Stability and posets

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RT^2_2 and CAC

- K_ω is the (countably) infinite graph in which every pair of nodes is connected.
- $\overline{K}_\omega$ is the infinite graph in which no pair of nodes is connected.

Theorem (Graph Version of Ramsey's Theorem for Pairs (RT^2_2))

Every infinite graph contains a copy of K_ω or $\overline{K}_\omega$.

Theorem (Chain–Antichain (CAC))

Every infinite poset has either an infinite chain or an infinite antichain.

In this talk, all chains and antichains are infinite.

Proving CAC from RT^2_2

For a poset P, define its *comparability graph* G_P by

- domain of G_P = domain of P
- a and b are connected in G_P iff a and b are comparable in P

Then,

- copies of K_ω in G_P are chains in P (and vice versa)
- copies of $\overline{K}_\omega$ in G_P are antichains in P (and vice versa)

So, a solution to RT^2_2 in G_P is a solution to CAC in P.

How hard is it to solve *CAC* for a computable poset?

By transferring his results on RT^2_2, Jockusch proved

- In the arithmetic hierarchy: Every computable poset has a Δ^0_2 chain, or a Δ^0_2 antichain, or both a Π^0_2 chain and a Π^0_2 antichain.
- In low hierarchy: Every computable poset has a low_2 chain or antichain.

Herrmann proved that you cannot improve these bounds.

- There is a computable poset with no Σ^0_2 chains or antichains.
- There is a computable poset with no low chains or antichains.

A clever idea of Cholak, Jockusch and Slaman

Split RT^2_2 into a *stable* version SRT^2_2 and a *cohesive* version CRT^2_2.

Definition

G is *stable* if for every $x \in G$, either x is connected to almost every other node or x is not connected to almost every node.

- SRT^2_2: Every infinite *stable* graph contains a copy of K_ω or $\overline{K}_\omega$.
- CRT^2_2: Every infinite graph has an infinite stable subgraph.
- $RT^2_2 \Leftrightarrow SRT^2_2 + CRT^2_2$
- CRT^2_2 is strictly weaker than RT^2_2
- Open question: Is SRT^2_2 strictly weaker than RT^2_2?

A clever idea of Hirschfeldt and Shore

Why not do the same thing for CAC?

To do this, they defined a notion of a *stable poset* (given later).

- *SCAC*: Every infinite stable poset has a chain or antichain.
- *CCAC*: Every infinite poset contains an infinite stable poset.
- $CAC \Leftrightarrow SCAC + CCAC$.
- Both *SCAC* and *CCAC* are strictly weaker than *CAC*.
- Analyzing *SCAC* and *CCAC*, they proved that *CAC* is strictly weaker than RT^2_2.

Stable posets

Definition

Fix an infinite poset P. An element $a \in P$ is

- *small* if $a <_P b$ for almost all $b \in P$
- *large* if $b <_P a$ for almost all $b \in P$
- *isolated* if a is incomparable with almost all $b \in P$

$S_P =$ the set of small elements in P

$L_P =$ the set of large elements in P

$I_P =$ the set of isolated elements in P

Definition (Hirschfeldt and Shore)

A poset P is *stable* if either $P = S_P \cup I_P$ or $P = L_P \cup I_P$.

Our work

Why restrict to $P = S_P \cup I_P$ or $P = L_P \cup I_P$ in definition of stability?

Definition

An infinite poset is *weakly stable* if $P = S_P \cup L_P \cup I_P$.

Note that

$$\text{stable} \;\Rightarrow\; \text{weakly stable}$$

but not conversely. For example, let P be the linear order $\omega + \omega^*$ viewed as a poset.

- $S_P =$ the elements in the ω part.
- $L_P =$ the elements in the ω^* part.
- $I_P = \emptyset$.

Therefore, P is weakly stable but not stable.

Definition (Comparability graph G_P of poset P)

$G_P = P$ with an edge between a and b if a and b are comparable.

$$P \text{ is a weakly stable poset } \Rightarrow G_P \text{ is a stable graph}$$

$$P \text{ is a weakly stable poset } \not\Leftarrow G_P \text{ is a stable graph}$$

For the linear order $\mathbb{Z}$ (viewed as a partial order), we have

- $G_{\mathbb{Z}} = K_\omega$ (and hence is a stable graph), but
- $S_{\mathbb{Z}} = L_{\mathbb{Z}} = I_{\mathbb{Z}} = \emptyset$ (and hence $\mathbb{Z}$ is not a weakly stable poset).

Notice that every copy $\mathcal{L}$ of $\mathbb{Z}$ has an infinite chain which is $\Delta^0_1(\mathcal{L})$.

Theorem (JKLLS)

If an infinite poset has a copy P such that no chain is $\Delta^0_1(P)$, then

$$P \text{ is weakly stable } \Leftrightarrow G_P \text{ is stable}$$

Assume G_P is stable but P is not weakly stable. Fix $a \notin S_P \cup L_P \cup I_P$.

- $a \notin I_P$ implies a is comparable with infinitely many (hence almost all) $p \in P$.
- $a \notin S_P \cup L_P$ implies there are infinitely many $p > a$ and infinitely many $p < a$.
- If $b \leq a$, then $b < p$ for infinitely many p and hence b is comparable with almost all $p \in P$. (Same for $b \geq a$.)
- Let $X \subseteq P$ consisting of elements comparable to a. X is $\Delta^0_1(P)$.
- Every element of X is comparable with almost every $p \in P$.
- There is a chain $C \in \Delta^0_1(X)$ and hence $C \in \Delta^0_1(P)$.

Reverse mathematics

These two notions of stability give rise to two different stable versions of *CAC*.

- *SCAC*: Every infinite *stable* poset has a chain or antichain.
- *WSCAC*: Every infinite *weakly stable* poset has a chain or antichain.

Theorem (JKLLS)

Over RCA_0, SCAC and WSCAC are equivalent.

Arithmetic hierarchy results

For a computable (weakly) stable P,

- each of S_P, L_P and I_P are Δ^0_2
- if P has chains, then P has Δ^0_2 chains
- if P has antichains, then P has Δ^0_2 antichains

For stable posets, we can do better than Δ^0_2.

Theorem (JKLLS)

Every computable stable poset has a computable chain or a Π^0_1 antichain.

However, the dual of this theorem fails.

Theorem (JKLLS)

There is a computable stable poset which has no Π^0_1 chain or computable antichain.

In the case of weakly stable posets, one cannot improve on Δ^0_2.

Theorem (JKLLS)

There is a computable weakly stable poset which has no Π^0_1 chains or Π^0_1 antichains.

Lowness hierarchy

Theorem (Hirschfeldt and Shore)

Every computable stable poset has a low chain or a computable antichain.

The dual of this theorem does hold

Theorem (JKLLS)

Every computable stable poset has a computable chain or a low antichain.

and it can be generalized to weakly stable posets.

Theorem (JKLLS)

Every computable weakly stable poset has a low chain or a computable antichain.

The dual of this theorem is open: Does a computable weakly stable poset have a computable chain or a low antichain?