The complexity of central series in nilpotent computable groups

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Two aspects of computable algebra

First aspect

Computable algebra is the study of computable model theory restricted to particular classes of structures.

From this point of view, one considers concepts such as

- computable dimension (relative computable dimension, **d**-dimension)
- degree spectra (of structures, of relations, ...)
- Scott rank
- index sets (for isomorphism problem, embedding problem, ...)

within the specified class.

Two aspects of computable algebra

Second aspect

Computable algebra is the study of the effectiveness of the basic theorems, constructions and structural properties within a specified class of structures.

From this point of view, one considers questions such as

- How complicated are the terms in the upper and lower central series of a nilpotent computable group?

Lower central series

Definition

Let G be a group and $x, y \in G$. The *commutator* of x and y is $[x, y] = x^{-1}y^{-1}xy$. If H and K are subgroups of G, then

$$[H, K] = \langle \{[h, k] \mid h \in H \text{ and } k \in K\} \rangle$$

Definition

The *lower central series of G* is

$$G = \gamma_1 G \geq \gamma_2 G \geq \gamma_3 G \geq \cdots$$

where $\gamma_{i+1} G = [\gamma_i G, G] = \langle \{ [x, g] \mid x \in \gamma_i G \text{ and } g \in G \} \rangle$.

- Any commutator $[x, g]$ is in $\gamma_2 G = [G, G]$.
- If $x = [u, v]$, then $[x, g] = [[u, v], g] \in \gamma_3 G$.
- More generally, we need to close under inverses and products so an element of $\gamma_{i+1} G$ looks like

$$[x_1, g_1]^{\alpha_1} \cdot [x_2, g_2]^{\alpha_2} \cdots [x_k, g_k]^{\alpha_k}$$

where $x_j \in \gamma_i G$, $g_j \in G$ and $\alpha_j \in \mathbb{Z}$.

Algebraic motivation for lower central series

- $\gamma_2 G$ is the smallest subgroup such that

$$\forall h, g \in G \, (gh = hg \text{ mod } \gamma_2 G)$$

- $\gamma_{i+1} G$ is the smallest subgroup such that

$$\forall h \in \gamma_i G,\ g \in G \, (gh = hg \text{ mod } \gamma_{i+1} G)$$

Upper central series

Definition

The *center* of a group G is

$$C(G) = \{g \in G \mid \forall h \in G(gh = hg)\}$$

$C(G)$ is a normal subgroup so there is a canonical projection map

$$\pi : G \to G/C(G) \text{ with } \pi(g) = gC(G)$$

If $g \in C(G)$, then $\pi(g) = 1_{G/C(G)}$ so $\pi(g) \in C(G/C(G))$. Therefore

$$C(G) \subseteq \pi^{-1}(C(G/C(G))$$

Definition

The *upper central series* of G is

$$1 = \zeta_0 G \leq \zeta_1 G \leq \zeta_2 G \leq \cdots$$

where $\zeta_{i+1} G = \pi^{-1}(C(G/\zeta_i G))$.

- $\zeta_1 G$ is the center of G.
- $h \in \zeta_{i+1} G \Leftrightarrow \forall g \in G(gh = hg \mod \zeta_i G)$
- $\zeta_{i+1} G$ is the largest subgroup of G such that

$$\forall h \in \zeta_{i+1} G,\ g \in G\,(gh = hg \text{ mod } \zeta_i G)$$

Nilpotent groups

Definition

A group G is *nilpotent* if the lower central series reaches 1 in finitely many steps, or equivalently, if the upper central series reaches G in finitely many steps.

The series are closely related. For example, $\gamma_{r+1}G = 1 \Leftrightarrow \zeta_r G = G$. Therefore, we say

$$G \text{ is class } r \text{ nilpotent} \Leftrightarrow \gamma_{r+1}G = 1 \Leftrightarrow \zeta_r G = G$$

More generally, in a class r nilpotent group,

$$\gamma_i G \leq \zeta_{r-i+1} G \text{ and in particular } \gamma_r G \leq \zeta_1 G$$

Mal'cev correspondence

Let R be a ring (with identity) and let G_R be all 3×3 matrices of form

$$\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in R$. G_R forms a class 2 nilpotent group.

