Notes on the Dual Ramsey Theorem

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1 Partitions and infinite variable words

The goal of these notes is to give a proof of the Dual Ramsey Theorem. This theorem was first proved in "A Dual Form of Ramsey's Theorem" by Tim Carlson and Steve Simpson, Advances in Mathematics, 1984. The proof presented here is the same proof given in the paper and follows much of the same notation and terminology. The main change is that we will explicitly use the notion of "infinite variable words" as used by Simpson in later discussions of this theorem. (For example, Simpson uses this language in his discussion of reverse mathematics and the Dual Ramsey Theorem in the Proceedings Volume from the Boulder conference.)

Before stating the Dual Ramsey Theorem, we need to introduce some notation and terminology. We let $\omega = \{0, 1, \ldots\}$ and A be a finite (possibly empty) alphabet which is disjoint from ω.

Definition 1.1. An A-*partition* is a collection of pairwise disjoint nonempty subsets of $A \cup \omega$ (called *blocks*) whose union is $A \cup \omega$ and such that each block contains at most one element of A. A block which is disjoint from A is called *free*.

Example 1.2. For many of the examples, we will work with the alphabet $A = \{a, b, c\}$. The A-partition P given by

$$\{a, 0, 2, 4, 6, \ldots\}, \{b, 1, 3, 5, \ldots\}, \{c\}$$

has no free blocks, while the A-partition Q given by

$$\{a, 0, 6, 12, 18, \ldots\}, \{b, 2, 8, 14, 20, \ldots\}, \{c, 4, 10, 16, 22, \ldots\}, \{1, 3\}, \{5, 7\}, \{9, 11\}, \ldots$$

has infinitely many free blocks. We think of the free blocks of Q as indexed by elements of ω, with the ordering determined by the least element of each block. That is, the 0-th free block of Q is $\{1, 3\}$, the 1-st free block of Q is $\{5, 7\}$ the 2-nd free block is $\{9, 11\}$ and so on.

We let $(\omega)^\omega_A$ denote the set of all A-partitions containing infinitely many free blocks and we let $(\omega)^k_A$ denote the set of all A-partitions containing exactly k many free blocks. Thus, in Example 1.2, $P \in (\omega)^0_A$ and $Q \in (\omega)^\omega_A$. If X is an A-partition, then we refer to the blocks of X as X-blocks. Note that if $A = \emptyset$, then an A-partition is just an ordinary partition of ω. In this case, we require that $k \geq 1$ (since $k = 0$ does not make sense if $A = \emptyset$), and we write

$(\omega)^\omega$ and $(\omega)^k$ in place of $(\omega)^\omega_\emptyset$ and $(\omega)^k_\emptyset$. Thus, $(\omega)^\omega$ denotes the set of all partitions of ω into infinitely many blocks and $(\omega)^k$ (for $k \geq 1$) denotes the set of all partitions of ω into k many blocks.

Definition 1.3. If X and Y are A-partitions, then Y *is coarser than* X if each X-block is contained in a Y-block. That is, we can obtain Y from X by collapsing X-blocks.

When we deal with coarsening in these notes, we always work within the collection of A-partitions. Thus, if X is an A-partition then we cannot collapse distinct nonfree blocks together because this would violate the condition that each block contains at most one element of A. For example, the only coarsening of the partition P in Example 1.2 is P because it has no free blocks. More generally, any element of $(\omega)^0_A$ cannot be properly coarsened.

Example 1.4. Let Q be as in Example 1.2. The A-partition R given by

$$\{a, 0, 6, 12, 18, \ldots\}, \{b, 2, 8, 14, 20, \ldots\}, \{c, 4, 10, 16, 22, \ldots\}, \{1, 3, 9, 11\},$$
$$\{5, 7, 13, 15\}, \{17, 19, 25, 27\}, \{21, 23, 29, 31\}, \ldots$$

is coarser than Q. We think of R as formed by collapsing the 0-th and 2-nd free blocks of Q, collapsing the 1-st and 3-rd free blocks of Q, collapsing the 4-th and 6-th free blocks of Q, collapsing the 5-th and 7-th free blocks of Q, and so on. Note that although no free blocks were collapsed into nonfree blocks in this example, that is also allowed by the definition of coarsening.

For $X \in (\omega)^\omega_A$, we write

$$(X)^\omega_A = \{Y \in (\omega)^\omega_A \mid Y \text{ is coarser than } X\}$$
$$(X)^k_A = \{Y \in (\omega)^k_A \mid Y \text{ is coarser than } X\}$$

Thus, in Example 1.4, $R \in (Q)^\omega_A$ since R is coarser than Q but still has infinitely many free blocks. As above, if $A = \emptyset$, the we drop the subscript A and write $(X)^\omega$ and $(X)^k$.

There are natural topologies on $(\omega)^k$ and $(\omega)^k_A$ which will be defined later and which give rise to the collection of Borel subsets of these spaces. We can now state the Dual Ramsey Theorem and the Generalized Dual Ramsey Theorem.

Theorem 1.5 (Dual Ramsey Theorem)**.** *For all $k, l \geq 1$, if $(\omega)^k = C_0 \cup C_1 \cup \cdots \cup C_{l-1}$, where each C_i is Borel, then there exists an $X \in (\omega)^\omega$ such that $(X)^k \subseteq C_i$ for some $i < l$.*

Theorem 1.6 (Generalized Dual Ramsey Theorem)**.** *Let A be a finite (possibly empty) alphabet, $l \geq 1$ and $k \geq 0$ (if A is nonempty, otherwise $k \geq 1$). If $(\omega)^k_A = C_0 \cup C_1 \cup \cdots \cup C_{l-1}$, where each C_i is Borel, then there exists an $X \in (\omega)^\omega_A$ such that $(X)^k_A \subseteq C_i$ for some $i < l$.*

The Dual Ramsey Theorem is the special case of the Generalized Dual Ramsey Theorem when $A = \emptyset$. The advantage of allowing nonempty alphabets comes in simplifying the induction in the proof. Our goal is to prove the Generalized Dual Ramsey Theorem. This proof breaks into two pieces. We will first cover what Carlson calls "the combinatorial core" of the proof which is a Ramsey-style statement about infinite variable words. Next we will give the topological half of the proof. Finally, I will explain why we cannot cross out the word "Borel" from the statement of the theorem.

For the remainder of this section and for the next section, we assume that A is a nonempty finite alphabet. We fix a countable collection of variables $\{x_i \mid i \in \omega\}$ disjoint from A and ω.

Definition 1.7. An *infinite variable word (over A)* is a function

$$X : \omega \to A \cup \{x_i \mid i \in \omega\}$$

such that

- for every variable x_j, there is an i such that $X(i) = x_j$ and
- if $j_0 < j_1$, i_0 is the least such that $X(i_0) = x_{j_0}$ and i_1 is the least such that $X(i_1) = x_{j_1}$, then $i_0 < i_1$.

That is, an infinite variable word is an ω-sequence in which each variable appears and the first occurrence of x_j comes before the first occurrence of x_{j+1}. (Note that x_j can appear after x_{j+1} as well.)

Example 1.8. For the purposes of examples, I will work with $A = \{a, b, c\}$. The word

$$bbx_0cx_0x_1ax_0ccbx_2x_3x_0ca\cdots$$

is the beginning of an infinite variable word, while the word

$$x_0aabx_2cx_1c\cdots$$

is not.

If X is an infinite variable word, then X can be thought of as an element of $(\omega)^\omega_A$ by associating the partition whose blocks are

$$B_a = \{a\} \cup \{n \in \omega \mid X(n) = a\} \text{ for each } a \in A$$
$$F_j = \{n \in \omega \mid X(n) = x_j\} \text{ for each variable } x_j$$

That is, the B_a blocks are the nonfree blocks and F_j is the j-th free block. As above, we count the free blocks starting with the 0-th free block and we order the free blocks by their least element. Thus, the restriction that the first occurrence of x_i comes before the first occurrence of x_{i+1} guarantees that the least element in F_i is less than the least element of F_{i+1}.

Example 1.9. For the infinite variable word starting

$$aabx_0cx_0x_1ax_0x_1x_2\cdots$$

we have that the blocks containing elements of A are $\{a, 0, 1, 7, \ldots\}$, $\{b, 2, \ldots\}$ and $\{c, 4, \ldots\}$, and there are free blocks $\{3, 5, 8, \ldots\}$, $\{6, 9, \ldots\}$ and $\{10, \ldots\}$.

Similarly, given an A-partition $P \in (\omega)^\omega_A$, there is an associated infinite variable word X given by $X(n) = a$ if n is in the same block as $a \in A$ and $X(n) = x_j$ if n is in the j-th free block.

Example 1.10. The infinite variable word associated with the A-partition Q from Example 1.2 is

$$ax_0bx_0cx_1ax_1bx_2cx_2ax_3bx_3cx_4ax_4\cdots$$

In this example, we have the special property that all the occurrences of x_i come before the first occurrence of x_{i+1}. This property is not required, but it will turn out to be useful to us later.

Because we can move back and forth between A-partitions and infinite variable words, we can work with either formalism. For the next couple of sections, we will work mainly with infinite variable words, but when we return to the Generalized Dual Ramsey Theorem later, it will be helpful to switch between perspectives.

Because of this connection, we use $(\omega)^\omega_A$ to denote the set of infinite variable words as well as the set of A-partitions. If X and Y are infinite variable words, we say Y is coarser than X, if the A-partition corresponding to Y is coarser than the A-partition corresponding to X. Therefore, the notation $(X)^\omega_A$ extends to infinite variable words as well.

Example 1.11. We can think of the process of coarsening on an infinite variable word in terms of substituting elements of A and variables in for other variables. For example, suppose

$$X = bbx_0ax_0x_1x_0bcbax_2x_3x_4bx_0x_2bb\cdots$$

To coarsen X, we first decide whether the free block containing 2 (i.e. the block represented by the variable x_0) will be collapsed into one of the nonfree blocks or whether it will remain free. Suppose we decide to collapse it into the block containing a. This corresponds to substituting a in for x_0 to get

$$bbaaax_1abcbax_2x_3x_4bax_2bb\cdots$$

We next decide what to do with the free block containing 5 (i.e. the block represented by x_1). We could again collapse this into a nonfree block or we could choose for it to remain free. Suppose we decide to keep this block free. In that case, since it is now the 0-th free block, we replace x_1 by x_0 to get

$$bbaaax_0abcbax_2x_3x_4bax_2bb\cdots$$

Next, we decide what to do with the block represented by x_2. Now, we have three options. We can collapse this block into a nonfree block (which corresponds to substituting an element of A in for x_2), or we can collapse it into the 0-th free block (which corresponds to substituting x_0 in for x_2), or we can let it remain free and disjoint from the 0-th free block. In the last case, since it is now the 1-st free block, we substitute x_1 in for x_2. Continuing in this manner, we can coarsen the original infinite variable word X. As long as all the variable occur at the end of this (potentially infinite) process, we will have coarsened X to another infinite variable word.

