

# Some descriptive set theory

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## 1 Polish spaces

**Definition 1.1.** Let  $(X, \tau)$  be a topological space. A subset  $D \subseteq X$  is called **dense** if  $D \cap O \neq \emptyset$  for every nonempty open set  $O \subseteq X$ .  $X$  is called **separable** if  $X$  has a countable dense subset.  $X$  is called **metrizable** if there is a metric  $d$  on  $X$  such that the topology  $\tau$  is the same as the topology induced by the metric. The metric is called **complete** if every Cauchy sequence converges in  $X$ . Finally,  $X$  is a **Polish space** if  $X$  is a separable topological space that is metrizable by a complete metric.

There are many natural examples of Polish spaces. For example,  $\mathbb{R}$  is a Polish space under the usual topology and metric because  $\mathbb{Q} \subseteq \mathbb{R}$  is a dense subset. Similarly,  $[0, 1]$  is a compact Polish space with the usual metric. Notice that  $(0, 1)$  is not a Polish space with the usual metric because the Cauchy sequence  $1, 1/2, 1/4, \dots$  in  $(0, 1)$  does not converge in  $(0, 1)$ . (We will show later that  $(0, 1)$  is a Polish space under a different metric!) The two most important examples of Polish spaces for our purposes will be *Cantor space*  $2^{\mathbb{N}}$  and *Baire space*  $\mathbb{N}^{\mathbb{N}}$ .

Before looking at Cantor space and Baire space in detail, we begin with some exercises about general properties of Polish spaces. For all of these exercises, let  $X$  be a Polish space with complete metric  $d$  and countable dense subset  $D \subseteq X$ .

**Exercise 1.2.** Prove that  $X$  is Hausdorff. That is, prove that for any two distinct points  $x$  and  $y$ , there are open sets  $U_x$  and  $U_y$  such that  $x \in U_x$ ,  $y \in U_y$  and  $U_x \cap U_y = \emptyset$ .

**Exercise 1.3.** Prove that  $X$  has a countable basis. That is, there is a countable collection of open sets which generate the topology  $\tau$  of open sets when you close under taking arbitrary unions. (Hint: Remember that the topology  $\tau$  is equivalent to the induced metric topology.)

**Exercise 1.4.** Let  $X$  be a Polish space with complete metric  $d$  and dense subset  $D \subseteq X$ . Prove that every element of  $X$  is the limit of a Cauchy sequence of elements from  $D$ .

**Exercise 1.5.** Prove that  $|X| \leq 2^{\aleph_0}$ .

**Exercise 1.6.** Let  $Y \subseteq X$  be closed and let  $a \in X \setminus Y$ . Let

$$i = \inf\{d(a, y) \mid y \in Y\}.$$

First, why is  $i$  well defined? Second, prove that  $i > 0$ . Third, prove that there is an element  $b \in Y$  such that  $d(a, b) = i$ . Because of these properties, it is reasonable to define the distance from  $a$  to  $Y$  as  $i$  and to denote this distance by  $d(a, Y)$ . (If  $a \in Y$ , we say that  $d(a, Y) = 0$  so that this “distance” is defined for all  $a \in X$ . Note that it is important that  $X$  is closed here!)

**Exercise 1.7.** Prove the following stronger version of the Hausdorff property. Let  $U$  be an open set in  $X$  and let  $a, b$  be distinct points in  $U$ . Prove that there are open sets  $U_a$  and  $U_b$  such that  $a \in U_a$ ,  $b \in U_b$ ,  $\overline{U_a} \subseteq U$ ,  $\overline{U_b} \subseteq U$ , and  $\overline{U_a} \cap \overline{U_b} = \emptyset$ .

**Exercise 1.8.** Prove that

$$\hat{d} = \frac{d(x, y)}{1 + d(x, y)}$$

is also a complete metric on  $X$  and that  $d$  and  $\hat{d}$  induce the same topology on  $X$ . Therefore, complete metrics on Polish spaces are not unique, and moreover, we can always assume that our complete metric satisfies  $d(x, y) < 1$  if it is useful. (This property is frequently denoted by the shorthand  $d < 1$ .)

Let us now turn to Baire space  $\mathbb{N}^{\mathbb{N}}$ . The elements of Baire space are functions from  $\mathbb{N}$  to  $\mathbb{N}$  and the metric is given by

$$d(f, g) = \frac{1}{2^{n+1}}$$

where  $n$  is the least element of  $\mathbb{N}$  such that  $f(n) \neq g(n)$ . (If  $f = g$ , then  $d(f, g) = 0$ .)

**Exercise 1.9.** Prove that the metric  $d$  on  $\mathbb{N}^{\mathbb{N}}$  is complete. Furthermore, show that the set

$$D = \{f \in \mathbb{N}^{\mathbb{N}} \mid \exists i \forall j \geq i (f(j) = 0)\}$$

is a countable dense subset of  $\mathbb{N}^{\mathbb{N}}$ .

A collection of basic open neighborhoods of  $\mathbb{N}^{\mathbb{N}}$  can be indexed by the strings  $\sigma \in \mathbb{N}^{<\mathbb{N}}$  as follows:

$$N_\sigma = \{f \in \mathbb{N}^{\mathbb{N}} \mid f \upharpoonright |\sigma| = \sigma\}.$$

**Exercise 1.10.** Show that the sets  $N_\sigma$  are both closed and open in  $\mathbb{N}^{\mathbb{N}}$  and that they form a basis for the topology.

Next, we show that the open and closed subsets of Baire space can be represented in particularly nice forms. The form for open sets follows almost immediately from the previous exercise.

**Exercise 1.11.** Prove that a subset  $U \subseteq \mathbb{N}^{\mathbb{N}}$  is open if and only if there is a set  $W \subseteq \mathbb{N}^{<\mathbb{N}}$  such that

$$U = \bigcup_{\sigma \in W} N_\sigma.$$

To give the nice form for closed sets, we need the definition for a subtree of  $\mathbb{N}^{<\mathbb{N}}$ .

**Definition 1.12.**  $T \subseteq \mathbb{N}^{<\mathbb{N}}$  is a **tree** if  $T$  is downward closed. That is, for all  $\sigma \in T$  and all  $\mu \subseteq \sigma$ ,  $\mu \in T$ . If  $T$  is a tree, then  $[T]$  denotes the set of all infinite paths through  $T$ . That is,  $f \in [T]$  if and only if  $f \upharpoonright n \in T$  for all  $n$ . We say that a tree  $T$  is **pruned** if for every  $\sigma \in T$ , there is an  $f \in [T]$  such that  $\sigma = f \upharpoonright |\sigma|$ . That is,  $T$  is pruned if every node on  $T$  can be extended to an infinite path through  $T$ .