Lemma

$\gamma_2 G_R = \zeta_1 G_R =$ *all matrices of form*

$$\begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Mal'cev correspondence

$$R \text{ (ring) } \longrightarrow G_R \text{ (class 2 nilpotent group)}$$

We can recover R from G_R. The center of G_R consists of the matrices

$$\begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $d \in R$. We recover addition in R by multiplication in G_R

$$\begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & e \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & d+e \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiplication can also be recovered effectively.

Applications of Mal'cev correspondence

Because the Mal'cev correspondence is highly effective, it can be used to transfer computable model theory results from the class of rings to the class of nilpotent groups.

Theorem (Goncharov, Molokov and Romanovskii)

For any $1 \leq n \leq \omega$, there is a computable nilpotent group with computable dimension n.

Theorem (Hirschfeldt, Khoussainov, Shore and Slinko)

- *Any degree spectra that can be realized by a countable graph can be realized by a countable nilpotent group.*
- *For each $1 \leq n \leq \omega$, there is a computably categorical nilpotent group G and an element $g \in G$ such that (G, g) has computable dimension n.*

Applications of Mal'cev correspondence

Theorem (Morozov (and others?))

There are computable nilpotent groups of Scott rank ω_1^{CK} *and* $\omega_1^{CK}+1$.

Theorem (various people)

The isomorphism problem for computable nilpotent groups is Σ_1^1*-complete.*

With regard to the general notions of computable model theory such as computable dimension, degree spectra, Scott rank, index sets and so on, we understand nilpotent groups very well.

Back to lower central series

Each lower central terms of a computable group is c.e.

Let G be a computable group. Recall $G = \gamma_1 G$ and $\gamma_{i+1} G = [\gamma_i G, G]$.

- $\gamma_1 G$ is computable (and hence c.e.)
- If $\gamma_i G$ is c.e., then so is $\{[h, g] \mid h \in \gamma_i G \text{ and } g \in G\}$.
- Therefore, $\gamma_{i+1} G$ is also c.e. since it is generated by this c.e. set.

Back to upper central series

Each upper central term of a computable group is co-c.e.

Let G be a computable group. Recall $1 = \zeta_0 G$ (and hence is co-c.e.) and

$$\begin{aligned} h \in \zeta_{i+1} G &\Leftrightarrow \forall g (gh = hg \bmod \zeta_i G) \\ &\Leftrightarrow \forall g ([g, h] \in \zeta_i G) \end{aligned}$$

Therefore, $\zeta_{i+1} G$ is a co-c.e. set.

Central question

Let G be computable group which is class r nilpotent. The terms

$$\gamma_1 G = \zeta_r G = G \text{ and } \gamma_{r+1} G = \zeta_0 G = 1$$

are trivially computable. The nontrivial terms

$$\gamma_2 G, \gamma_3 G, \ldots, \gamma_r G$$
$$\text{and } \zeta_1 G, \zeta_2 G, \ldots, \zeta_{r-1} G$$

have c.e. degree. What more can we say about the degrees of the nontrivial terms?

Main theorem

In fact, the nontrivial terms are computationally independent.

Theorem (Csima and Solomon)

Fix $r \geq 2$ and c.e. degrees $\mathbf{d_1}, \ldots, \mathbf{d_{r-1}}$ and $\mathbf{e_2}, \ldots, \mathbf{e_r}$. There is a computable group G which is class r nilpotent with $deg(\zeta_i G) = \mathbf{d_i}$ and $deg(\gamma_i G) = \mathbf{e_i}$.

Moreover, the group G in this theorem is torsion free and admits a computable order. Therefore, the computational independence result holds in the class of computable ordered groups as well.

What happens in Mal'cev correspondence?

Fix a ring R and the corresponding group G_R. Recall that

$$\zeta_1 G_R = \gamma_2 G_R$$

In any computable presentation of G_R, one of these sets is c.e. and the other is co-c.e. Therefore, both are computable.

Proving the main theorem

Lemma

For any groups G and H, $\gamma_i(G \times H) = \gamma_i G \times \gamma_i H$ and $\zeta_i(G \times H) = \zeta_i G \times \zeta_i H$.

Therefore, it suffices to prove the following theorem.