Of course, the formalism of words also works for A-partitions with k many free blocks. The only change is that we think of our variable words as being over $A \cup \{x_i \mid i < k\}$. That is, rather than using infinitely many variables, we only use k many variables. In this context, the A-partitions with no free blocks correspond to element of A^ω.

We will also be interested in finite initial segments of these infinite variable words, or in the context of A-partitions, A-partitions of initial segments of ω. An A-*segment* is a finite word s over $A \cup \{x_j \mid j \in \omega\}$ which is an initial segment of some element of $(\omega)^\omega_A$. If $n = \text{length}(s)$, then s can be thought of as an A-partition of $A \cup \{0, 1, \ldots, n-1\}$, and conversely, any A-partition of $A \cup \{0, 1, \ldots, n-1\}$ corresponds to an A-segment. We write $\#(s)$ for the number of variables appearing in s (or equivalently, the number of free blocks in the corresponding partition). If s and t are A-segments and $X \in (\omega)^\omega_A$, then we write $s \preceq t$ (or $s \prec X$) if s is an initial segment of t (or X respectively).

We write $s \leq t$ to means that s and t have the same length and that the A-partition corresponding to s is coarser than the A-partition corresponding to t. That is, we can obtain the A-partition corresponding to s by collapsing some of the blocks in the A-partition corresponding to t. As finite variable words, this means that we can obtain the word s from the word t by doing a finite number of substitutions of either elements of A for variables or variables for other variables, being careful to renumber the variables according to the order in which they appear.

Example 1.12. Let $s = abbx_0bcx_0x_0bx_1$ and $t = ax_0bx_1x_0cx_2x_1x_0x_3$. Then s corresponds to the partition of $A \cup \{0, \ldots, 9\}$

$$\{a, 0\}, \{b, 1, 2, 4, 8\}, \{c, 5\}, \{3, 6, 7\}, \{9\}$$

and t corresponds to the partition

$$\{a, 0\}, \{b, 2\}, \{c, 5\}, \{1, 4, 8\}, \{3, 7\}, \{6\}, \{9\}$$

By collapsing $\{b, 2\}$ with $\{1, 4, 8\}$ and collapsing $\{3, 7\}$ with $\{6\}$, we can get from the t partition to the s partition. Therefore, $s \leq t$. Alternately, we start with t

$$ax_0bx_1x_0cx_2x_1x_0x_3$$

substitute b for x_0 (since the 0-th free block is collapsed into the nonfree block containing b) to get

$$abbx_1bcx_2x_1bx_3$$

substitute x_0 for x_1 (since the 1-st free block in t remains free and hence is now the 0-th free block) to get

$$abbx_0bcx_2x_0bx_3$$

substitute x_0 for x_2 (since the 2-nd free block in t remains free but is collapsed into what is now the 0-th free block) to get

$$abbx_0bcx_0x_0bx_3$$

and finally substitute x_1 for x_3 (since the last free block in t remains free and is not collapsed into another free block, and hence becomes the 1-st free block) to get

$$abbx_0bcx_0x_0bx_1$$

which is exactly s.

Similarly, we write $s \leq X$ if $s \leq X[\text{length}(s)]$ where $X[n]$ denotes the initial segment of X of length n (i.e. the word $X(0)X(1)\cdots X(n-1)$).

We extend our notations about coarsening from infinite variable words to A-segments as follows. Let $(n)^k_A$ denote the set of all A-segments (finite words) s such that $\text{length}(s) = n$ and $\#(s) = k$. That is, $(n)^k_A$ is the set of all A-segments (finite words) of length n in which only the variables $x_0, \ldots, x_{k-1}$ appear. (Alternately, $(n)^k_A$ is the set of all A-partitions of $A \cup \{0, 1, \ldots, n\}$ which have exactly k many free blocks.) If $t \in (n)^k_A$ and $l < k$, then

$$(t)^l_A = \{s \in (n)^l_A \mid s \leq t\}$$

That is, $(t)^l_A$ is the set of all A-segments which can be obtained from t by a finite series of substitutions (and renumbering of variables) as above and which contain exactly the variables $x_0, \ldots, x_{l-1}$. In particular, $(t)^0_A$ is the set of all substitution instances of t (i.e. elements of A^n which can be obtained by substituting an element of A for each variable in t). Also, $(t)^1_A$ is the set of all A-segments obtained by first substituting elements of A in for some (but not all) of the variables in t and then replacing the remaining variables by x_0.

Example 1.13. Let $A = \{a, b\}$ and $t = x_0ax_1x_0b$. (We use a smaller alphabet to keep the number of substitution instances smaller.) We obtain the elements of $(t)^0_A$ by running through each possible combination of plugging in a and b for x_0 and x_1. That is, we plug in a for both x_0 and x_1, we plug in a for x_0 and b for x_1, and so on, to obtain

$$(t)^0_A = \{aaaab, aabab, baabb, babbb\}$$

We obtain $(t)^1_A$ in a similar fashion except we have to leave one variable free (or one block free). Thus, we can plug in x_0 for x_1 (i.e. collapse the two free blocks into one), we can leave x_0 as a variable (i.e. leave its block free) and plug in an element of A for x_1 (i.e. collapse its free block into a nonfree block), or plug in an element of A for x_0 (i.e. collapse its free block into a nonfree block) and plug in x_0 for x_1 (i.e. leave its block free, but renumber it as the 0-th free block) to obtain

$$(t)^1_A = \{x_0ax_0x_0b, x_0aax_0b, x_0abx_0b, aax_0ab, bax_0bb\}$$

We need one last notion before stating our Ramsey-style result. If s is an A-segment (viewed as a word), then $s^* = sx_{\#(s)}$. That is, we take s and attached the first variable not appearing in s on the end. When viewing s as an A-partition of $A \cup \{0, 1, \ldots, \text{length}(s) - 1\}$, this amounts to defining s^* as the partition of $A \cup \{0, 1, \ldots, \text{length}(s)\}$ by taking the s-blocks and adding a new free block $\{\text{length}(s)\}$ (i.e. the new free block contains only the number $\text{length}(s)$). In other words, s^* is the unique word such that $\text{length}(s^*) = \text{length}(s) + 1$, $s \preceq s^*$ and $\#(s^*) = \#(s) + 1$.

Example 1.14. Consider the A-segment t from Example 1.13. As a finite word, $t^* = x_0ax_1x_0bx_2$. As a finite A-partition, t partitions $\{a, b\} \cup \{0, 1, 2, 3, 4\}$ into the blocks

$$\{a, 1\}, \{b, 4\}, \{0, 3\}, \{2\}$$

Therefore, t^* partitions $\{a, b\} \cup \{0, 1, 2, 3, 4, 5\}$ into the blocks

$$\{a, 1\}, \{b, 4\}, \{0, 3\}, \{2\}, \{5\}$$

If $X \in (\omega)^\omega_A$, then $(X)^*_A$ is the set of all A-segments s such that $\#(s) = 0$ and $s^* \leq X$. In other words, $s \in (X)^*_A$ if and only if there is an A-segment t such that $t \prec X$, the next element after the initial segment t in the infinite variable word X is the variable $x_{\#(t)}$ (i.e. the first variable not appearing in t) and $s \in (t)^0_A$. To give one more description, a string $s \in A^{<\omega}$ is in $(X)^*_A$ if and only if s is formed by cutting off X just before the first occurrence of a variable and substituting all the variables in this initial segment by elements of A.

Example 1.15. Suppose that $A = \{a, b\}$ and $X = abx_0bx_1x_0x_2\cdots$. We form $(X)^*_A$ as follows. First, we can cut off X just before the first occurrence of x_0 to get ab. Since there are no variables to substitute, we have $ab \in (X)^*_A$. Next, we can cut off X just before the first occurrence of x_1 to get abx_0b. Plugging in a and b for x_0 gives us the strings $abab, abbb \in (X)^*_A$. Next, we can cut off X just before the first occurrence of x_2 to get $abx_0bx_1x_0$. Plugging in all combinations of a and b for x_0 and x_1 gives us the strings $ababaa, ababba, abbbab, abbbbb \in (X)^*_A$. Continuing in this manner, we have

$$(X)^*_A = \{ab, abab, abbb, ababaa, ababba, abbbab, abbbbb, \ldots\}$$

One particular case of note is $(\omega)^*_A$. Here we regard ω as the partition $\{0\}, \{1\}, \ldots$, or equivalently, the word $x_0x_1x_2\cdots$. It follows directly from the definitions that $(\omega)^*_A = A^{<\omega}$.

We can now state the first version of the combinatorial core theorem.

Theorem 1.16. *Let A be a finite nonempty alphabet. If $(\omega)^*_A = C_0 \cup \cdots \cup C_{l-1}$, then there exists an $X \in (\omega)^\omega_A$ such that $(X)^*_A \subseteq C_i$ for some $i < l$.*

Before proving this theorem, we state a corollary which is the real statement we will need later in the proof of the Generalized Dual Ramsey Theorem.

Corollary 1.17. *Let A be a finite nonempty alphabet. If $Y \in (\omega)^\omega_A$ and $(Y)^*_A = C'_0 \cup \cdots \cup C'_{l-1}$, then there exists $Z \in (Y)^\omega_A$ such that $(Z)^*_A \subseteq C'_i$ for some $i < l$.*

Proof. Fix $Y \in (\omega)^\omega_A$. There is a canonical bijection from $(\omega)^\omega_A$ onto $(Y)^\omega_A$. To see why, it is easiest to think in terms of A-partitions, viewing ω as the A-partition

$$\{a_0\}, \ldots, \{a_n\}, \{0\}, \{1\}, \ldots$$

where $A = \{a_0, \ldots, a_n\}$. That is, both ω and Y consist of a single block for each $a \in A$ and then infinitely many free blocks which we can think of as ordered by comparing their least elements. An element $W \in (\omega)^\omega_A$ is a coarsening of the A-partition ω which still contains infinitely many free blocks. That is, we form W by performing a sequence of choices about collapsing the free blocks. First, we decide whether to collapse the 0-th free block with a nonfree block or allow it to remain free. Second, we decide whether to collapse the 1-st free block with a nonfree block, collapse it with the 0-th free block (if it remained free after the first choice) or do not collapse it with a previous block. Continuing in this manner (assuming we retain infinitely many free blocks in the end) yields W. The image of W under the bijection just performs the same sequence of choices on the corresponding free blocks in Y.