**Exercise 1.13.** Show that  $C \subseteq \mathbb{N}^{\mathbb{N}}$  is closed if and only if there is a (pruned) tree  $T \subseteq \mathbb{N}^{<\mathbb{N}}$  such that  $C = [T]$ . *Hint:* If  $C$  is closed, then define  $T$  by setting  $\sigma \in T$  if and only if  $N_\sigma \cap C \neq \emptyset$ . Prove that  $T$  is a tree and that  $C = [T]$ .

The basic definitions and results for Cantor space  $2^{\mathbb{N}}$  are essentially the same as those for Baire space. The standard metric is  $d(f, g) = 2^{-(n+1)}$  where  $n$  is the least element of  $\mathbb{N}$  such that  $f(n) \neq g(n)$ , the set  $D$  defined for Baire space but restricted to  $2^{\mathbb{N}}$  is dense, and the basic open (in fact clopen) neighborhoods are  $N_\sigma$  defined as before except restricted to  $2^{\mathbb{N}}$ . The main difference between Baire space and Cantor space is that Cantor space is compact while Baire space is not.

**Exercise 1.14.** Verify that Baire space is not compact while Cantor space is compact.

In the concrete contexts of Cantor space and Baire space, it is not hard to show that every uncountable closed set has size  $2^{\aleph_0}$ . For the next two exercises, consider an uncountable closed set  $C \subseteq \mathbb{N}^{\mathbb{N}}$ .

**Exercise 1.15.** Assume that  $\sigma \in \mathbb{N}^{<\mathbb{N}}$  is such that  $N_\sigma \cap C$  is uncountable. Show that there are two extensions  $\tau_0$  and  $\tau_1$  of  $\sigma$  such that  $N_{\tau_0} \cap N_{\tau_1} = \emptyset$  and both  $N_{\tau_0} \cap C$  and  $N_{\tau_1} \cap C$  are uncountable.

**Exercise 1.16.** Use the previous exercise to build a sequence of basic clopen sets  $U_\sigma$  in  $\mathbb{N}^{\mathbb{N}}$  indexed by  $\sigma \in 2^{<\mathbb{N}}$  with the following properties. (That is, each  $U_\sigma$  will be a basic clopen neighborhood of the form  $N_\tau$  for some  $\tau \in \mathbb{N}^{<\mathbb{N}}$ .)

1.  $U_\emptyset = \mathbb{N}^{\mathbb{N}} = N_\emptyset$ ,
2.  $U_\sigma \subseteq U_\mu$  for all  $\sigma \supseteq \mu$ ,
3.  $U_\sigma \cap C$  is uncountable for each  $\sigma$ ,
4.  $U_{\sigma*0} \cap U_{\sigma*1} = \emptyset$ .

Show that for any  $f \in 2^{\mathbb{N}}$ ,

$$\left| \bigcap_n U_{f \upharpoonright n} \right| = 1$$

and that

$$\bigcap_n U_{f \upharpoonright n} \subseteq C.$$

Therefore, the map  $\phi : 2^{\mathbb{N}} \rightarrow C$  given by

$$\phi(f) = \bigcap_n U_{f \upharpoonright n}$$

is well defined. Show that  $\phi$  is one-to-one and therefore  $|C| \geq 2^{\aleph_0}$ . Since  $|C| \leq 2^{\aleph_0}$  by Exercise 1.5, you know  $|C| = 2^{\aleph_0}$ . Finally, show that  $\phi$  is continuous by showing that for each  $\tau \in \mathbb{N}^{<\mathbb{N}}$ ,  $\phi^{-1}(N_\tau)$  is open in  $2^{\mathbb{N}}$ .

**Definition 1.17.** Let  $P \subseteq \mathbb{N}^{\mathbb{N}}$  and let  $f \in P$ . We say  $f$  is **isolated** in  $P$  if there is a basic clopen set  $N_\sigma$  such that  $P \cap N_\sigma = \{f\}$ .  $P$  is called **perfect** if  $P$  is closed and contains no isolated points.

The same style of argument in the previous two exercises works in the case when  $P \subseteq \mathbb{N}^{\mathbb{N}}$  is a nonempty perfect set. For the next two exercises, let  $P$  be a nonempty perfect subset of  $\mathbb{N}^{\mathbb{N}}$ .

**Exercise 1.18.** Assume that  $\sigma \in \mathbb{N}^{<\mathbb{N}}$  is such that  $N_\sigma \cap P$  is nonempty. Show that there are two extensions  $\tau_0$  and  $\tau_1$  of  $\sigma$  such that  $N_{\tau_0} \cap N_{\tau_1} = \emptyset$  and both  $N_{\tau_0} \cap P$  and  $N_{\tau_1} \cap P$  are nonempty.

**Exercise 1.19.** Use the previous exercise to build a sequence of basic clopen sets  $U_\sigma$  in  $\mathbb{N}^{\mathbb{N}}$  indexed by  $\sigma \in 2^{<\mathbb{N}}$  with the following properties. (That is, each  $U_\sigma$  will be a basic clopen neighborhood of the form  $N_\tau$  for some  $\tau \in \mathbb{N}^{<\mathbb{N}}$ .)

1.  $U_\emptyset = \mathbb{N}^{\mathbb{N}} = N_\emptyset$ ,
2.  $U_\sigma \subseteq U_\mu$  for all  $\sigma \supseteq \mu$ ,
3.  $U_\sigma \cap P$  is nonempty for each  $\sigma$ ,
4.  $U_{\sigma*0} \cap U_{\sigma*1} = \emptyset$ .

Show that for any  $f \in 2^{\mathbb{N}}$ ,

$$\left| \bigcap_n U_{f \upharpoonright n} \right| = 1$$

and that

$$\bigcap_n U_{f \upharpoonright n} \subseteq P.$$

(Recall that  $P$  is closed!) Therefore, the map  $\phi : 2^{\mathbb{N}} \rightarrow P$  given by

$$\phi(f) = \bigcap_n U_{f \upharpoonright n}$$

is well defined. Show that  $\phi$  is one-to-one and therefore  $|P| = 2^{\aleph_0}$ . Finally, show that  $\phi$  is continuous.

Before leaving this section of basic exercises on Polish spaces and in particular on Cantor space and Baire space, I want you to prove that our Polish topology on  $\mathbb{N}^{\mathbb{N}}$  is the same as the usual product topology on  $\mathbb{N}^{\mathbb{N}}$  when  $\mathbb{N}$  is given the discrete topology. This equivalence is contained in the remaining exercises in this section. (Similarly, the Polish topology on Cantor space is the same as the product topology on  $2^{\mathbb{N}}$  when  $2 = \{0, 1\}$  is given the discrete topology.)