Theorem

For any $r \geq 2$ and c.e. degree $\mathbf{d}$, there exists

- *a class r nilpotent computable group G such that each $\zeta_i G$ is computable, each $\gamma_i G$ for $1 \leq i \leq r-1$ is computable and $deg(\gamma_r G) = \mathbf{d}$*
- *and a class r nilpotent computable group H such that each $\gamma_i H$ is computable, each $\zeta_i H$ for $0 \leq i \leq r-2$ is computable and $deg(\zeta_{r-1} G) = \mathbf{d}$.*

The value of speaking in Novosibirsk

Theorem (Latkin)

Fix $r \geq 2$ and c.e. degrees $\mathbf{e_2}, \ldots, \mathbf{e_r}$. There is a computable group G which is class r nilpotent with deg$(\gamma_i G) = \mathbf{e_i}$.

Free nilpotent groups

Definition

Extend the definition of commutators inductively by

$$[x_1, x_2 \ldots, x_{n+1}] = [[x_1, x_2, \ldots, x_n], x_{n+1}]$$

Characterizing nilpotent groups as a variety

A group G is class r nilpotent if and only if $[x_1, x_2, \ldots, x_{r+1}] = 1$ for all $x_1, \ldots, x_{r+1} \in G$.

The free class r nilpotent group on generators X is F/N where F is the free group on X and

$$N = \langle \{[g_1, g_2, \ldots, g_{r+1}] \mid g_1, \ldots, g_{r+1} \in F\} \rangle$$

Basic commutators

Let F be a free class r nilpotent group on X. Define the *basic commutators of weight k* by

- Each $x \in X$ is basic with $w(x) = 1$.
- Assume basic commutators of weight $\leq k$ are defined with order $\leq_k$. $[c, d]$ is a basic commutator of weight $k+1$ if
  - c and d are basic commutators of weight $\leq k$ and $w(c) + w(d) = k + 1$
  - $d <_k c$
  - if $c = [u, v]$, then $v \leq_k d$.

Let $\leq_{k+1}$ be an end-extension of $\leq_k$ with $[c, d] \leq_{k+1} [u, v]$ if $\langle c, d \rangle \leq_k^{lex} \langle u, v \rangle$.

This process stops with weight r basic commutators

Normal form theorem

Theorem (Marshall Hall Jr.)

Each $y \in F$ can be uniquely written as a finite product

$$c_0^{m_0} \cdot c_1^{m_1} \cdots c_l^{m_l}$$

where each c_i is a basic commutator, $c_i <_r c_{i+1}$ and $m_i \in \mathbb{Z} \setminus \{0\}$. Furthermore, $y \in \gamma_i F$ if and only if the normal form contains only basic commutators of weight $\geq i$.

Commutator relations

Definition

Define $[x, y^{(n)}]$ for $n \in \mathbb{N}$ by

$$[x, y^{(n)}] = [x, y, y, \ldots, y]$$

where y appears n many times. (So, $[x, y^{(0)}] = x$.)

Lemma

The following commutator relations holds in any nilpotent group

- $x \cdot y = y \cdot x \cdot [x, y]$
- $x^{-1} \cdot y = y \cdot [x, y]^{-1} \cdot x^{-1}$
- $x \cdot y^{-1} = y^{-1} \cdot x \cdot [x, y^{(2)}] \cdot [x, y^{(4)}] \cdots [x, y^{(3)}]^{-1} \cdot [x, y]^{-1}$
- $x^{-1} \cdot y^{-1} = y^{-1} \cdot [x, y] \cdot [x, y^{(3)}] \cdots [x, y^{(4)}]^{-1} \cdot [x, y^{(2)}]^{-1} \cdot x^{-1}$

Proof sketch

Fix $r \geq 2$ and a c.e. set D. Construct a computable group G such that

- G is class r nilpotent,
- every lower central term $\gamma_i G$ is computable,
- every upper central term $\zeta_i G$ is computable except $\zeta_{r-1} G$, and
- $\zeta_{r-1} G \equiv_T D$.

Build an auxiliary group H

Generators:

$$d < y_0 < y_1 < y_2 < \cdots$$

Nontrivial basic commutators:

$$d \quad \text{and} \quad [y_i, d^{(l)}] \text{ for } 0 \leq l < r$$

That is,

$$d, y_0, y_1, \ldots, [y_0, d], [y_1, d], \ldots, [y_0, d^{(2)}], [y_1, d^{(0)}], \ldots,$$

$$\ldots, [y_0, d^{(r-1)}], [y_1, d^{(r-1)}] \ldots$$

We make all other basic commutators, such as $[y_i, y_j]$, equal to 1.