How does this work at the level of words? Suppose we want to find the image of the following element of $(\omega)_A^\omega$ where $A = \{a, b, c\}$.

$$abx_0x_1x_0cax_1x_0x_2\cdots$$

We can think of this word as being obtained by the following sequence of substitutions: $a \mapsto x_0$ (i.e. substitute a in for x_0), $b \mapsto x_1$, $x_0 \mapsto x_2$, $x_1 \mapsto x_3$, $x_0 \mapsto x_4$ and so on. Perform the same sequence of substitutions on Y to obtain the image of this word in $(Y)_A^\omega$.

The same idea works at the level of $(\omega)_A^*$ and $(Y)_A^*$. That is, an element of $(\omega)_A^*$ is formed by chopping off the infinite word $x_0x_1\cdots$ just before a new variable and substituting all the variables by elements of A in this initial segment. We map this element of $(\omega)_A^*$ to the element of $(Y)_A^*$ formed by chopping off Y just before the same variable and performing the same substitutions. This gives a bijection from $(\omega)_A^*$ onto $(Y)_A^*$.

Fix the coloring $(Y)_A^*$. Color $(\omega)_A^* = C_0 \cup \cdots \cup C_{l-1}$ by assigning each $s \in (\omega)_A^*$ the color C_i if its image in $(Y)_A^*$ is colored C_i'. Applying Theorem 1.16, there is an $X \in (\omega)_A^\omega$ and an $i < l$ such that $(X)_A^* \subseteq C_i$. Let $Z \in (Y)_A^\omega$ be the image of X under the first bijection. It follows that $(Z)_A^* \subseteq C_i'$ as required. □

We make one more simplification (actually strengthening) of Theorem 1.16 before giving the proof.

Definition 1.18. An infinite variable word $X \in (\omega)_A^\omega$ is an *ordered infinite variable word* if for all $i \in \omega$, every occurrence of x_i comes before the first occurrence of x_{i+1}. (Note that in particular, each variable occurs only finitely often,)

We let $\langle\omega\rangle_A^\omega$ denote the set of all ordered infinite variable words. Some of our other notation will be recast in terms of ordered words as follows.

- For $X \in \langle\omega\rangle_A^\omega$, we let $\langle X\rangle_A^\omega$ denote the set of all $Y \in \langle\omega\rangle_A^\omega$ which are coarser than X. That is, $\langle X\rangle_A^\omega = (X)_A^\omega \cap \langle\omega\rangle_A^\omega$.
- An *ordered A-segment* is an A-segment s such that $s \prec X$ for some $X \in \langle\omega\rangle_A^\omega$. That is, s satisfies the same ordering restriction of its variables.

Example 1.19. Consider the A-partitions Q and R from Example 1.4. The infinite variable word version of Q is

$$Q = ax_0bx_0cx_1ax_1bx_2cx_2ax_3bx_3cx_4ax_4\cdots$$

which is an element of $\langle\omega\rangle_A^\omega$. The infinite variable word version of R is

$$R = ax_0bx_0cx_1ax_1bx_0cx_0ax_1bx_1cx_2ax_2bx_3cx_3ax_2bx_2cx_3ax_3\cdots$$

which is not an element of $\langle \omega \rangle_A^\omega$. (That is, to get R, we collapsed the 0-th and 2-nd free blocks of Q and the 1-st and 3-rd free blocks of Q, and thus creating an instance of x_0 after an instance of x_1.) Thus, despite the fact that $R \in (Q)_A^\omega$, we do not have $R \in \langle Q \rangle_A^\omega$. On the other hand, the infinite variable word S given by

$$S = ax_0bx_0cx_0ax_0bx_1cx_1ax_1bx_1cx_2ax_2bx_2cx_2 \cdots$$

which corresponds to the A-partition

$$\{a, 0, 6, 12, 18, \ldots\}, \{b, 2, 8, 14, 20, \ldots\}, \{c, 4, 10, 16, 22, \ldots\}, \{1, 3, 5, 7\},$$
$$\{9, 11, 13, 15\}, \{17, 19, 21, 23\} \ldots$$

is an element of $\langle Q \rangle_A^\omega$. Note that S is formed by collapsing the 0-th and 1-st free blocks of Q, collapsing the 2-nd and 3-rd free blocks of Q, collapsing the 4-th and 5-th blocks of Q and so on.

Because of the nature of the proofs to come, it is more convenient to change some of the notation in the ordered case. This is unfortunate, but I will follow the notation in the Carlson and Simpson paper.

- For $m \in \omega$, we let $\langle \omega \rangle_A^m$ denote the set of all ordered A-segments s such that $\#(s) = m$. In particular, $\langle \omega \rangle_A^0 = (\omega)_A^* = A^{<\omega}$ since there are no variables in these sets and hence the ordering condition does not apply.
- For $X \in \langle \omega \rangle_A^\omega$, we let $\langle X \rangle_A^m$ denote the set of all ordered A-segments $s \in \langle \omega \rangle_A^m$ such that $s^* \leq X$. That is, $s \in \langle X \rangle_A^m$ if there is an initial segment $t \preceq X$ such that the next element in the word X is a new variable, $s \in (t)_A^m$ and s is ordered. In particular, $\langle X \rangle_A^0 = (X)_A^*$ because we have substituted out all the variables in both of these sets so the ordering conditions on the variables does not apply.
- We let $\langle \omega \rangle_A^{<\omega} = \cup_{m \in \omega} \langle \omega \rangle_A^m$ and $\langle X \rangle_A^{<\omega} = \cup_{m \in \omega} \langle X \rangle_A^m$

The real combinatorial core theorem we will prove is the following strengthening of Theorem 1.16. (To see why Theorem 1.16 follows immediately from Theorem 1.20, recall from above that $\langle \omega \rangle_A^0 = (\omega)_A^* = A^{<\omega}$ and $\langle X \rangle_A^0 = (X)_A^*$. Thus, we are merely strengthening the conclusion to require that the infinite variable word X is ordered.)

Theorem 1.20 (Combinatorial Core)**.** *Let A be a finite nonempty alphabet. If*

$$\langle \omega \rangle_A^0 = C_0 \cup \cdots C_{l-1}$$

then there exists $X \in \langle \omega \rangle_A^\omega$ such that $\langle X \rangle_A^0 \subseteq C_i$ for some $i < l$.

2 Combinatorial core

The key notion for proving Theorem 1.20 is density.

Definition 2.1. For $X \in \langle\omega\rangle_A^\omega$ and $D \subseteq A^{<\omega}$, we say D is *dense in* $\langle X\rangle_A^0$ if $\langle Y\rangle_A^0 \cap D \neq \emptyset$ for all $Y \in \langle X\rangle_A^\omega$. That is, no matter how you refine X to an ordered infinite variable word Y, there is some variable such that if you cut off Y just before the first occurrence of this variable, it is possible to perform a substitution instance on the remaining variables in this initial segment of Y to get a string in D.

You should think of D being dense as a largeness property. The key lemma we will establish is the following.

Lemma 2.2. *If D is dense in $\langle X\rangle_A^0$, then there exists a $W \in \langle X\rangle_A^\omega$ such that $\langle W\rangle_A^0 \subseteq D$.*

Before proving Lemma 2.2, we show how to use it to establish Theorem 1.20.

Theorem 2.3 (Combinatorial Core)**.** *Let A be a finite nonempty alphabet. If*

$$A^{<\omega} = \langle\omega\rangle_A^0 = C_0 \cup \cdots C_{l-1}$$

then there exists $X \in \langle\omega\rangle_A^\omega$ such that $\langle X\rangle_A^0 \subseteq C_i$ for some $i < l$.

Proof. We proceed by induction on l. The case of $l = 1$ is trivial. Consider the case when $l = 2$. In this case, we have a coloring $\langle\omega\rangle_A^0 = C_0 \cup C_1$. Since $\langle\omega\rangle_A^0 = A^{<\omega}$, we know $C_0 \subseteq A^{<\omega}$. We break into two cases.

First, suppose that C_0 is dense in $\langle\omega\rangle_A^0$. Then by Lemma 2.2, there is a $W \in \langle\omega\rangle_A^\omega$ such that $\langle W\rangle_A^0 \subseteq C_0$ and we are done.

Otherwise, C_0 is not dense in $\langle\omega\rangle_A^0$. By definition, this means that there is a $Y \in \langle\omega\rangle_A^\omega$ such that $\langle Y\rangle_A^0 \cap C_0 = \emptyset$. But then $\langle Y\rangle_A^0 \subseteq C_1$ and we are done. Therefore, we have established the case when $l = 2$.

Suppose $l > 2$ and $\langle\omega\rangle_A^0 = C_0 \cup \cdots \cup C_{l-1}$. As above, if C_0 is dense in $\langle\omega\rangle_A^0$, then we are done by Lemma 2.2. If C_0 is not dense in $\langle\omega\rangle_A^0$, then there is a $Y \in \langle\omega\rangle_A^\omega$ such that $\langle Y\rangle_A^0 \subseteq C_1 \cup \cdots \cup C_{l-1}$. However, as in the proof of Corollary 1.17, there is a bijection between $\langle\omega\rangle_A^\omega$ and $\langle Y\rangle_A^\omega$. Therefore, we can shift this coloring of $\langle Y\rangle_A^0$ (with one fewer colors) to $\langle\omega\rangle_A^0$, apply the inductive hypothesis and pull the resulting ordered variable word back into $\langle Y\rangle_A^\omega$. □

For the remainder of this section, we work toward a proof of Lemma 2.2. At a key point, we will use one result from standard Ramsey Theory, the Hales-Jewett Theorem. Let A^n denote the set of all length n words over A. For $s \in A^n$, we denote the element of A in the i-th position in s (for $0 \leq i < n$) by $s(i)$. Assume that

$$A = \{a_0, \ldots, a_{k-1}\}$$

A *line in* A^n is a sequence $s_0, \ldots, s_{k-1}$ of elements of A^n such that for each coordinate $0 \leq i < n$ either

(1) $s_j(i) = s_{j'}(i)$ for all $0 \leq j, j' < k$, or

(2) $s_j(i) = a_j$ for all $0 \leq j < k$

and the situation in Condition (2) happens at least once. (Note that we are implicitly assuming there is a fixed order on the set A.)