As a reminder, the discrete topology on  $\mathbb{N}$  is the topology under which every subset of  $\mathbb{N}$  is open. Notice that  $\mathbb{N}$  with the discrete topology is separable since  $\mathbb{N}$  is countable and hence forms a countable dense subset of itself!

**Exercise 1.20.** Prove that the discrete topology on  $\mathbb{N}$  has a countable basis consisting of the singleton subsets of  $\mathbb{N}$ .

**Exercise 1.21.** Show that the discrete topology on  $\mathbb{N}$  is the same as the metric topology generated by the complete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

This exercise includes showing that  $d$  is a complete metric.

You now know a quite different looking example of a Polish space –  $\mathbb{N}$  with the discrete topology! We next need to introduce the usual product topology. Let  $X_i$  for  $i \in I$  be topological spaces and consider the product set

$$\prod_I X_i = \{f : I \rightarrow \bigcup X_i \mid \forall i \in I (f(i) \in X_i)\}.$$

There are natural projection functions  $\pi_i : \prod X_i \rightarrow X_i$  for each  $i \in I$  defined by  $\pi_i(f) = f(i)$ . The product topology on  $\prod_I X_i$  is generated as follows. For each  $i \in I$  and each basic open  $O_i \subseteq X_i$ , the subset  $\pi_i^{-1}(O_i)$  is open in the product topology. The collection of all such subsets forms a subbasis for the product topology on  $\prod_I X_i$ . We typically omit the subscript  $I$  when it is understood from context.

The set  $\pi_i^{-1}(O_i)$  consists of all functions  $f \in \prod X_i$  such that  $f(i) \in O_i$ . That is, we restrict the  $i$ -th component of  $f$  to lie within  $O_i$  and allow the other components  $f(j)$  for  $j \neq i$  to take on any values in  $X_j$ . Taking finite intersections of these subbasic open sets means that a basic open set is formed by specifying a finite number of components  $i_0, i_1, \dots, i_k$  and basic open sets  $O_{i_0}, O_{i_1}, \dots, O_{i_k}$  from  $X_{i_0}, X_{i_1}, \dots, X_{i_k}$  and taking

$$\pi_{i_0}^{-1}(O_{i_0}) \cap \pi_{i_1}^{-1}(O_{i_1}) \cap \dots \cap \pi_{i_k}^{-1}(O_{i_k}).$$

In other words, we restrict finitely many values  $f(i)$  to lie within specified basic open subsets  $O_i$  of  $X_i$  and allow the other values  $f(j)$  to range over all of  $X_j$ .

The most important property of the product topology is that if you look at any map  $\phi : Y \rightarrow \prod X_i$  where  $Y$  is a topological space and  $\prod X_i$  is given the product topology, then  $\phi$  is continuous if and only if it is continuous when restricted to each component. That is,  $\phi$  is continuous if and only if each map  $\phi_i : Y \rightarrow X_i$  defined by  $\phi_i \circ \phi$  is continuous.

**Exercise 1.22.** Consider  $\mathbb{N}^{\mathbb{N}}$  as the product space  $\prod_{\mathbb{N}} \mathbb{N}$  where each copy of  $\mathbb{N}$  is given the discrete topology. Given that a basic open set in  $\mathbb{N}$  is a singleton, what do the subbasic open sets in  $\prod_{\mathbb{N}} \mathbb{N}$  look like? What do the basic open sets in  $\prod_{\mathbb{N}} \mathbb{N}$  look like?

**Exercise 1.23.** Prove that  $\prod_{\mathbb{N}} \mathbb{N}$  with the product topology is homeomorphic to Baire space  $\mathbb{N}^{\mathbb{N}}$  with our Polish topology.

In Lemma 3.1 we will extend this exercise to show that countable products of Polish spaces are always Polish. The way that the metric is defined in this extension does not quite match the way that the metric was defined on  $\mathbb{N}^{\mathbb{N}}$ . The following exercise gives a different metric on Baire space which you will show generates the same topology as the metric we have already defined on this space. It is the metric from Exercise 1.24 that we will use to show that countable products of Polish spaces are Polish.

**Exercise 1.24.** Let  $d$  denote our previous metric on Baire space: for  $f \neq g \in \mathbb{N}^{\mathbb{N}}$ ,  $d(f, g) = 2^{-(n+1)}$  where  $n$  is the least number such that  $f(n) \neq g(n)$ . Let  $d_{\mathbb{N}}$  denote the metric on  $\mathbb{N}$  with the discrete topology from Exercise 1.21. Define a new metric  $\hat{d}$  on  $\mathbb{N}^{\mathbb{N}}$  as follows: for  $f \neq g \in \mathbb{N}^{\mathbb{N}}$

$$\hat{d}(f, g) = \sum_{i=0}^{\infty} \frac{1}{2^{n+1}} d_{\mathbb{N}}(f(i), g(i)).$$

Prove that  $\hat{d}$  is a metric and that  $d$  and  $\hat{d}$  generate the same topology on Baire space. (Therefore  $\hat{d}$  is a complete metric and it generates the product topology when Baire space is viewed as a product space.)

## 2 Basic properties

If  $X$  is a Polish space and  $A \subseteq X$ , we let

$$\text{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}.$$

Notice that in general,  $\text{diam}(A)$  does not have to be finite. Two of the fundamental facts about Polish spaces are contained in the following lemmas.

**Lemma 2.1.** *Let  $X$  be a Polish space and  $X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$  be a nested sequence of nonempty closed subsets such that  $\lim_{n \rightarrow \infty} \text{diam}(X_n) = 0$ . Then, there is an  $x \in X$  such that  $\bigcap_{n \in \mathbb{N}} X_n = \{x\}$ .*

*Proof.* For each  $n \in \mathbb{N}$ , fix an element  $x_n \in X$ . Because  $\text{diam}(X_n) \rightarrow 0$ , the sequence  $x_n$ ,  $n \in \mathbb{N}$ , is a Cauchy sequence (or more precisely, contains a Cauchy subsequence). Let  $x \in X$  be the limit of this sequence. For each  $n$ , the “tail”  $x_n, x_{n+1}, x_{n+2}, \dots$  is in  $X_n$ , so because  $X_n$  is closed,  $x \in X_n$ . Therefore,  $x \in \bigcap X_n$ . Since  $\text{diam}(X_n) \rightarrow 0$ , if  $y \in \bigcap X_n$ ,  $y = x$ . Therefore,  $\bigcap X_n = \{x\}$ .  $\square$

**Lemma 2.2.** *Let  $X$  be a Polish space,  $U \subseteq X$  be open, and  $\epsilon > 0$ . Then, there are nonempty open sets  $U_0, U_1, \dots$  such that  $U = \bigcup U_k = \bigcup \overline{U_k}$  and  $\text{diam}(U_k) < \epsilon$  for all  $k$ .*

*Proof.* Fix a countable dense subset  $D \subseteq X$ . Let  $U_0, U_1, \dots$  list all subsets of  $X$  of the form  $B_{1/n}(d)$  where  $d \in D$ ,  $1/n < \epsilon/2$  and  $\overline{B_{1/n}(d)} \subseteq U$ . We check that  $U_k$  has the required properties. It is clear that  $\text{diam}(B_{1/n}(d)) \leq 2/n < \epsilon$  and that  $\bigcup U_k \subseteq \bigcup \overline{U_k} \subseteq U$ .

