Because T_0 is finitely branching, there must be a node $\delta \in T_0$ at level n such that for all $m \geq n$, there is an extension

σ of δ at level m such that $E_\sigma = E_0$. Fix such a δ . Define S'' to be the family such that $S''(k)$ is the set of all E_1 such that there is a node $\sigma \in T_0$ extending δ at level $n+k$ such that $E_\sigma = E_0$ and $S_\sigma = E_0 \cup E_1$. That is, $S''(k)$ is the set of all “right halves” of the splits of elements of $S'(n+k)$ for which E_0 was the “left half”.

We claim that S'' is the desired family. We check the required properties. (Note that Property (A1) in Definition 4.3 is immediate.)

- $S'' \leq S$: Fix $E_1 \in S''(k)$. Then, $E_0 \cup E_1 \in S'(n+k)$ and hence $E_0 \cup E_1 \in S(n+k)$. Therefore, $E_1 \subseteq E$ for some $E \in S(n+k)$.
- S'' is infinite: Fix k and we show that $S''(k)$ is nonempty. By assumption, there is a node $\sigma \in T_0$ extending δ at level $n+k$ with $E_\sigma = E_0$. Therefore, $S_\sigma \setminus E_0 \in S''(k)$.
- Property (A2) holds: Let $E_1 \in S''(k+1)$ and fix $\sigma \in T_0$ extending δ at level $n+k+1$ be such that $E_0 \cup E_1 = S_\sigma$. Let τ be the predecessor of σ on T_0 . Since $\delta \subseteq \tau \subseteq \sigma$ and $E_\delta = E_\sigma = E_0$, we have $E_\tau = E_0$. Therefore, $E'_1 = S_\tau \setminus E_0 \in S''(k)$. By construction, $S_\tau = E_0 \cup E'_1$ is the predecessor of $S_\sigma = E_0 \cup E_1$ in S' and in S . Therefore, $E'_1 \subseteq E_1$ and the elements of $E_1 \setminus E'_1 = (E_0 \cup E_1) \setminus (E'_1 \cup E_0)$ are greater than $\max(S(n+k))$ and hence greater than $\max(S''(k))$.
- (F, I, S'') settles $\mathcal{K}^{X, A^*(c), B^*(c)}$: Fix $E_1 \in S''(k)$ such that E_1 is on an infinite path through S'' . Let $\sigma \in T_0$ be the node witnessing that $E_1 \in S''(k)$. Then σ is on an infinite path through T_0 and $E_0 \cup E_1 = S_\sigma$. By the main claim, for all transitive $F' \subseteq E_1$, $F' \cup F' \notin \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, \infty)}$.

For the second case, assume that for any level n and any $E_0 = E_\sigma$ for some $\sigma \in T_0$ at level n , there is a level $m > n$ such that $E_\tau \neq E_0$ for all $\tau \in T_0$ at level m . Define $S''(k)$ inductively as follows. $S''(0)$ is the set of all E_0 such that $E_0 = E_\sigma$ for some $\sigma \in T_0$ at level 0. Assume that $S''(k)$ has been defined as the set of all E_0 such that $E_0 = E_\sigma$ for some $\sigma \in T_0$ at level ℓ_k . Let ℓ_{k+1} be the first level in T_0 such that for every $E_0 \in S''(k)$ and every $\tau \in T_0$ at level ℓ_{k+1} , $E_\tau \neq E_0$. By our case assumption and the fact that T_0 is finitely branching, ℓ_{k+1} is defined. Let $S''(k+1)$ be the set of all E_0 such that $E_0 = E_\tau$ for some $\tau \in T_0$ at level ℓ_{k+1} .

We claim that S'' is the desired family. We check the required properties. (Again, (A1) holds trivially.)