Theorem 2.4 (Hales-Jewett Theorem)**.** *Let A be a finite nonempty alphabet. For every l, there is an n such that for all $n' \geq n$, if $A^{n'}$ is l-colored, then there is a monochromatic line.*

To translate this statement into our notation, recall that $(n)^0_A = A^n$. If $t \in (n)^1_A$, then t is a length n word over A and a single variable. Therefore, $(t)^0_A$ is the set of size $|A|$ whose elements are obtained by substituting an element of A in for the single variable in t. We can order this set (from an ordering of $A = \{a_0, \ldots, a_{k-1}\}$) as $s_0, \ldots, s_{k-1}$ where s_i is the result of substituting a_i in for the variable in t. Therefore, a line in A^n has the form $(t)^0_A$ for some $t \in (n)^1_A$. We can state the Hales-Jewett Theorem as follows.

Theorem 2.5. *Let A be a finite nonempty alphabet. For each l, there is an n such that for all $n' \geq n$, if $(n')^0_A = C_0 \cup \cdots \cup C_{l-1}$, then there exists a $t \in (n')^1_A$ such that $(t)^0_A \subseteq C_i$ for some $i < l$.*

We need one further restatement of the Hales-Jewett Theorem to get to the version we will apply.

Theorem 2.6. *Let A be a finite nonempty alphabet. For each l, there is an n such that for all $u \in \langle\omega\rangle^n_A$ and any coloring $(u)^0_A = C_0 \cup \cdots \cup C_{l-1}$, there exists a $v \in (u)^1_A$ such that $(v)^0_A \subseteq C_i$ for some $i < l$.*

Proof. By Theorem 2.5, let n be large enough that for any coloring $(n)^0_A = C_0 \cup \cdots \cup C_{l-1}$, then there exists a $t \in (n)^1_A$ such that $(t)^0_A \subseteq C_i$ for some $i < l$. If $u \in \langle\omega\rangle^n_A$, then u looks like the word $x_0 x_1 \cdots x_{n-1}$ with extra variables and elements of A interspersed, subject to the restriction that u is an ordered A-segment. That is, if $n = 3$, then u could be the word $x_0 a b x_0 x_1 x_1 b b x_2 a x_2$.

There is an obvious bijection between $(u)^0_A$ and $(n)^0_A$ which allows us to shift the coloring of $(u)^0_A$ to a coloring of $(n)^0_A$. (That is, any element of $(u)^0_A$ is obtained by substituting elements of A for the variables $x_0, \ldots, x_{n-1}$ in u. Apply the same substitutions to the word $x_0 x_1 \cdots x_{n-1}$ to obtain the image in $(n)^0_A$.) Apply Theorem 2.5 to the coloring of $(n)^0_A$ to get $t \in (n)^1_A$ such that $(t)^0_A$ is monochromatic. Since t is formed from the word $x_0 x_1 \cdots x_{n+1}$ by substituting in elements of A for some (but not all) of the variables and substituting x_0 in for the remaining variables, we can perform the same sequence of substitutions on u to get $v \in (u)^1_A$. Since $(t)^0_A$ is monochromatic, it follows that $(v)^0_A$ is monochromatic. □

For the remainder of this section, let A be an arbitrary finite nonempty alphabet, X be an arbitrary element of $\langle\omega\rangle^\omega_A$ and D be an arbitrary subset of $A^{<\omega}$ which is dense in $\langle X\rangle^0_A$. The following lemmas hold for any such objects.

Lemma 2.7. *There is an $s \in \langle X\rangle^{<\omega}_A$ such that $(t)^0_A \cap D \neq \emptyset$ for all $t \in \langle X\rangle^{<\omega}_A$ such that $s \preceq t$.*

Proof. First, we claim that if $\langle X\rangle_A^0 \subseteq D$, then the lemma follows. Suppose that $\langle X\rangle_A^0 \subseteq D$. Consider any $t \in \langle X\rangle_A^{<\omega}$. The set $(t)_A^0$ contains all the strings in $A^{|t|}$ formed by substituting elements of A in for all the variables in t. By the definition of $\langle X\rangle_A^0$, $(t)_A^0 \subseteq \langle X\rangle_A^0$. Since $\langle X\rangle_A^0 \subseteq D$, we have $(t)_A^0 \subseteq D$ and hence $(t)_A^0 \cap D \neq \emptyset$ for all $t \in \langle X\rangle_A^0$. In this case, we can let $s \preceq X$ be the finite word obtained by cutting off X before the first occurrence of x_0 and the lemma follows.

To proceed in general, we assume for a contradiction that the conclusion of the lemma fails. By the first paragraph, we know that there is a $t_0 \in \langle X\rangle_A^0$ such that $t_0 \notin D$. Fix t_0. Notice that $(t_0)_A^0 = \{t_0\}$ since there are no variables to substitute in for. Therefore, $(t_0)_A^0 \cap D = \emptyset$.

Assume we have been given $t_m \in \langle X\rangle_A^m$ such that $(t_m)_A^0 \cap D = \emptyset$. Recall that t_m is formed by cutting off X just before the first occurrence of a variable x_i and possibly refining it (by plugging in elements of A for variables or variables for other variables in a manner consistent with maintaining an ordered A-segment) to have only m variables left. Fix $s_{m+1} \in \langle X\rangle_A^{m+1}$ such that $t_m^* \preceq s_{m+1}$. (For example, extend t_m by adding on the part of X beginning with the first occurrence of x_i and stopping just before the first occurrence of x_{i+1}. Renaming the variable x_i by x_m yields such a string s_{m+1}.) Since the conclusion of the lemma fails with s_{m+1}, there is a $t' \in \langle X\rangle_A^{<\omega}$ such that $s_{m+1} \preceq t'$ and $(t')_A^0 \cap D = \emptyset$. Let t_{m+1} be the result of substituting x_m in for any variable x_j in t' with index $j > m$ (if such variables occur in t'). Thus we have $t_{m+1} \in \langle X\rangle_A^{m+1}$ such that $s_{m+1} \preceq t_{m+1}$, $t_{m+1} \in \langle X\rangle_A^{m+1}$ and $(t_{m+1})_A^0 \subseteq (t')_A^0$. Hence, $(t_{m+1})_A^0 \cap D = \emptyset$.

Altogether, we have a sequence of words $t_0^* \preceq t_1^* \preceq t_2^* \preceq \cdots$ such that $t_m \in \langle X\rangle_A^m$ and $(t_m)_A^0 \cap D = \emptyset$. Let Y be the limit of these words. That is, $Y \in \langle X\rangle_A^\omega$ and $t_m^* \preceq Y$ for each m. We claim that $\langle Y\rangle_A^0 = \cup(t_m)_A^0$. Consider $s \in \langle Y\rangle_A^0$. The string s is formed by cutting off Y immediately before the first occurrence of some variable x_m and substituting elements of A for each variable in the resulting initial segment of Y. But, cutting off Y before the variable x_m yields t_m and hence $s \in (t_m)_A^0$. Conversely, since each $t_m^* \prec Y$, we have that each $s \in (t_m)_A^0$ is in $\langle Y\rangle_A^0$. Thus, $\langle Y\rangle_A^0 = \cup(t_m)_A^0$ and hence $\langle Y\rangle_A^0 \cap D = \emptyset$, contradicting the fact that D is dense in $\langle X\rangle_A^\omega$. $\square$

For the next lemma, we need to define a method of concatenating two A-segments. If s and t are A-segments (viewed as finite words), then $s \oplus t$ is the finite word obtained by concatenating s and t and renumbering the variables in t by replacing x_i by $x_{i+\#(s)}$. That is, we concatenate the finite words and renumber the variables so that the variables in s and the renumbered variables in t do not overlap.

Example 2.8. If $s = ax_0x_1bcx_1$ and $t = bx_0x_1x_1cx_2$, then $s \oplus t = ax_0x_1bcx_1bx_2x_3x_3cx_4$.

Lemma 2.9. *There is a $t \in \langle X\rangle_A^1$ such that $(t)_A^0 \subseteq D$.*

Proof. Fix s as in Lemma 2.7. Let $l = |(s)_A^0|$ and let $(s)_A^0 = \{s_0, \ldots, s_{l-1}\}$. By Theorem 2.6, let n be large enough so that for every $u \in \langle\omega\rangle_A^n$, and any coloring $(u)_A^0 = C_0 \cup \cdots \cup C_{l-1}$, there exists a $v \in (u)_A^1$ such that $(v)_A^0 \subseteq C_i$ for some $i < l$.

Pick $u \in \langle\omega\rangle_A^n$ such that $s \oplus u \in \langle X\rangle_A^{<\omega}$. That is, let u be the part of X after s extending through the next n many variables and stopping immediately before the $(n+1)$-st variable.

Consider an element $w \in (u)^0_A$. Then $s \oplus w$ is a refinement of $s \oplus u$ (that is, $s \oplus w \leq s \oplus u$) obtained by substituting elements of A for all the variables in the u part of $s \oplus u$ and leaving the variables in the s part alone. Therefore, $s \oplus w \in \langle X \rangle^{<\omega}_A$ and $s \preceq s \oplus w$. By Lemma 2.7, $(s \oplus w)^0_A \cap D \neq \emptyset$. Therefore, there is a refinement s_i of s (that is, $s_i \leq s$) with $\#(s_i) = 0$ obtained by substituting elements of A for all the variables in s such that $s_i \oplus w \in D$.

We color $(u)^0_A$ as follows. Let

$$C_i = \{w \in (u)^0_A \mid s_i \oplus w \in D\}$$

By the previous paragraph $(u)^0_A = C_0 \cup \cdots \cup C_{l-1}$. By the choice of n, there is a $v \in (u)^1_A$ and $i < l$ such that $(v)^0_A \subseteq C_i$. In other words, $(s_i \oplus v)^0_A \subseteq D$. Set $t = s_i \oplus v$. Then $t \in \langle X \rangle^1_A$ and $(t)^0_A \subseteq D$. □

For the next lemma, we extend our $\oplus$ notation for $s \in \langle \omega \rangle^{<\omega}_A$ and $Y \in \langle \omega \rangle^{\omega}_A$ by defining $s \oplus Y$ to be the infinite ordered variable word formed by concatenating s and Y and renumbering all the variables in Y by sending x_i to $x_{i+\#(s)}$.

Example 2.10. If $s = ax_0x_0bcx_1$ and $Y = abx_0x_1bcbx_1x_2\cdots$, then

$$s \oplus Y = ax_0x_0bcx_1abx_2x_3bcbx_3x_4\cdots$$

Lemma 2.11. *There exists $s \in \langle \omega \rangle^1_A$ and $Y \in \langle \omega \rangle^{\omega}_A$ such that $s \oplus Y \in \langle X \rangle^{\omega}_A$ and the set*

$$\{t \in A^{<\omega} \mid (s \oplus t)^0_A \subseteq D\}$$

is dense in $\langle Y \rangle^0_A$.