It remains to check that  $U \subseteq \bigcup U_k$ . Fix  $x \in U$ . Because  $U$  is open, there is a  $\delta$  such that  $B_\delta(x) \subseteq U$  and  $\delta < \epsilon/2$ . Because  $D$  is dense, there is an  $a \in D$  and an  $n \in \mathbb{N}$  such that  $d(x, a) < 1/n < \delta/3$ . It follows that  $x \in B_{1/n}(a)$ ,  $1/n < \epsilon/2$  and (by the triangle inequality)  $\overline{B_{1/n}(a)} \subseteq B_\delta(x) \subseteq U$ . Therefore,  $B_{1/n}(a)$  is one of our  $U_k$  sets and  $x \in \bigcup U_k$  as required.  $\square$

One of the first applications of these fundamental lemmas is the following result.

**Theorem 2.3.** *Let  $X$  be a Polish space. There is a continuous surjection  $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow X$ .*

*Proof.* Applying Lemma 2.2, we can obtain a “tree” of open subsets  $V_\sigma \subseteq X$  indexed by  $\sigma \in \mathbb{N}^{<\mathbb{N}}$  that has the following properties:

1.  $V_\emptyset = X$ ,
2.  $V_\sigma$  is open,
3.  $\text{diam}(V_\sigma) < 1/|\sigma|$ ,
4.  $\overline{V_\mu} \subseteq V_\sigma$  for all  $\sigma \subset \mu$ , and
5.  $V_\sigma = \bigcup_{n \in \omega} V_{\sigma * n}$ .

How do we get such subsets? Start with  $V_\emptyset = X$ . Assume we have defined  $V_\sigma$  and we show how to define  $V_{\sigma * k}$  for  $k \in \mathbb{N}$ . Apply Lemma 2.2 with  $U = V_\sigma$  (which by the induction hypothesis is open) and  $\epsilon = 1/(|\sigma| + 1)$  to get a sequence of open subsets  $U_0, U_1, \dots$  such that  $\text{diam}(U_k) < \epsilon$  and  $V_\sigma = U = \bigcup U_k = \bigcup \overline{U_k}$ . Set  $V_{\sigma * k} = U_k$ . Notice that each  $V_{\sigma * k}$  is open,  $\overline{V_{\sigma * k}} \subseteq V_\sigma$ ,  $V_\sigma = \bigcup V_{\sigma * k}$ , and

$$\text{diam}(V_{\sigma * k}) < \epsilon = \frac{1}{|\sigma| + 1} = \frac{1}{|\sigma * k|}.$$

Therefore, we have the desired properties of our tree of open subsets of  $U$ .

We want to use this tree of subsets to define the map  $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow X$ . Fix any  $f \in \mathbb{N}^{\mathbb{N}}$ . Let  $W_n = \overline{V_{f|n}}$ . By the properties of our sets  $V_\sigma$ , we have that the  $W_n$  sets form a nested sequence of closed sets

$$X = W_0 \supseteq W_1 \supseteq W_2 \supseteq \dots$$

with  $\text{diam}(W_n) \rightarrow 0$ . By Lemma 2.1, there is a unique element  $x \in \bigcap W_n$ . We define  $\phi(f) = x$ .

It remains to show that the map  $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow X$  is onto and is continuous. To see that  $\phi$  is onto, fix any element  $x \in X$ . We build a sequence  $\sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \dots$  such that  $|\sigma_n| = n$  and  $x \in V_{\sigma_n}$  for all  $n$ . Set  $\sigma_0 = \emptyset$  and notice that  $x \in X = V_\emptyset = V_{\sigma_0}$ . By induction, assume that we have defined  $\sigma_n$  such that  $|\sigma_n| = n$  and  $x \in V_{\sigma_n}$ . Since  $V_{\sigma_n} = \bigcup_{k \in \omega} V_{\sigma_n * k}$ , there is a  $k$  such that  $x \in V_{\sigma_n * k}$ . Fix such a  $k$  and set  $\sigma_{n+1} = \sigma_n * k$ .

Because  $\sigma_0 \subset \sigma_1 \subset \sigma_2 \subset \dots$ , there is a unique function  $f$  such that  $f = \bigcup \sigma_n$ . (That is,  $f$  is defined so that  $f \upharpoonright n = \sigma_n$ .) Because  $x \in V_{\sigma_n} \subseteq \overline{V_{\sigma_n}} = \overline{V_{f \upharpoonright n}}$  for all  $n$ , we have that  $\phi(f) = x$  as required.

Finally, we check that  $\phi$  is continuous. Fix an open ball  $B_\epsilon(x) \subseteq X$ . Let

$$S = \{\sigma \in \mathbb{N}^{<\mathbb{N}} \mid \overline{V_\sigma} \subseteq B_\epsilon(x)\}$$

and let  $O = \bigcup_{\sigma \in S} N_\sigma$ .  $O$  is open in  $\mathbb{N}^{\mathbb{N}}$  since it is a union of open sets. We claim that  $\phi^{-1}(B_\epsilon(x)) = O$  (which suffices to prove that  $\phi$  is continuous).

First we show that  $O \subseteq \phi^{-1}(B_\epsilon(x))$ . Consider  $f \in O$  and fix  $\sigma \in S$  such that  $f \in N_\sigma$ . Since  $f \in N_\sigma$  implies that  $\phi(f) \in \overline{V_\sigma}$ , we have that  $\phi(f) \in B_\epsilon(x)$ . Therefore,  $O \subseteq \phi^{-1}(B_\epsilon(x))$ .