- A word over the basic commutators is a sequence

$$c_1^{\alpha_1} c_2^{\alpha_2} \cdots c_k^{\alpha_k}$$

with each c_i a basic commutator and $\alpha_i \in \mathbb{Z} \setminus \{0\}$.
- A word over the basic commutators is an *H-normal form* if $c_i < c_{i+1}$ in the ordering of the basic commutators.
- The elements of H are the H-normal forms.

Multiplication rules

(R1)

$$[y_i, d^{(k)}]\,[y_j, d^{(l)}] = [y_j, d^{(l)}]\,[y_i, d^{(k)}]$$

(R2)

$$[y_i, d^{(l)}]\,d = d\,[y_i, d^{(l)}]\,[y_i, d^{(l+1)}]$$

(R3)

$$[y_i, d^{(l)}]^{-1}\,d = d\,[y_i, d^{(l)}]^{-1}\,[y_i, d^{(l+1)}]^{-1}$$

(R4)

$$[y_i, d^{(l)}]\,d^{-1} = d^{-1}\,[y_i, d^{(l)}]\prod_{k=1}^{r-l-1}[y_i, d^{(l+k)}]^{(-1)^k}$$

(R5)

$$[y_i, d^{(l)}]^{-1}\,d^{-1} = d^{-1}\,[y_i, d^{(l)}]^{-1}\prod_{k=1}^{r-l-1}[y_i, d^{(l+i)}]^{(-1)^{k+1}}$$

To multiply two elements of H

$$(d^\beta c_0^{\alpha_0} c_1^{\alpha_1} \cdots c_k^{\alpha_k}) \cdot (d^\mu d_0^{\delta_0} d_1^{\delta_1} \cdots d_l^{\delta_l})$$

use the multiplication rules to pass d^μ left across $c_k^{\alpha_k}$, then $c_{k-1}^{\alpha_{k-1}}$ and so on, until it combines with d^β to form $d^{\beta+\mu}$. Then rearrange the remaining basic commutators in order since they commute.

Lemma

With these multiplication rules, H is a group. Moreover, for $x \in H$, we have $x \in \gamma_{j+1}H$ if and only if x contains only basic commutators $[y_i, d^{(l)}]$ with $l \geq j$.

Coding one number from D

- Fix a 1-to-1 function f with range D.
- To code whether $k \in D$, let $H_k = H$ and

$$S_k = \{[y_i, d^{(r-1)}] \mid \neg \exists j \leq i \, (f(j) = k)\} \subseteq \gamma_r H_k \leq \zeta_1 H_K$$

  $S_k \subseteq C(H_k)$, so $\langle S_k \rangle$ is normal.
- Let $G_k = H_k / \langle S_k \rangle$.
- If $k \notin D$, then $\langle S_k \rangle = \gamma_r H_k$ and hence G_k is a class $r-1$ nilpotent group which means $d \in \zeta_{r-1} G_k$.
- If $k \in D$, then G_k is properly class r and $d \notin \zeta_{r-1} G_k$.

The lower central terms in G_k are computable.

$$x \in \gamma_{j+1} G_k \Leftrightarrow \text{ all commutators in } x \text{ are } [y_i, d^{(l)}] \text{ for } l \geq j$$

For $0 \leq j \leq r-2$, $\zeta_j G_k$ is computable.

$$x \in \zeta_j G_k \Leftrightarrow \text{ all commutators in } x \text{ are } [y_i, d^{(l)}] \text{ for } l \geq r-j$$
$$\text{or } [y_i, d^{(l)}] \text{ for } l \geq r-j-1 \text{ with } \neg\exists u \leq i(f(u) = k)$$

Putting it all together

Let $G = \bigoplus_{k \in \omega} G_k$ and denote the generator of G_k by

$$d_k, \, y_{0,k}, \, y_{1,k}, \ldots$$

Since the lower central terms in G_k and the upper central terms (except $\zeta_{r-1} G_k$) are uniformly computable, the corresponding terms in G are computable.

$\zeta_{r-1} G \equiv_T D$ because

$$k \in D \Leftrightarrow d_k \notin \zeta_{r-1} G$$