Proof. Assume not and we derive a contradiction. By our assumption, for all $s \oplus Y \in \langle X \rangle^{\omega}_A$, the set $\{t \mid (s \oplus t)^0_A \subseteq D\}$ is not dense in $\langle Y \rangle^0_A$. That is, there is a $Z \in \langle Y \rangle^0_A$ such that

$$\langle Z \rangle^0_A \cap \{t \mid (s \oplus t)^0_A \subseteq D\} = \emptyset$$

In other words, for all $t \in \langle Z \rangle^0_A$, $(s \oplus t)^0_A \not\subseteq D$. Or stated one more way, for every $t \in \langle Z \rangle^0_A$, there is a substitution instance (of the single variable in s) s' of s such that $t \oplus s' \notin D$.

To derive a contradiction, we will define $W \in \langle X \rangle^{\omega}_A$ such that for all $u \in \langle W \rangle^1_A$, $(u)^0_A \not\subseteq D$. This contradicts Lemma 2.9. (Note that if D is dense in $\langle X \rangle^0_A$ and $W \in \langle X \rangle^{\omega}_A$, then D is also dense in $\langle W \rangle^0_A$.) We construct W by constructing a sequence $s_0, s_1, \ldots$ such that each $s_i \in \langle \omega \rangle^1_A$ and setting $W = s_0 \oplus s_1 \oplus s_2 \oplus \cdots$.

Fix $s_0 \in \langle \omega \rangle^1_A$ and Y_0 such that $X = s_0 \oplus Y_0$. That is, let s_0 be the initial segment of X including all instances of x_0 and stopping right before the first instance of x_1. Let Y_0 be the remainder of X with the variables renumbered by replacing x_i by x_{i-1} so that $X = s_0 \oplus Y_0$ In the future, we will not mention the renumbering of variables and leave it implied by the context. Let Z_0 be such that $Z_0 \in \langle Y_0 \rangle^{\omega}_A$ and $(s \oplus t)^0_A \not\subseteq D$ for all $t \in \langle Z_0 \rangle^0_A$.

To continue the induction, fix $s_1 \in \langle \omega \rangle^1_A$ and $Y_1 \in \langle Z_0 \rangle^{\omega}_A$ such that $Z_0 = s_1 \oplus Y_1$. Notice that $s_0 \oplus s_1 \oplus Y_1 \in \langle X \rangle^{\omega}_A$. Consider the finite set $(s_0 \oplus s_1)^1_A$ and list this set as $v_0, \ldots, v_k$. That is, $s_0 \oplus s_1$ contains the variables x_0 and x_1, so there are finitely many ways to perform substitutions to reduce this word to having only one variable. We define a sequence $V_0, \dots, V_k$ such that each $V_i \in \langle Y_1 \rangle_A^\omega$ as follows.

Since $v_0 \oplus Y_1 \in \langle X \rangle_A^\omega$ and $v_0 \in \langle \omega \rangle_A^1$, there is (by the assumption in the first paragraph of this proof) a $V_0 \in \langle Y_1 \rangle_A^\omega$ such that $(v_0 \oplus t)_A^0 \not\subseteq D$ for all $t \in \langle V_0 \rangle_A^0$.

By induction, assume that we have defined $V_i \in \langle Y_1 \rangle_A^\omega$ such that $(v_j \oplus t)_A^0 \not\subseteq D$ for all $j \leq i$ and all $t \in \langle V_i \rangle_A^0$. Since $v_{i+1} \oplus V_i \in \langle X \rangle_A^\omega$ and $v_{i+1} \in \langle \omega \rangle_A^1$, there is a $V_{i+1} \in \langle V_i \rangle_A^\omega$ such that $(v_{i+1} \oplus t)_A^0 \not\subseteq D$ for all $t \in \langle V_{i+1} \rangle_A^0$. Note the following facts.

- $V_{i+1} \in \langle Y_1 \rangle_A^\omega$ since $V_{i+1} \in \langle V_i \rangle_A^\omega$ and $V_i \in \langle Y_1 \rangle_A^\omega$.
- For any $t \in \langle V_{i+1} \rangle_A^0$, we have $t \in \langle V_i \rangle_A^0$ since $V_{i+1} \in \langle V_i \rangle_A^\omega$.
- Fix $t \in \langle V_{i+1} \rangle_A^0$ and $j \leq i$. Since $t \in \langle V_i \rangle_A^0$, we have by induction that $(v_j \oplus t)_A^0 \not\subseteq D$.

We have now established that $V_{i+1} \in \langle Y_1 \rangle_A^\omega$ and for all $j \leq i+1$ and all $t \in \langle V_{i+1} \rangle_A^0$, $(v_j \oplus t)_A^0 \not\subseteq D$. Therefore, we have established the required induction hypothesis to continue defining the sequence $V_0, \dots, V_k$.

Set $Z_1 = V_k$. Notice that $Z_1 \in \langle Y_1 \rangle_A^\omega$ and for all $s \in (s_0 \oplus s_1)_A^1$, we have $(s \oplus t)_A^0 \not\subseteq D$ for all $t \in \langle Z_1 \rangle_A^0$. Write $Z_1 = s_2 \oplus Y_2$ where $s_2 \in \langle \omega \rangle_A^1$. Since $s_0 \oplus s_1 \oplus Y_1 \in \langle X \rangle_A^\omega$ and $Z_1 \in \langle Y_1 \rangle_A^\omega$, we have that $s_0 \oplus s_1 \oplus s_2 \oplus Y_2 \in \langle X \rangle_A^\omega$.

The induction continues in the same way. Suppose we have defined $s_0, \dots, s_n$ and Y_n so that $s_0 \oplus \cdots \oplus s_n \oplus Y_n \in \langle X \rangle_A^\omega$. Let $Z_n \in \langle Y_n \rangle_A^\omega$ be such that $(s \oplus t)_A^0 \not\subseteq D$ for all $s \in (s_0 \oplus \cdots \oplus s_n)_A^1$ and all $t \in \langle Z_n \rangle_A^0$. Write $Z_n = s_{n+1} \oplus Y_{n+1}$ to continue the induction.

Let $W = s_0 \oplus s_1 \oplus \cdots \in \langle X \rangle_A^\omega$. Recall that our desired contradiction is to show that $(u)_A^0 \not\subseteq D$ for all $u \in \langle W \rangle_A^1$. Fix any $u \in \langle W \rangle_A^1$. The finite word u is formed by cutting off W before the first occurrence of a variable x_{n+1} and performing substitutions to reduce this initial segment to having only one variable. That is, we substitute elements of A for some (but not all) the variables in the initial segment and substitute x_0 for the remaining variables. Because u must contain a variable, we cannot obtain the initial segment of X by cutting off before x_0. Therefore, we can denote the variable used to obtain the initial segment of X by x_{n+1}.

It follows that u has the form $u \in (s_0 \oplus \cdots \oplus s_n \oplus t)_A^1$ where $t \in A^{<\omega}$ is the initial segment of s_{n+1} formed by cutting off s_{n+1} just before the occurrence of the (only) variable in s_{n+1}. However, since s_{n+1} was defined such that $Z_n = s_{n+1} \oplus Y_{n+1}$, we have that $t \prec Z_n$, and more specifically, t is the initial segment of Z_n formed by cutting off Z_n before the first occurrence of the first variable in Z_n. Therefore, $t \in \langle Z_n \rangle_A^0$. But, then our construction of Z_n implies that $(u)_A^0 \not\subseteq D$, giving the necessary contradiction. □

Lemma 2.12. *There exists $s \in \langle \omega \rangle_A^1$ and $Y \in \langle \omega \rangle_A^\omega$ such that $s \oplus Y \in \langle X \rangle_A^\omega$, the set*

$$\{t \in A^{<\omega} \mid (s \oplus t)_A^0 \subseteq D\}$$

is dense in $\langle Y \rangle_A^0$ and there is an $r \in D$ such that $r^* \preceq s$.

Proof. We apply Lemma 2.11 repeatedly to define three sequences $s_0, s_1, \dots$ with $s_i \in \langle \omega \rangle_A^1$, $Y_0, Y_1, \dots$ with $Y_i \in \langle \omega \rangle_A^\omega$ and $D_0, D_1, \dots$ such that $D_n \subseteq A^{<\omega}$ is dense in $\langle Y_n \rangle_A^0$. Let s_0 and Y_0

be as in the conclusion of Lemma 2.11. That is, $s_0 \oplus Y_0 \in \langle X\rangle_A^\omega$ and $D_0 = \{t \mid (s_0 \oplus t)_A^0 \subseteq D\}$ is dense in $\langle Y_0\rangle_A^0$.

Given Y_n and D_n (constructed inductively) such that D_n is dense in $\langle Y_n\rangle_A^0$, apply Lemma 2.11 to obtain s_{n+1} and Y_{n+1} such that $s_{n+1} \oplus Y_{n+1} \in \langle Y_n\rangle_A^\omega$ and $D_{n+1} = \{t \mid (s_{n+1} \oplus t)_A^0 \subseteq D_n\}$ is dense in $\langle Y_{n+1}\rangle_A^0$. By induction on n, we have the following two properties.

- $s_0 \oplus \cdots \oplus s_n \oplus Y_n \in \langle X\rangle_A^\omega$
- $D_n = \{t \mid (s_0 \oplus \cdots \oplus s_n \oplus t)_A^0 \subseteq D\}$

Set $W = s_0 \oplus s_1 \oplus \cdots \in \langle X\rangle_A^\omega$. Since D is dense in $\langle X\rangle_A^0$ and $W \in \langle X\rangle_A^\omega$, there is an $r \in D$ such that $r \in \langle W\rangle_A^0$. Fix $n \in \omega$ such that $\text{length}(r) < \text{length}(s_0 \oplus \cdots \oplus s_n)$. Since $r \in \langle W\rangle_A^0$, there is an $s \in (s_0 \oplus \cdots \oplus s_n)_A^1$ such that $r^* \preceq s$. Fix s and set $Y = Y_n$. Since $r^* \preceq s$, it remains to show the items in the conclusion of Lemma 2.11.