Second we show that  $O \supseteq \phi^{-1}(B_\epsilon(x))$ . Consider  $y \in B_\epsilon(x)$  and fix any  $g \in \mathbb{N}^{\mathbb{N}}$  such that  $\phi(g) = y$ . We need to show that  $g \in O$ . Since  $y \in B_\epsilon(x)$ , we can fix a  $\delta$  such that  $B_\delta(y) \subseteq B_\epsilon(x)$ . Since  $\phi(g) = y$ , we know that setting  $W_n = \overline{V_{g \upharpoonright n}}$  gives a nested sequence of closed sets such that  $\text{diam}(W_n) \rightarrow 0$  and  $\bigcap W_n = \{y\}$ . Therefore, there must be an  $n$  such that  $W_n \subseteq B_\delta(y) \subseteq B_\epsilon(x)$ . In particular, setting  $\sigma = g \upharpoonright n$ , we have

$$W_n = \overline{V_{g \upharpoonright n}} = \overline{V_\sigma} \subseteq B_\epsilon(x)$$

and hence  $\sigma \in S$ . Since  $\sigma = g \upharpoonright n$ , we have that  $g \in N_\sigma \subseteq O$  as required.  $\square$

We will see this technique of building a map by using a “tree” of subsets several more times in these notes. Before moving to the next section, we show one more property of the map  $\phi$  from the proof of Theorem 2.3 that is often true of maps defined in this way. The map  $\phi$  is actually an open map – that is, it maps open sets in  $\mathbb{N}^{\mathbb{N}}$  to open sets in  $X$ . (In general  $\phi$  need not be one-to-one, so it is not a homeomorphism.) To show that  $\phi$  is open, it suffices to show that  $\phi$  maps every basic open set in  $\mathbb{N}^{\mathbb{N}}$  to an open set in  $X$ . Fix  $\sigma \in \mathbb{N}^{<\mathbb{N}}$  and we show that  $\phi(N_\sigma) = V_\sigma$ , which by definition is open in  $X$ .

First we show that  $\phi(N_\sigma) \subseteq V_\sigma$ . Fix  $x \in \phi(N_\sigma)$  and let  $f \in N_\sigma$  be such that  $\phi(f) = x$ . Because  $f \in N_\sigma$ , we know that  $f \upharpoonright |\sigma| = \sigma$ . By the definition of  $\phi$ ,  $\phi(f) = x$  implies that

$$x = \phi(f) \in \overline{V_{\sigma * f \upharpoonright (|\sigma|)}} \subseteq V_\sigma.$$

Therefore,  $x \in V_\sigma$  as required.

Second we show that  $\phi(N_\sigma) \supseteq V_\sigma$ . (This proof is just a “translation” to the cone  $V_\sigma$  of the proof that  $\phi$  is onto.) Fix  $x \in V_\sigma$ . We define a sequence  $\sigma_0 \subseteq \sigma_1 \subseteq \dots$  such that  $\sigma_0 = \sigma$ ,  $x \in V_{\sigma_n}$  for all  $n$ , and  $|\sigma_n| = |\sigma| + n$ . Assume that  $\sigma_n$  has been defined so that  $x \in V_{\sigma_n}$ . Fix  $k$  such that  $x \in V_{\sigma_n * k}$  and set  $\sigma_{n+1} = \sigma_n * k$ . It is clear that this sequence has the stated properties. Set  $f = \bigcup \sigma_n$ . Since  $\sigma_0 = \sigma$ , we have that  $f \upharpoonright |\sigma| = \sigma$  and hence  $f \in N_\sigma$ . Because  $\phi(f) = \bigcap \overline{V_{f \upharpoonright n}}$  (by the definition of  $\phi$ ) and  $x \in \overline{V_{f \upharpoonright n}}$  for all  $n$  (by the definition of the sequence  $\sigma_n$ ), we have  $\phi(f) = x$  as required.

### 3 Building new Polish spaces

We will develop various tools for obtaining new Polish spaces from old Polish spaces. One simple way to do this is to take countable products. In the next lemma, we show how to



define a complete metric on a countable product of Polish spaces. The metric we use is a generalization of the metric from Exercise 1.24. After the lemma, you will show in an exercise that the topology generated by this complete metric is actually the product topology on the product space.

**Lemma 3.1.** *Let  $X_0, X_1, \dots$  be a countable sequence of Polish spaces. The product space  $\prod X_n$  is also a Polish space.*

*Proof.* Fix complete metrics  $d_n$  on each  $X_n$  and assume that  $d_n(x, y) < 1$  for all  $x, y \in X_n$ . Define  $\hat{d}$  on  $\prod X_n$  by

$$\hat{d}(f, g) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} d_n(f(n), g(n)).$$

We need to show that  $\hat{d}$  is a complete metric on  $\prod X_n$  and that the topology induced by this metric is separable. First, we show that  $\hat{d}$  is complete. (**Check:** this is a metric.)

Fix a Cauchy sequence  $f_0, f_1, \dots$  in  $\prod X_n$ . For any fixed  $n \in \mathbb{N}$ , look at the sequence  $f_0(n), f_1(n), f_2(n), \dots$  in  $X_n$  and we calculate a bound on  $d_n(f_i(n), f_{i+1}(n))$ . Since  $\hat{d}(f_i, f_{i+1}) \leq 2^{-i}$  by our Cauchy assumption, we have that

$$\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} d_n(f_i(n), f_{i+1}(n)) \leq 2^{-i}.$$

Since the terms in this sum are all positive, this inequality implies that

$$\frac{1}{2^{n+1}} d_n(f_i(n), f_{i+1}(n)) \leq 2^{-i}$$

or in other words,  $d_n(f_i(n), f_{i+1}(n)) \leq 2^{n+1-i}$ . Since  $n$  is fixed, it follows that the sequence  $f_0(n), f_1(n), \dots$  is a Cauchy sequence in  $X_n$ . Therefore, because  $d_n$  is a complete metric on  $X_n$ ,  $\lim_{i \rightarrow \infty} f_i(n)$  exists for each  $n$ . Define

$$g(n) = \lim_{i \rightarrow \infty} f_i(n).$$

**Check:**  $g$  is the limit of the Cauchy sequence  $f_0, f_1, \dots$  in  $\prod X_n$ . Therefore,  $\hat{d}$  is a complete metric on  $\prod X_n$ .

It remains to show that the topology induced by  $\hat{d}$  on  $\prod X_n$  is separable. To see this fact, let  $a_0^n, a_1^n, a_2^n, \dots$  be a dense subset of  $X_n$ . For each string  $\sigma \in \mathbb{N}^{<\mathbb{N}}$ , define  $f_\sigma \in \prod X_n$  by  $f_\sigma(n) = a_{\sigma(n)}^n$  for  $n < |\sigma|$  and  $f_\sigma(n) = a_0^n$  for  $n \geq |\sigma|$ . (For the intuition behind this definition, look back at Exercise 1.9.)