- $S'' \leq S$: If $E_0 \in S''(k)$, then $E_0 = E_\sigma$ for some $\sigma \in T_0$ at level ℓ_k . Let $E_1 = S_\sigma \setminus E_0$. Then, $E_0 \cup E_1 \in S'(\ell_k)$ and $E_0 \cup E_1 \in S(\ell_k)$. Therefore, $E_0 \subseteq E$ for some $E \in S(\ell_k)$.
- S'' is infinite: By our case assumption, ℓ_k is defined for all k . Since T_0 is infinite, $S''(k) \neq \emptyset$ for all k .
- Property (A2): Let $E_0 \in S''(k+1)$ and fix $\sigma \in T_0$ at level ℓ_{k+1} such that $E_\sigma = E_0$. Let $\tau \in T_0$ at level ℓ_k be such that $\tau \subseteq \sigma$. Then $E_\tau \in S''(k)$ and $E_\tau \subsetneq E_\sigma$. The elements in $E_\sigma \setminus E_\tau$ are all greater than the elements in $S(\ell_k)$ and hence are all greater than the elements in $S''(k)$.

- (F, I, S'') settles $\mathcal{K}^{X, A^*(c), B^*(c)}$: Fix $E_0 \in S''(k)$ such that E_0 is on an infinite path through S'' . Let $\sigma \in T_0$ at level ℓ_k be such that $E_0 = E_\sigma$. Because σ is on an infinite path through T_0 , it follows by the main claim that all transitive $F' \subseteq E_0$ satisfy $F \cup F' \notin \mathcal{K}^{X, a_{\mathcal{K}}^X(F), (x, \infty)}$.

□

4.2 Ground forcing

We now carry out the ground level forcing to produce the coloring c . Our forcing conditions are triples (c, A^*, B^*) where c is coloring of two element subsets of a finite domain $[0, |c|]$, A^* and B^* are subsets of $[0, |c|]$ and $A^* \cap B^* = \emptyset$. We say $(c, A^*, B^*) \leq (c_0, A_0^*, B_0^*)$ if:

- $c_0 \subseteq c$,
- $A_0^* \subseteq A^*$,
- $B_0^* \subseteq B^*$,
- Whenever $a \in A_0^*$ and $x > |c_0|$, $c(a, x) = 0$,
- Whenever $b \in B_0^*$ and $x > |c_0|$, $c(b, x) = 1$.

Clearly the set of (c, A^*, B^*) such that $i \in A^* \cup B^*$ is dense, so we may ensure that the coloring given by a generic is stable. We need to ensure that our generic coloring does not compute a solution to itself. We say $(c, A^*, B^*) \Vdash (\Phi_e^G \text{ is finite})$ if

$$\exists k \forall (c_0, A_0^*, B_0^*) \leq (c, A^*, B^*) \forall x (\Phi_e^{c_0}(x) = 1 \rightarrow x \leq k).$$

We say $(c, A^*, B^*) \Vdash (\Phi_e^G \not\subseteq A^*(G) \wedge \Phi_e^G \not\subseteq B^*(G))$ if

$$\exists a \in A^* \exists b \in B^* (\Phi_e^c(a) = \Phi_e^c(b) = 1).$$

Lemma 4.25. *For each index e , the set of conditions which either force Φ_e^G is finite or force $\Phi_e^G \not\subseteq A^*(G) \wedge \Phi_e^G \not\subseteq B^*(G)$ is dense.*

Proof. Fix an index e and a condition (c, A^*, B^*) . If some extension of (c, A^*, B^*) forces Φ_e^G is finite, then we are done. Otherwise, since (c, A^*, B^*) does not force Φ_e^G is finite, there is an $x > |c|$ and a condition (c_0, A^*, B^*) extending (c, A^*, B^*) such that $\Phi_e^{c_0}(x) = 1$. (Without loss of generality, only the coloring changes.) Since $(c_0, A^* \cup \{x\}, B^*) \leq (c, A^*, B^*)$, $(c_0, A^* \cup \{x\}, B^*)$ does not force Φ_e^G is finite. So there is a $y > |c_0|$ and a condition $(c_1, A^* \cup \{x\}, B^*)$ extending $(c_0, A^* \cup \{x\}, B^*)$ such that $\Phi_e^{c_1}(y) = 1$. The condition $(c_1, A^* \cup \{x\}, B^* \cup \{y\})$ extends (c, A^*, B^*) and forces $\Phi_e^G \not\subseteq A^*(G) \wedge \Phi_e^G \not\subseteq B^*(G)$. □

Finally, we need to force the requirements $\mathcal{K}^{G, A^*(G), B^*(G)}$ for any generic G to be uniformly dense in \mathbb{Q}_e^G . Fix an index e and a potential iterated forcing condition $p = (F_p, I_p, S_p)$, where F_p is a finite set, I_p is a pair of elements in F_p and S_p is the index for a potential family of subtournaments of T_e^G . We define the following forcing notions.