- $s \oplus Y \in \langle X\rangle_A^\omega$ follows since $s_0 \oplus \cdots \oplus s_n \oplus Y_n \in \langle X\rangle_A^\omega$ and we have both $Y = Y_n$ and $s \in (s_0 \oplus \cdots \oplus s_n)_A^1$.
- We know that $\{t \mid (s_0 \oplus \cdots s_n \oplus t)_A^0 \subseteq D\}$ is dense in $\langle Y\rangle_A^0$. Since $s \in (s_0 \oplus \cdots \oplus s_n)_A^1$, we have that $(s \oplus t)_A^0 \subseteq (s_0 \oplus \cdots \oplus s_n \oplus t)_A^0$ for every t. Therefore

  $$\{t \mid (s_0 \oplus \cdots \oplus s_n \oplus t)_A^0 \subseteq D\} \subseteq \{t \mid (s \oplus t)_A^0 \subseteq D\}$$

  so $\{t \mid (s \oplus t)_A^0 \subseteq D\}$ is dense in $\langle Y\rangle_A^0$.

□

Finally, we can prove Lemma 2.2 which we restate below for convenience.

Lemma 2.13. *If D is dense in $\langle X\rangle_A^0$, then there is a $W \in \langle X\rangle_A^\omega$ such that $\langle W\rangle_A^0 \subseteq D$.*

Proof. The proof is very similar to the proof of Lemma 2.12, except that we apply Lemma 2.12 at each step instead of Lemma 2.11. That is, we define sequences

- $s_0, s_1, \ldots \in \langle \omega\rangle_A^1$,
- $Y_0, Y_1, \ldots \in \langle \omega\rangle_A^\omega$
- $D_0, D_1, \ldots$ such that D_n is dense in $\langle Y_n\rangle_A^0$,
- $r_0, r_1, \ldots$ such that $r_i^* \preceq s_i$, $r_0 \in D$, $r_{n+1} \in D_n$.

Let r_0, s_0 and Y_0 be as in the conclusion of Lemma 2.12. That is, $r_0 \in D$, $s_0 \oplus Y_0 \in \langle X\rangle_A^\omega$ and $D_0 = \{t \mid (s_0 \oplus t)_A^0 \subseteq D\}$ is dense in $\langle Y_0\rangle_A^0$.

Given Y_n and D_n (constructed inductively) such that D_n is dense in $\langle Y_n\rangle_A^0$, apply Lemma 2.12 to obtain r_{n+1}, s_{n+1} and Y_{n+1} such that $r_{n+1} \in D_n$, $r_{n+1}^* \preceq s_{n+1}$, $s_{n+1} \oplus Y_{n+1} \in \langle Y_n\rangle_A^\omega$ and $D_{n+1} = \{t \mid (s_{n+1} \oplus t)_A^0 \subseteq D_n\}$ is dense in $\langle Y_{n+1}\rangle_A^0$. By induction on n, we have

- $s_0 \oplus \cdots \oplus s_n \oplus Y_n \in \langle X\rangle_A^\omega$

- $D_n = \{t \mid (s_0 \oplus \cdots \oplus s_n \oplus t)^0_A \subseteq D\}$

Let $W = s_0 \oplus s_1 \oplus \cdots \in \langle X \rangle^\omega_A$. It remains to show that $\langle W \rangle^0_A \subseteq D$. Fix any $r \in \langle W \rangle^0_A$. If $\text{length}(r) < \text{length}(s_0)$, then $r = r_0 \in D$ (because we have to obtain r by cutting off W right before the first occurrence of the first variable). If

$$\text{length}(s_0 \oplus \cdots \oplus s_n) \leq \text{length}(r) < \text{length}(s_0 \oplus \cdots \oplus s_{n+1})$$

then we obtain r by cutting off W before the first occurrence of the variable in s_{n+1} and substituting in for the variables in $s_0 \oplus \cdots \oplus s_n$. Since $r^*_{n+1} \preceq s_{n+1}$, we have

$$r \in (s_0 \oplus \cdots \oplus s_n \oplus r_{n+1})^0_A$$

But, $r_{n+1} \in D_n$ and hence $(s_0 \oplus \cdots \oplus s_n \oplus r_{n+1})^0_A \subseteq D$. Therefore, $r \in D$ as required. $\square$

3 Dual Ramsey Theorem

Recall the statement of the Dual Ramsey Theorem.

Theorem 3.1 (Dual Ramsey Theorem)**.** *Let $k \geq 1$. If*

$$(\omega)^k = C_0 \cup C_1 \cup \cdots \cup C_{l-1}$$

where each C_i is Borel, then there exists an $X \in (\omega)^\omega$ such that $(X)^k \subseteq C_i$ for some C_i.

Remember that $(\omega)^k$ denotes the set of all partitions of ω into exactly k blocks. Note that unlike the notation $\langle \omega \rangle^k_A$ which denoted *finite* ordered variable words with k many variables (which correspond to A-partitions of finite initial segments of ω), we are back to looking at partitions of all of ω. Similarly, $(\omega)^k_A$ denotes the set of all A-partitions of ω with exactly k many free blocks.

To define the topology used here, notice that we can view each partition P in $(\omega)^\omega$ or $(\omega)^k$ as a relation $P \subseteq \omega \times \omega$ for which $P(n,m)$ holds if and only if n and m are in the same partition block. Therefore, each partition P is an element of $2^{\omega \times \omega}$. This space is given the product topology (and $\{0,1\}$ is given the discrete topology), so $(\omega)^k$ and $(\omega)^\omega$ inherit a subspace topology. The same idea holds for a finite alphabet A and the sets $(\omega)^k_A$ and $(\omega)^\omega_A$. These spaces inherit a subspace topology from $2^{(A \cup \omega) \times (A \cup \omega)}$.

To give a little more intuition about the topology on $(\omega)^k_A$, consider the basic open sets in $2^{(A \cup \omega) \times (A \cup \omega)}$. To specify a basic open set in this space, we list finitely many conditions of the form u and v are related (i.e. in the same block), or u and v are not related (i.e. not in the same block) for elements $u, v \in A \cup \omega$. This finite list of conditions determines the basic open set of all relations on $A \cup \omega$ which satisfy these conditions. Of course, not all such sets of conditions can be satisfied by an A-partition. For example, if we specify that 0 and 1 are related, 1 and 2 are related, but 0 and 2 are not related, then this finite information cannot be satisfied by an A-partition. Similarly, if $a, b \in A$ and we specify that a and b are related, then this information cannot be accommodated by an A-partition. However, restricting the finite list of conditions to actual sets of A-partitions is taken care of by the subspace topology (i.e. by the restriction to the intersection with the set $(\omega)^k_A$).

In this section, we will prove the following generalization of the Dual Ramsey Theorem. The Dual Ramsey Theorem is the special case when $A = \emptyset$.

Theorem 3.2. *Let A be a finite (possibly empty) alphabet and $k \in \omega$ (with $k \geq 1$ if $A = \emptyset$). If $(\omega)^k_A = C_0 \cup C_1 \cup \cdots \cup C_{l-1}$ where each C_i is Borel, then there exists an $X \in (\omega)^\omega_A$ such that $(X)^k_A \subseteq C_i$ for some $i < l$.*

Theorem 3.2 follows from three lemmas.

Lemma 3.3. *Let $k \in \omega$ and let A be a finite alphabet such that $|A| = 1$. If Theorem 3.2 holds for $(\omega)^k_A$, then it also holds for $(\omega)^{k+1}_\emptyset$ (i.e. for $(\omega)^{k+1}$).*

Lemma 3.4. *Let A be a finite nonempty alphabet. If*

$$(\omega)^0_A = C_0 \cup C_1 \cup \cdots \cup C_{l-1}$$

where each C_i is Borel, then there exists a $Y \in (\omega)^\omega_A$ such that $(Y)^0_A \subseteq C_i$ for some $i < l$.

Lemma 3.5. *Let A be a finite nonempty alphabet and let a be a new symbol not in A. If Theorem 3.2 holds for $(\omega)^k_{A\cup\{a\}}$, then it also holds for $(\omega)^{k+1}_A$.*

Notice that Theorem 3.2 follows immediately from these three lemmas. Fix a finite (possible empty) alphabet A and $k \in \omega$. If $k = 0$ (and hence A is nonempty), then we are done by Lemma 3.4. If $k \geq 1$, then let $A + k$ denote an alphabet of size $|A| + k$. If A is nonempty, then Theorem 3.2 follows by applying Lemma 3.5 k many times. If A is empty, then we apply Lemma 3.5 $k-1$ times to obtain Theorem 3.2 for $(\omega)^{k-1}_B$ where B is a singleton set. Then we apply Lemma 3.3 once to obtain Theorem 3.2 for $(\omega)^k$.

Therefore, it remains to prove Lemmas 3.3 – 3.5. The proof of Lemma 3.3 is trivial because $(\omega)^k_{\{a\}} \cong (\omega)^{k+1}$. That is, in $(\omega)^k_{\{a\}}$, the lone nonfree block corresponding to the single element a acts like any other block and hence $(\omega)^k_{\{a\}}$ is really the set of all partitions of ω into exactly $k+1$ blocks. More specifically, we obtain an element of $(\omega)^{k+1}$ by collapsing blocks in the partition

$$\{0\}, \{1\}, \{2\}, \ldots$$

of ω until we are left with exactly $k+1$ many blocks. Similarly, we obtain an element of $(\omega)^k_{\{a\}}$ by collapsing blocks in the partition

$$\{a\}, \{0\}, \{1\}, \ldots$$

of $\{a\} \cup \omega$ until we have the block containing a and exactly k many free blocks. However, notice that there are no restrictions on blocks which can be collapsed with the nonfree block containing a. Thus, the nonfree block containing a acts just like a free block. Therefore, Lemma 3.3 is established.

Notice that this reasoning cannot be extended. That is, $(\omega)^k_{\{a,b\}}$ is not the same as $(\omega)^{k+2}$. When collapsing blocks in the partition

$$\{a\}, \{b\}, \{0\}, \{1\}, \ldots$$

to obtain an A-partition with k free blocks, the nonfree blocks containing a and b do not act like arbitrary free blocks because they cannot be collapsed together. This is why we need to work harder to prove our generalization of the Dual Ramsey Theorem.

Since we are now working with partitions instead of words, let me review some of the definitions in this context. We say s is a segment of an A-partition X and write $s \prec X$ to mean $s = X[n]$ for some n, where we identify n with $\{0, 1, \ldots, n-1\}$ and

$$X[n] = \{x \cap (A \cup n) \mid x \in X\} \setminus \emptyset$$

That is, we simply restrict the X-blocks to the initial segment n of ω and remove any blocks whose intersection with this initial segment is empty. Thus, we obtain exactly the A-partition of n formed by restricting the A-partition X.