We claim that  $D = \{f_\sigma \mid \sigma \in \mathbb{N}^{<\mathbb{N}}\}$  is a dense subset of  $\prod X_n$ . It is enough to show that every basic open ball  $B_\epsilon(g)$  in  $\prod X_n$  contains an element of  $D$ . Fix  $k$  such that

$$\sum_{n=k}^{\infty} \frac{1}{2^{n+1}} < \epsilon$$

and let  $\sigma = g \upharpoonright k$ . Because  $f_\sigma(n) = g(n)$  for all  $n < k$ , we have

$$\sum_{n=0}^{k-1} d_n(f_\sigma(n), g(n)) = 0.$$

Because of our assumption that  $d_n(x, y) < 1$  for all  $x, y \in X_n$ , we have that

$$\sum_{n=k}^{\infty} \frac{1}{2^{n+1}} d_n(f_\sigma(n), g(n)) = \sum_{n=k}^{\infty} \frac{1}{2^{n+1}} d_n(a_0^n, g(n)) \leq \sum_{n=k}^{\infty} \frac{1}{2^{n+1}} < \epsilon.$$

Therefore,

$$\hat{d}(f_\sigma, g) = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} d_n(f_\sigma(n), g(n)) < \epsilon$$

and  $f_\sigma \in D$  as required.  $\square$

**Exercise 3.2.** Prove that the complete metric  $\hat{d}$  we defined on  $\prod X_n$  generates the usual product topology on this space.

A second way to get new Polish spaces is by taking certain subspaces of known Polish spaces. The simplest examples are closed subsets of Polish spaces because the restriction of a complete metric on  $X$  to a closed subset is still a complete metric.

**Exercise 3.3.** Verify that every closed subspace of a Polish space is a Polish space (with the subspace topology).

Dealing with open subsets of a Polish space is not so straightforward. For example, the interval  $(0, 1)$  is an open subset of  $\mathbb{R}$ , but the restriction of the usual metric on  $\mathbb{R}$  to  $(0, 1)$  is not a complete metric. (It is a metric, it is just not complete.) However, perhaps surprisingly, open subsets of Polish spaces are in fact still Polish! Of course, they require a modified metric, but they do admit a complete metric which is compatible with the subspace topology.

**Lemma 3.4.** *If  $X$  is a Polish space and  $U \subseteq X$  is open, the  $U$  (with the subspace topology) is a Polish space.*

*Proof.* Fix a complete metric  $d$  on  $X$  such that  $d$  is compatible with the topology on  $X$  and  $d < 1$ . Define the following metric on  $U$ :

$$\hat{d}(x, y) = d(x, y) + \left| \frac{1}{d(x, X \setminus U)} - \frac{1}{d(y, X \setminus U)} \right|.$$

Notice that because  $U$  is open,  $X \setminus U$  is closed, and hence the expression  $d(x, X \setminus U)$  makes sense and is positive by Exercise 1.6. **Check:**  $\hat{d}$  is a metric on  $U$ .

We need to show that  $\hat{d}$  is compatible with the subspace topology on  $U$  and that  $\hat{d}$  is a complete metric. First, we show that  $\hat{d}$  is compatible with the subspace topology. This means showing that every  $d$ -open ball (in the subspace topology) contains a  $\hat{d}$ -open ball and vice

versa. To make the notation clear, we will use  $B_{\epsilon,d}(x)$  for the  $d$ -open ball of radius  $\epsilon$  around  $x$  and we will use  $B_{\epsilon,\hat{d}}(x)$  for the  $\hat{d}$ -open ball of radius  $\epsilon$  around  $x$ .

Consider a  $d$ -open ball  $B_{\epsilon,d}(x)$  for  $x \in U$ . Since  $\hat{d}(x,y) \geq d(x,y)$  for all  $y$ , we have that  $B_{\epsilon,\hat{d}}(x) \subseteq B_{\epsilon,d}(x)$ . Therefore, one half of the compatibility check is trivial.

Next, consider a  $\hat{d}$ -open ball  $B_{\epsilon,\hat{d}}(x)$  for  $x \in U$ . We need to find a  $\delta > 0$  such that  $B_{\delta,d}(x) \subseteq B_{\epsilon,\hat{d}}(x)$ . By Exercise 1.6  $d(x, X \setminus U) > 0$ , so we can fix a positive real  $r$  such that  $d(x, X \setminus U) = r > 0$ . For any  $\delta > 0$ , it follows from the triangle inequality that if  $d(x,y) < \delta$ , then  $d(y, X \setminus U) > r - \delta$ . Therefore, for any  $0 < \delta < r$ , we have that if  $d(x,y) < \delta$ , then

$$\left| \frac{1}{d(x, X \setminus U)} - \frac{1}{d(y, X \setminus U)} \right| \leq \left| \frac{1}{r} - \frac{1}{r - \delta} \right| = \left| \frac{-\delta}{r(r - \delta)} \right|.$$

It remains to choose an appropriate  $\delta$  to control the fraction  $\frac{\delta}{r(r - \delta)}$ . Choose  $\delta$  such that  $0 < \delta < r$  and  $\delta + \frac{\delta}{r(r - \delta)} < \epsilon$ . (This choice is possible since  $\lim_{x \rightarrow 0^+} f(x) = 0$  where  $f(x) = x + \frac{x}{r(r - x)}$  for  $0 < x < r$ .) Finally, suppose that  $y \in B_{\delta,d}(x)$ . Then  $d(x,y) < \delta$  and by the calculations above and the choice of  $\delta$ , we have

$$\hat{d}(x,y) \leq \delta + \left| \frac{1}{r} - \frac{1}{r - \delta} \right| = \delta + \left| \frac{-\delta}{r(r - \delta)} \right| < \epsilon.$$

Therefore  $B_{\delta,d}(x) \subseteq B_{\epsilon,\hat{d}}(x)$  as required. This completes the proof that the metric  $\hat{d}$  on  $U$  is compatible with the subspace topology on  $U$ .

It remains to show that the metric  $\hat{d}$  is complete on  $U$ . Fix a  $\hat{d}$ -Cauchy sequence  $x_0, x_1, \dots$  in  $U$ . Because  $d(x_i, x_{i+1}) \leq \hat{d}(x_i, x_{i+1})$ , this sequence is also a  $d$ -Cauchy sequence and therefore has a limit  $x$  in  $X$ . We need to show that this limit  $x$  is actually in  $U$ .