- $(c, A^*, B^*) \Vdash (F_p \text{ is a transitive subtournament of } T_e^G)$ if for every $u, v \in F_p$, $\Phi_e^c(u, v)$ converges and the induced structure on F_p makes it a transitive tournament.
- $(c, A^*, B^*) \Vdash (F_p \text{ is a not transitive subtournament of } T_e^G)$ if there is no extension $(c_0, A_0^*, B_0^*) \leq (c, A^*, B^*)$ forcing F_p is a transitive subtournament of T_e^G .
- $(c, A^*, B^*) \Vdash (S_q^G \text{ is not total})$ if there is an ℓ such that for every $(c_0, A_0^*, B_0^*) \leq (c, A^*, B^*)$, $S_q^{c_0}(\ell)$ diverges.
- $(c, A^*, B^*) \Vdash (p \notin \mathbb{Q}_e^G)$ if any of the following conditions hold
 - $(c, A^*, B^*) \Vdash (F_p \text{ is not a transitive subtournament of } T_e^G)$; or
 - $(c, A^*, B^*) \Vdash (F_p \text{ is a transitive subtournament of } T_e^G)$ but I_p is not a minimal interval in F_p ; or
 - $(c, A^*, B^*) \Vdash (S_q^G \text{ is not total})$; or
 - $S_q^c(0) = \{\tilde{F}_0, \dots, \tilde{F}_k\}$ and $F \not\subseteq \tilde{F}_j$ for some j ; or
 - there is an n such that $S_q^c(n)$ and $S_q^c(n+1)$ both converge, but they violate the (A2) condition.
- $(c, A^*, B^*) \Vdash (\mathcal{K}^G \text{ is not essential below } p)$ if for every completion $(\tilde{c}, A^*(\tilde{c}), B^*(\tilde{c}))$ of (c, A^*, B^*) to a stable 2-coloring of ω for which $\mathcal{K}^{\tilde{c}, A^*(\tilde{c}), B^*(\tilde{c})}$ is a requirement and $p \in \mathbb{Q}_e^{\tilde{c}}$, $\mathcal{K}^{\tilde{c}}$ is not essential below p .

Lemma 4.26. *Let $\mathcal{K}^{G, A^*(G), B^*(G)}$ be a potential requirement given by the indices i and i' . Then for any potential iterated forcing condition p , there is a dense set of conditions (c, A^*, B^*) such that:*

- $(c, A^*, B^*) \Vdash p \notin \mathbb{Q}_e^G$; or
- $(c, A^*, B^*) \Vdash \mathcal{K}^G \text{ is not essential below } p$; or
- *there is a level n such that $S_p^c(n)$ converges and whenever $E \in S_p^c(n)$ and $E = E_0 \cup E_1$ is a partition, there is an $i \in \{0, 1\}$ and a transitive $F' \subseteq E_i$ such that*

$$\exists a \in A^* \exists b \in B^* (\Phi_i^c(F_p \cup F', a, b) = 1).$$

Proof. Fix a condition (c, A^*, B^*) and a potential iterated forcing condition $p = (F_p, I_p, S_p)$. If there is any $(c', A', B') \leq (c, A^*, B^*)$ forcing that $p \notin \mathbb{Q}_e^c$ then we are done, so assume not.