In this section, we will freely use the language of both variable words and A-partitions since these notions are equivalent (as explained in the first section). Recall that we no longer have the restriction on variable words that all occurrences of x_i come before the first occurrence of x_{i+1}. We do retain the restriction that the first occurrence of x_i comes before the first occurrence of x_{i+1}. (The point of this restriction was to have a one-to-one correspondence between variable words and A-partitions.) Recall that $s \leq X$ means $s \leq X[n]$, or equivalently, $s \preceq Y[n]$ for some $Y \in (X)^\omega_A$. For $s \leq X$, we write

$$(s, X)^\omega_A = \{Y \in (X)^\omega_A \mid s \prec Y\}$$
$$(s, X)^k_A = \{Y \in (X)^k_A \mid s \prec Y\}$$

We can now prove Lemma 3.4, which is restated here for convenience.

Lemma 3.6. *Let A be a finite nonempty alphabet. If $(\omega)^0_A = C_0 \cup C_1 \cup \cdots \cup C_{l-1}$, where each C_i is Borel, then there exists a $Y \in (\omega)^\omega_A$ such that $(Y)^0_A \subseteq C_i$ for some $i < l$.*

Proof. Note that $(\omega)^0_A = A^\omega$ and hence is a compact Hausdorff space with basic open sets

$$T_s = \{Y \in A^\omega \mid s \prec Y\}$$

where $s \in A^{<\omega}$. Pictorially, you can think of A^ω as the set of infinite paths through the finitely branching tree $A^{<\omega}$. The basic open set T_s is the set of all infinite paths passing through the node s on $A^{<\omega}$.

We will use two topological facts without proof. The first topological fact is that Borel subsets of A^ω have the Property of Baire. That is, each Borel subset of A^ω differs from an open set by a set of first category (i.e. by a countable union of nowhere dense sets). Since each color C_i is Borel, this means that for each $i < l$, there is open set O_i and a sequence of dense open sets $D_{i,n}$ such that

$$C_i \triangle O_i \subseteq \bigcup_{n \in \omega} \overline{D_{i,n}}$$

or equivalently

$$C_i \triangle O_i \subseteq A^\omega \setminus \bigcap_{n \in \omega} D_{i,n}$$

Since $C_0 \cup \cdots \cup C_{l-1} = A^\omega$, we have that $O_0 \cup \cdots \cup O_{l-1}$ contains all the points in A^ω except possibly the points in each $C_i \setminus O_i$. Since

$$\bigcup_{i<l}(C_i \setminus O_i) \subseteq \bigcup_{i<l, n\in\omega} \overline{D_{i,n}}$$

we have

$$A^\omega \setminus \bigcup_{i<l, n\in\omega} \overline{D_{i,n}} \subseteq O_0 \cup \cdots \cup O_{l-1}$$

and hence

$$\bigcap_{i<l, n\in\omega} D_{i,n} \subseteq O_0 \cup \cdots \cup O_{l-1}.$$

The second topological fact we will use is the Baire Category Theorem which says that a countable intersection of dense open sets in A^ω remains dense. Thus, the set $\cap_{i,n} D_{i,n}$ is dense and hence $O_0 \cup \cdots \cup O_{l-1}$ is a dense open set. Therefore, at least one of the O_i sets is nonempty, so we can fix an $i < l$ and a $t_0 \in A^{<\omega}$ such that $T_{t_0} \subseteq C_i$.

We want to continue to define a sequence $t_0, t_1, t_2, \ldots$ such that $t_n \prec t_{n+1}$, $\#(t_{n+1}) = n+1$ and for all $s \leq t_{n+1}$ with $\#(s) = 0$, $T_s \subseteq D_{i,n}$. In other words, for $n \geq 1$, t_n is a finite variable word with n many variables such that for each s obtained by substituting elements of A for all the variables in t_n, the basic open set T_s is contained in $D_{i,n-1}$. Our desired infinite word Y will be the limit of the finite t_n words.

How do we define t_1? Recall that $D_{i,0}$ is a dense open set. Therefore, it intersects T_{t_0} and the intersection is open (and hence contains a basic open set). Fix $t_0' \in A^{<\omega}$ such that $t_0 \preceq t_0'$ and $T_{t_0'} \subseteq D_{i,0}$. Let $t_1 = t_0'^* = t_0' x_0$. Clearly, $\#(t_1) = 1$ as required and any $s \leq t_1$ with $\#(s) = 0$ satisfies $t_0' \preceq s$ and hence $T_s \subseteq T_{t_0'} \subseteq D_{i,0}$.

How do we define t_2? For simplicity, suppose $A = \{a, b\}$ and recall that $t_1 = t_0' x_0$. Let $s_a = t_0' a$ be the result of substituting a in for x_0 in t_1. Thus $s_a \in A^{<\omega}$ determines a basic open set T_{s_a}. Since $D_{i,1}$ is a dense open set, there is a $s_a' \in A^{<\omega}$ such that $s_a \preceq s_a'$ and $T_{s_a'} \subseteq D_{i,1}$. Fix $u \in A^{<\omega}$ such that $s_a' = s_a u = t_0' a u$.

Let $s_b = t_0' b u \in A^{<\omega}$. That is, replace the occurrence of the symbol a in s_a' which was originally substituted for x_0 in t_1 to form s_a by the symbol b. Again, since $D_{1,n}$ is an open dense set, there is a string $s_b' \in A^{<\omega}$ such that $s_b \preceq s_b'$ and $T_{s_b'} \subseteq D_{i,1}$. Fix $v \in A^{<\omega}$ such that

$$s_b' = s_b v = t_0' b u v$$

and let t_2 be defined by

$$t_2 = t_0' x_0 u v x_1$$

Notice that $\#(t_2) = 2$ as required. Also, if $s \leq t_2$ and $\#(s) = 0$ (so $s \in A^{<\omega}$), then either $s_a' \preceq s$ (if a is substituted in for x_0) or $s_b' \preceq s$ (if b is substituted in for x_0). In either case, we have that $T_s \subseteq D_{i,1}$ are required.

Containing in this way, we get a sequence $t_0 \prec t_1 \prec t_2 \prec \cdots \prec t_{n+1} \prec \cdots$ such that $\#(t_{n+1}) = n+1$ and for all $s \leq t_{n+1}$ with $\#(s) = 0$, $T_s \subseteq D_{i,n}$.

Let Y be the limit of the finite t_n partitions. That is, Y is the unique element of $(\omega)^\omega_A$ such that $t_n \prec Y$ for all n. The fact that $t_0 \prec Y$ implies that $(Y)^0_A \subseteq O_i$. That is, an arbitrary element of $(Y)^0_A$ is formed by substituting elements of A in for the (infinitely many) variables of Y. Since $t_0 \prec Y$ and t_0 contains no variables, no matter how this substitution is done, the resulting string contains t_0 as an initial segment. Therefore, an arbitrary element of $(Y)^0_A$ has t_0 as an initial segment, and hence is contained in T_{t_0}, which is contained (by construction) in O_i.

The fact that $t_{n+1} \prec Y$ implies that $(Y)^0_A \subseteq D_{i,n}$. That is, since $t_{n+1} \prec Y$, any element of $(Y)^0_A$ must contain some $s \leq t_{n+1}$ with $\#(s) = 0$ as an initial segment. By construction, for such s, we have $T_s \subseteq D_{i,n}$. Hence, each element of $(Y)^0_A$ is in $D_{i,n}$. Since $(Y)^0_A \subseteq D_{i,n}$ for every n, we have

$$(Y)^0_A \subseteq \bigcap_{n \in \omega} D_{i,n}$$

Since

$$C_i \bigtriangleup O_i \subseteq A^\omega \setminus \bigcap_{n \in \omega} D_{i,n}$$

we have $(Y)^0_A \cap (C_i \bigtriangleup O_i) = \emptyset$. This means that $(Y)^0_A$ is either contained in both C_i and O_i or in neither C_i nor O_i. Since $(Y)^0_A \subseteq O_i$, it follows that $(Y)^0_A \subseteq C_i$ as required. $\square$

Before proving Lemma 3.5, we need an important observation. Assume that the conclusion of Theorem 3.2 holds for $(\omega)^k_{A\cup\{a\}}$. Our goal is to show that this conclusion also holds for $(\omega)^{k+1}_A$. The idea will be to transfer instances of $(\omega)^{k+1}_A$ to instances of $(\omega)^k_{A\cup\{a\}}$. The difficulty, as noted before, is that in $(\omega)^\omega_{A\cup\{a\}}$, the block containing the element of a does not behave like an arbitrary free block because it cannot be collapsed with a block containing an element of A. Therefore, there is a crucial difference between $(\omega)^{k+1}_A$ and $(\omega)^k_{A\cup\{a\}}$ even though the number of partition blocks in each case is the same.

What we need is a way of "fixing" a free block in $(\omega)^\omega_A$ that is not collapsed into a block containing an element of A, so that this free block acts like the block containing a in $(\omega)^\omega_{A\cup\{a\}}$. The idea for doing this procedure will be to iterate the following lemma. Recall that for $X \in (\omega)^\omega_A$ and $s \leq X$, $(s, X)^k_A$ denotes the set of all $Y \in (X)^k_A$ such that $s \prec Y$. In terms of infinite words, $(s, X)^k_A$ is the set of all coarsenings Y of the infinite variable word X such that Y is an infinite word with exactly k variables and Y contains s as an initial segment.

Lemma 3.7. *Assume that Theorem 3.2 holds for $(\omega)^k_{A\cup\{a\}}$ and fix a Borel coloring*

$$(\omega)^{k+1}_A = C_0 \cup \cdots \cup C_{l-1}$$

For any $X \in (\omega)^\omega_A$, $s \in (X)^*_A$ and $X' \in (X)^\omega_A$ such that $X'[|s^*|] = X[|s^*|]$, there is an $X'' \in (X')^\omega_A$ such that $X''[|s^*|] = X'[|s^*|]$ and $(s^*, X'')^{k+1}_A \subseteq C_i$ for some $i < l$.

Proof. Fix $X \in (\omega)^\omega_A$, $s \in (X)^*_A$ and $X' \in (X)^\omega_A$ such that $X'[|s^*|] = X[|s^*|]$. That is, there is a finite word t such that $t \prec X$, the first symbol in X after t is the variable $x_{\#(t)}$ (i.e. $t^* \preceq X$), $s \in (t)^0_A$ (i.e. s is the result of substituting elements of A in for all the variables in t) and (since $|s^*| = |sx_0| = |t^*|$), $t^* \prec X'$. In terms of a picture, we have fixed a coarsening s of an initial segment t of X, where the first symbol in X after the initial segment t is a new variable, and X' is an infinite variable word coarsening X which starts with this fixed initial segment t^*.