Because  $x_0, x_1, \dots$  is a  $\hat{d}$ -Cauchy sequence, we have that

$$\left| \frac{1}{d(x_i, X \setminus U)} - \frac{1}{d(x_{i+1}, X \setminus U)} \right| \leq \frac{1}{2^i}.$$

Therefore, the sequence

$$\frac{1}{d(x_0, X \setminus U)}, \frac{1}{d(x_1, X \setminus U)}, \frac{1}{d(x_2, X \setminus U)}, \dots$$

is a Cauchy sequence in  $\mathbb{R}$  and must approach a limit  $r \in \mathbb{R}$ . Because  $d < 1$ , each term in this sequence is  $> 1$  and hence  $r \geq 1$ . The important point is that  $r > 0$  and therefore,  $\lim_{i \rightarrow \infty} d(x_i, X \setminus U) = 1/r > 0$  and each term  $\frac{1}{d(x_i, X \setminus U)}$  is bounded away from 0 by some fixed positive  $\epsilon$ . Thus,  $d(x, X \setminus U) > 0$  and  $x \in U$  as desired.  $\square$

This lemma can be improved to show that every  $G_\delta$  subset of a Polish space is a Polish space with the subspace topology. However, the  $G_\delta$  sets are the best that one can do in terms of the Borel hierarchy with such a result. That is, if  $X$  is a Polish space and  $Y \subseteq X$  is a Polish space with the subspace topology, then  $Y$  is a  $G_\delta$  subset of  $X$ . (Later we will see that any Borel subset of a Polish space can be made into a Polish space, but this requires changing the topology so that we are no longer using the subspace topology.)

We end this section with one more example of building new Polish spaces from old ones.

**Lemma 3.5.** *Let  $X$  and  $Y$  be disjoint Polish spaces. The disjoint union  $X \sqcup Y$  is the space  $X \cup Y$  where  $U \subseteq X \cup Y$  is open if and only if  $U \cap X$  is open (in  $X$ ) and  $U \cap Y$  is open (in  $Y$ ). The disjoint union  $X \sqcup Y$  is a Polish space.*

*Proof.* Let  $d_X$  be the complete compatible metric on  $X$  and let  $d_Y$  be the complete compatible metric on  $Y$ . Assume that  $d_X < 1$  and  $d_Y < 1$ . Define the metric  $d$  on  $X \sqcup Y$  by

$$d(a, b) = \begin{cases} d_X(a, b) & \text{if } a, b \in X \\ d_Y(a, b) & \text{if } a, b \in Y \\ 2 & \text{otherwise.} \end{cases}$$

Notice that  $X$  and  $Y$  are both clopen in  $X \sqcup Y$  and that any open set  $U \subseteq X \sqcup Y$  is the union of an open set in  $X$  and an open set in  $Y$ . **Check:**  $X \sqcup Y$  is separable and  $d$  is a complete metric on  $X \sqcup Y$ . (For completeness, notice that any Cauchy sequence in  $X \sqcup Y$  must eventually be inside  $X$  or inside  $Y$ .)  $\square$

## 4 Borel sets

**Definition 4.1.** Let  $X$  be any set. A  $\sigma$ -algebra on  $X$  is a collection of subsets of  $X$  which is closed under taking complements and countable unions.

**Exercise 4.2.** Show that any  $\sigma$ -algebra on  $X$  is also closed under taking countable intersections.

**Definition 4.3.** Let  $X$  be a Polish space. The class of **Borel sets** on  $X$ , denoted  $\mathcal{B}(X)$ , is the smallest  $\sigma$ -algebra on  $X$  containing the open sets. We introduce the following notations (called the Borel hierarchy) to classify the complexity of individual Borel sets on  $X$  by induction on ordinals  $\alpha < \omega_1$ :

- $\Sigma_1^0(X)$  is the collection of all open subsets of  $X$ .
- $\Pi_\alpha^0(X)$  is the collection of all subsets of  $X$  of the form  $X \setminus A$  where  $A \in \Sigma_\alpha^0(X)$ .
- For  $\alpha > 0$ ,  $\Sigma_\alpha^0(X)$  is the collection of all subsets of  $X$  of the form  $\bigcup_{i \in \mathbb{N}} A_i$  where each  $A_i \in \Pi_{\beta_i}^0(X)$  for some  $\beta_i < \alpha$ .

We say that a subset  $A \subseteq X$  is  $\Delta_\alpha^0(X)$  if and only if  $A \in \Sigma_\alpha^0(X) \cap \Pi_\alpha^0(X)$ .

**Exercise 4.4.** Prove that  $\mathcal{B}(X)$  is the smallest collection of subsets of  $X$  containing the open sets and closed under taking complements and countable intersections. (That is, we can replace “closing under countable unions” with “closing under countable intersections” and still obtain the Borel sets.)

The classes lower down in the Borel hierarchy have more standard names:  $\Pi_1^0(X)$  is exactly the closed subsets of  $X$ ,  $\Sigma_2^0(X)$  is exactly the  $F_\sigma$  subsets of  $X$  (the countable unions of closed sets), and  $\Pi_2^0(X)$  is exactly the  $G_\delta$  subsets of  $X$  (the countable intersections of open sets).

Notice that here we are using a boldface type on  $\Sigma$ ,  $\Pi$  and  $\Delta$  as opposed to the lightface type  $\Sigma$ ,  $\Pi$  and  $\Delta$  that you have seen in recursion theory or in the study of random sets. When reading these symbols, one typically reads  $A \in \Sigma_\alpha^0$  by saying “ $A$  is boldface Sigma-0-alpha.”

What is the difference between the boldface and lightface notions? The difference is whether one places computability restrictions on the definition of the classes. In the study of randomness, we say that  $U \subseteq 2^\mathbb{N}$  is  $\Sigma_1^0$  (“ $U$  is lightface Sigma-0-1”) if and only if there is a computably enumerable set  $W \subseteq 2^{<\mathbb{N}}$  such that  $U = \bigcup_{\sigma \in W} N_\sigma$ . Notice that  $U$  is an open subset of  $2^\mathbb{N}$ , but furthermore, it is effectively open in the sense that it is the union of a computably enumerable sequence of basic open neighborhoods.

On the other hand,  $V \in \Sigma_1^0(2^\mathbb{N})$  if and only if  $V$  is an open subset of  $2^\mathbb{N}$ , which you know is equivalent to the existence of a set  $Y \subseteq 2^{<\mathbb{N}}$  such that  $V = \bigcup_{\sigma \in Y} N_\sigma$ . That is, there is no restriction here on the complexity of the set  $Y$  of basic open neighborhoods making up  $V$ . Thus, the difference between the lightface and boldface notions is whether one places computability restrictions on the definitions or not.

We will not be going into a detailed analysis of the Borel hierarchy, but there are a few easy facts to check that are frequently useful to know.