Suppose there is an extension $(c', A^*, B^*) \leq (c, A^*, B^*)$ such that $a_{\mathcal{K}}^{c'}(F_p)$ converges, a finite set $B > \max(A^* \cup B^* \cup \{a_{\mathcal{K}}^{c'}(F_p)\})$ and an n such that $S_p^{c'}(n)$ converges and whenever $E \in S_p^{c'}(n)$ and $E = E_0 \cup E_1$ is a partition, there is an $i \in \{0, 1\}$ and a transitive $F' \subseteq E_i$ such that

$$\exists b \in B (\Phi_i^{c'}(F_p \cup F', a_{\mathcal{K}}^{c'}(F_p), b) = 1)$$

i.e. $F_p \cup F' \in \mathcal{K}^{c', a_{\mathcal{K}}^{c'}(F_p), B}$. Without loss of generality, we may assume $a_{\mathcal{K}}^{c'}(F_p) \in A^*$ or $a_{\mathcal{K}}^{c'}(F_p) \in B^*$. For simplicity, we assume $a_{\mathcal{K}}^{c'}(F_p) \in A^*$ as the other case is symmetric. Then $(c', A^*, B^* \cup B)$ is the desired condition.

Suppose there is no such (c', A^*, B^*) . Then we claim that (c, A^*, B^*) already forces that \mathcal{K}^G is not essential below p . Let \tilde{c} be any completion of c to a stable 2-coloring on ω , and suppose \mathcal{K}^c were essential below p . Then there would be some $B > \max(A^* \cup B^* \cup \{a_{\mathcal{K}^c}(F_p)\})$ and an n such that $S_p^c(n)$ converges and whenever $E \in S_p^c(n)$, every partition is as described above. In particular, there would be some finite initial segment of \tilde{c} witnessing the necessary computations, contradicting our assumption. \square

The ideal to prove Theorem 1.15 is constructed in the same way as the ideal to prove Theorem 1.7.

5 Conclusion

We end with a few questions for future investigation. [4] introduced two other principles we have not discussed:

(CRT₂²) Cohesive Ramsey's Theorem for pairs: Every 2-coloring c of $[\mathbb{N}]^2$ has an infinite set S such that $\forall x \in S \exists y \forall z \in S [z > y \rightarrow c(x, y) = c(x, z)]$

(CADS) Cohesive ADS: Every linear order has either an infinite subset which has order-type ω , ω^* , or $\omega + \omega^*$.

They show that COH implies CRT₂² which in turn implies CADS. Both reversals remain open:

Question 5.1. Does COH imply CRT₂²? Does CADS imply CRT₂²?

While [3] recently showed that SRT₂² does not imply RT₂² using a nonstandard model, the following remains open:

Question 5.2. Does SRT₂² imply RT₂² in ω -models?

Their method requires constructing the entire Turing ideal satisfying SRT₂² simultaneously in a way similar to the method used in this paper. There is an essential difference, however: in the construction in [3], the instance of RT₂² and all solutions are low. Since the low sets are all known in advance, they can construct an instance of RT₂² by diagonalizing against all low sets, and then separately construct a Turing ideal satisfying SRT₂² and consisting only of low sets.

In this paper, on the other hand, we constructed a Turing ideal by staying inside a collection of sets which could only be identified after a particular instance of SCAC or SRT₂² had been chosen. This required that the diagonalization step anticipate what may happen in the construction of the Turing ideal (by forcing certain sets to be generic in a future forcing notion). This is unusual—most similar arguments in reverse mathematics carry out a diagonalization step against a collection of sets which have an a priori definition. If this is necessary, it indicates a difference in kind between the non-implications in this paper and others in the literature. To make this precise, we need to decide what we mean by an a priori collection of sets. One way is the following:

Question 5.3. Is there a set of Turing degrees \mathcal{S} definable without parameters (in first-order logic) from \leq and jump, together with an instance (M, \preceq_M) of SCAC belonging to \mathcal{S} such that:

- No element of \mathcal{S} is a solution to (M, \preceq_M) , and
- There is a Turing ideal $\mathcal{I} \subseteq \mathcal{S}$ containing (M, \preceq_M) and satisfying ADS?

All other known non-implications between combinatorial principles weaker than or similar to RT_2^2 have positive solutions to the corresponding variant of this question.

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