We need to show that there is a further coarsening X'' of X' which retains the initial segment t^* but which has the property that any way we coarsen X'' down to an infinite word with exactly $k+1$ many variables which begins with s^*, we land in a fixed color C_i. Notice that by restricting ourselves to coarsenings of X'' which begin with s^*, we have guaranteed that the variable x_0 occurs as the $|s|$-th symbol in the coarsening. In terms of partitions, the free block containing the number $|s|$ is not collapsed into a nonfree block in any element of $(s^*, X'')_A^{k+1}$.

To prove the existence of X'', we claim that there is a canonical homeomorphism between $(s^*, X')_A^{k+1}$ and $(\omega)_{A\cup\{a\}}^k$. To see why, consider how we obtain elements of these two spaces. Let $A = \{a_0, \ldots, a_k\}$. To form an element of $(\omega)_{A\cup\{a\}}^k$, we start with the trivial $A \cup \{a\}$-partition with nonfree blocks

$$B_{a_0} = \{a_0\}, \ldots, B_{a_k} = \{a_k\}, B_a = \{a\}$$

and free blocks

$$F_0 = \{0\}, F_1 = \{1\}, F_2 = \{2\}, \ldots$$

First, we choose whether to collapse the free block F_0 into one of the nonfree blocks or to allow it to remain free. Second, we choose whether to collapse F_1 into one of the nonfree blocks, to collapse it with the free block F_0 (if F_0 remains free), or to allow it to remain as a new free block. This process continues through the free blocks as we choose in stage n whether to collapse F_n into a nonfree block, to collapse it with a free block of lower index (which remains free), or to allow it to remain a new free block. To get an element of $(\omega)_{A\cup\{a\}}^k$, we must retain exactly k many free blocks at the end of this process.

To get an element of $(s^*, X')_A^{k+1}$, we perform a similar process. Let $Y \in (\omega)_A^\omega$ be such that $t \oplus Y = X'$ and let $Z = s \oplus Y$. That is, Z is the result of doing the coarsening prescribed by s to X'. Note that $s^* \prec Z$ since $t^* \prec X'$. Thus $(s^*, X')_A^{k+1} = (s^*, Z)_A^{k+1}$ and (since $s \in A^{<\omega}$) the least element of ω which is in a free block in Z is $|s|$. Therefore, the 0-th free block of Z contains the number $|s|$.

To form an element of $(s^*, X')_A^{k+1}$, we start with the A-partition given by Z which consists of nonfree blocks

$$B'_{a_1}, B'_{a_2}, \ldots, B'_{a_k}$$

and free blocks

$$F'_0 = \{|s|, \ldots\}, F'_1, F'_2, \ldots$$

We proceed by making choices in stages about collapsing the free blocks F'_n exactly as in the case of $(\omega)_{A\cup\{a\}}^k$ with the important exception that we are forced to choose to retain F'_0 as a free block (since we consider only coarsenings that begin with s^*). Thus F'_0 is a fixed free block that remains free in all the coarsenings considered. So, we can think of elements of $(s^*, X')_A^{k+1}$ as constructed in stages where at stage n, we decide what to do with the free block F'_{n+1} (since there is no choice for F_0).

We can now describe the homeomorphism between $(s^*, X')_A^{k+1}$ and $(\omega)_{A\cup\{a\}}^k$. For an element $W \in (\omega)_{A\cup\{a\}}^k$ consider the list of choices made in the stages constructing W. Construct an element $V \in (s^*, X')_A^{k+1}$ by (at stage n) collapsing F'_{n+1} into B'_{a_i} if F_n is collapsed into B_{a_i}, collapsing F'_{n+1} into F'_0 if F_n is collapsed into B_a, collapsing F'_{n+1} into F'_{i+1} if F_n is collapsed into F_i (for $i < n$), and leaving F'_{n+1} as a new free block if F_n is left as a new free block. This gives a bijection between these spaces which is a homeomorphism because it respects finite lists of conditions specifying which free blocks are (or are not) collapsed into nonfree blocks and which free blocks are (or are not) collapsed together.

It remains to use this homeomorphism to extract an appropriate X''. Fix a Borel coloring of $(\omega)^{k+1}_A$. This restricts to a Borel coloring of $(s^*, X')^{k+1}_A$ and via the homeomorphism gives a Borel coloring of $(\omega)^k_{A\cup\{a\}}$. Since Theorem 3.2 holds for $(\omega)^k_{A\cup\{a\}}$, there is a $U \in (\omega)^\omega_{A\cup\{a\}}$ such that $(U)^k_{A\cup\{a\}}$ is monochromatic.

We want to push this solution back to a solution for $(s^*, X')^{k+1}_A$. Notice that the correspondence we have described between $(\omega)^k_{A\cup\{a\}}$ and $(s^*, X')^{k+1}_A$ also works as a correspondence between $(\omega)^\omega_{A\cup\{a\}}$ and $(s^*, X')^\omega_A$. That is, rather than mapping coarsening with exactly k (or $k+1$) free blocks, we look at coarsenings which retain infinitely many free blocks. The obvious map preserves these. Therefore, we can map $U \in (\omega)^\omega_{A\cup\{a\}}$ to $V \in (s^*, X')^\omega_A$. Furthermore, because the method of coarsening is the same in both spaces (modulo the shift of $+1$ in the indices on the free blocks and the use of F'_0 in place of B_a), we have that if $(U)^k_{A\cup\{a\}}$ is monochromatic in the induced coloring on $(\omega)^k_{A\cup\{a\}}$, then $(s^*, V)^{k+1}_A$ is monochromatic in the coloring on $(s^*, X')^{k+1}_A$.

To get X'' so that $X''[|s^*|] = X'[|s^*|] = X[|s^*|] = t^*$, we replace the initial segment $s^* \prec V$ by t^*. Since $s \leq t$, we have $(s^*, X'')^{k+1}_A = (s^*, V)^{k+1}_A$ is monochromatic (i.e. contained in some C_i) as required. $\square$

We can now give our proof of Lemma 3.5, which is restated here for convenience.

Lemma 3.8. *Let A be a finite nonempty alphabet and let a be a new symbol not in A. If Theorem 3.2 holds for $(\omega)^k_{A\cup\{a\}}$, then it also holds for $(\omega)^{k+1}_A$.*

Proof. We construct a sequence $X_0, X_1, \ldots$ of elements of $(\omega)^\omega_A$ and a sequence of A-segments $t_0, t_1, \ldots$ such that $t^*_0 \preceq t^*_1 \preceq t^*_2 \preceq \cdots$ as follows. Let $X_0 \in (\omega)^\omega_A$ be arbitrary. Given $X_n \in (\omega)^\omega_A$, let t_n be the unique A-segment such that $t^*_n \prec X_n$ and $\#(t_n) = n$. In terms of words, let t_n be the initial segment of X_n ending just before the first occurrence of x_n. We claim that there is an $X_{n+1} \in (t^*_n, X_n)^\omega_A$ such that for each $s \leq t_n$ with $\#(s) = 0$, there is a color C_i (depending on s) such that $(s^*, X_{n+1})^{k+1}_A \subseteq C_i$. (We prove this below.) Let X_{n+1} be such an A-partition to continue the recursive definition. Note that since $X_{n+1} \in (t^*_n, X_n)^\omega_A$, the definition of t_{n+1} yields $t^*_n \preceq t^*_{n+1}$.

Why is there such an X_{n+1}? Let $\{s_j \mid j \leq m\}$ enumerate all $s \leq t_n$ such that $\#(s) = 0$. Set $X^0_n = X_n$. Note that $t^*_n \prec X^0_n$ and $s_0 \leq t_n$. Therefore, $s^*_0 \in (X^0_n)^*_A$. Assume we have defined X^j_n for a fixed $j \leq m$ such that $t^*_n \prec X^j_n$ and hence $s_j \in (X^j_n)^*_A$. Apply our observation above to X^j_n and s_j to get a color C_i and a partition $X^{j+1}_n \in (t^*_n, X^j_n)^\omega_A$ such that $(s^*_j, X^{j+1}_n)^{k+1}_A \subseteq C_i$ and

$$X^{j+1}_n[\text{length}(s^*_j)] = X^j_n[\text{length}(s^*_j)] = X_n[\text{length}(s^*_j)] = t^*_n$$

Finally, let $X_{n+1} = X^{m+1}_n$.

Let Y be the limit of the t^*_n segments. That is, $Y \in (\omega)^\omega_A$ is the unique element such that $t^*_n \prec Y$ for all n. Consider any $s \in (Y)^*_A$. By definition, s is formed by cutting off Y just before the first occurrence of a variable x_n and substituting elements of A for all the variables in this initial segment. However, since $t_n^* \preceq Y$ and t_n^* ends with the first occurrence of x_n, we have that the initial segment of Y formed by cutting off before the first occurrence of x_n is exactly t_n. Therefore, if $s \in (Y)_A^*$, then $s \leq t_n$ for some $n \in \omega$ and $\#(s) = 0$. Therefore, by construction $(s^*, X_{n+1})_A^{k+1} \subseteq C_i$ for some i depending on s. However, $Y \in (t_n^*, X_{n+1})_A^\omega$ since Y is the limit of the process of coarsening of the X_i partitions. Therefore, $(s^*, Y)_A^{k+1} \subseteq (s^*, X_{n+1})_A^{k+1} \subseteq C_i$.

Define a coloring $(Y)_A^* = C_0^* \cup \cdots \cup C_{l-1}^*$ as follows. For $s \in (Y)_A^*$, we put $s \in C_i^*$ if and only if $(s^*, Y)_A^{k+1} \subseteq C_i$. By Corollary 1.17 (which is a form of our Combinatorial Core Theorem), there is a $Z \in (Y)_A^\omega$ such that $(Z)_A^* \subseteq C_i^*$ for some $i < l$. That is, there is a fixed $i < l$ such that $(s^*, Z)_A^{k+1} \subseteq C_i$ for all $s \in (Z)_A^*$.

We claim that $(Z)_A^{k+1} \subseteq C_i$ which finishes the proof. Consider an arbitrary $U \in (Z)_A^{k+1}$. Let $s \in A^{<\omega}$ be such that $s^* \prec U$ (i.e. s is the initial segment of U obtained by cutting off before the first occurrence of x_0 in U). Because U is a coarsening of Z, the first occurrence of x_0 in U corresponds to the first occurrence of some variable in Z. Therefore, $s \in (Z)_A^*$ and hence $(s^*, Z)_A^{k+1} \subseteq C_i$. However, $U \in (s^*, Z)_A^{k+1}$ and hence $U \in C_i$ as required. $\square$