**Exercise 4.5.** Prove (by induction on  $\alpha$ ) that  $\Sigma_\alpha^0(X) \cup \Pi_\alpha^0(X) \subseteq \Delta_{\alpha+1}^0(X)$  for all  $\alpha < \omega_1$ . (Really, start with  $\alpha = 1$  since we have not defined  $\Sigma_0^0(X)$ . The same comment applies below – I will just ignore that  $\alpha = 0$  case. Sometimes  $\Sigma_0^0(X)$  and  $\Pi_1^0(X)$  are defined to be equal to the basic open sets, which has the advantage of letting one work with  $\alpha = 0$ , but it breaks the symmetry of taking complements between the  $\Sigma$  and  $\Pi$  sides.)

**Exercise 4.6.** Prove that  $\mathcal{B}(X) = \bigcup_{\alpha < \omega_1} \Sigma_\alpha^0(X)$ . For the  $\supseteq$  containment, proceed by induction on  $\alpha$ . For the  $\subseteq$  containment, show that  $\bigcup_{\alpha < \omega_1} \Sigma_\alpha^0(X)$  contains the open sets and is closed under complementation and countable unions.

## 5 Changing the topology

In this section, we will show that if  $X$  is a Polish space and  $Y \subseteq X$  is Borel, then there is a topology we can put on  $Y$  that will make  $Y$  into a Polish space. In general, we cannot just use the subspace topology on  $Y$  (the way we did when  $Y$  was closed or open) because not all Borel subsets  $Y \subseteq X$  are Polish spaces with the subspace topology. (As commented earlier,  $Y \subseteq X$  is a Polish space with the subspace topology if and only if  $Y$  is  $G_\delta$ .) However, what we can do is to change to topology on  $X$  so that  $X$  is still a Polish space (with the new topology) but  $Y$  becomes closed (even clopen) in the new topology. Thus,  $Y$  with the new subspace topology is a Polish space by Exercise 3.3!

We begin with the case when  $Y \subseteq X$  is closed and show how to put a new Polish topology on  $X$  so that  $Y$  becomes clopen, but the class of Borel sets remains the same. Because we will be changing the topology, we explicitly denote the (original) topology on  $X$  by  $\tau$ .

**Lemma 5.1.** *Let  $(X, \tau)$  be a Polish space and let  $Y \subseteq X$  be closed. There is a Polish topology  $\tau_1$  on  $X$  such that  $Y$  is clopen in  $\tau_1$ . Furthermore,  $\tau \subseteq \tau_1$  and the Borel sets in  $(X, \tau)$  and  $(X, \tau_1)$  are the same.*

*Proof.* By Exercise 3.3, we know that  $Y$  is a Polish space with the subspace topology from  $\tau$ . Furthermore, since  $X \setminus Y$  is open, we know from Lemma 3.4 that  $X \setminus Y$  is a Polish space with the subspace topology from  $\tau$ . Let  $\tau_1$  be the topology on the disjoint union  $Y \cup (X \setminus Y)$  from Lemma 3.5, so  $(X, \tau_1)$  is a Polish space. As pointed out in the proof of Lemma 3.5,  $Y$  is clopen in the Polish topology  $\tau_1$  and the open sets in  $\tau_1$  are intersections of open sets in  $Y$  and  $X \setminus Y$  (with the subspace topologies from  $\tau$ ). Therefore, every open set in  $(X, \tau)$  is open in  $(X, \tau_1)$ . Furthermore, any open set in  $(X, \tau_1)$  is Borel in  $\tau$ , and hence the Borel sets in  $\tau$  and  $\tau_1$  are the same.  $\square$

**Theorem 5.2.** *Let  $(X, \tau)$  be a Polish space and let  $Y \subseteq X$  be Borel. There is a Polish topology  $\tau^*$  on  $X$  such that  $Y$  is clopen in  $(X, \tau^*)$ . Furthermore,  $\tau \subseteq \tau^*$  and  $(X, \tau)$  and  $(X, \tau^*)$  have the same Borel sets.*

*Proof.* Let  $\Omega$  be the collection of all  $B \in \mathcal{B}(X)$  such that there is a Polish topology on  $X$  which makes  $B$  clopen, which refines  $\tau$ , and which has the same Borel sets as  $\tau$ . To show  $\Omega = \mathcal{B}(X)$ , it suffices to show that  $\Omega$  contains all of the  $\tau$ -open sets and that  $\Omega$  is closed under complementation and countable intersections. (Recall Exercise 4.4.)

To see that  $\Omega$  is closed under complementation, suppose that  $B \in \Omega$  and let  $\tau^*$  be the Polish topology on  $X$  witnessing the fact that  $B \in \Omega$ . Because  $B$  is clopen in the topology  $\tau^*$ , we have that  $\overline{B}$  is also clopen in  $\tau^*$ . Therefore,  $\tau^*$  also witnesses that  $\overline{B} \in \Omega$ .

By Lemma 5.1,  $\Omega$  contains all of the  $\tau$ -closed sets. Since  $\Omega$  is closed under complementation, it follows that  $\Omega$  contains all  $\tau$ -open sets.

Finally, we need to show that  $\Omega$  is closed under countable intersections. Fix  $A_0, A_1, \dots \in \Omega$  and let  $B = \bigcap A_i$ . Fix Polish topologies  $\tau_i$  witnessing that  $A_i \in \Omega$ . Consider the Polish space  $\prod (X, \tau_i)$  and the diagonal map  $\phi : X \rightarrow \prod (X, \tau_i)$  defined by  $\phi(x) = f_x$  where  $f_x : \mathbb{N} \rightarrow X$  is  $f_x(i) = x$ . That is,  $\phi(x) = (x, x, x, \dots) \in \prod (X, \tau_i)$ . Unfortunately, as a map from  $(X, \tau)$  to the product space,  $\phi$  need not be continuous. Therefore, it is useful to think of  $X$  as just a set rather than as a topological space. We will define  $\tau^*$  on  $X$  by pulling  $\tau^*$  back from the product space.

We have projection functions  $\pi_i : \prod (X, \tau_i) \rightarrow X$  defined by  $\pi_i(f) = f(i)$  for each  $f \in \prod (X, \tau_i)$ . Recall that we get a subbasis for the product space  $\prod (X, \tau_i)$  by taking subsets of  $\prod (X, \tau_i)$  of the form  $\pi_i^{-1}(U)$  where  $i \in \mathbb{N}$  and  $U$  is a basic open set in  $(X, \tau_i)$ . That is, we restrict one component of the product space to a basic open set  $U \in (X, \tau_i)$  and let the other components equal  $X$ .

First, we show that  $\overline{\phi(X)}$  is closed in the product space  $\prod (X, \tau_i)$ . One way to do this by showing that  $\overline{\phi(X)}$  is open in  $\prod (X, \tau_i)$ . By our definition of  $\phi$ ,  $f \in \overline{\phi(X)}$  if and only if there are  $i \neq j \in \mathbb{N}$  such that  $f(i) \neq f(j)$ .

Consider any tuple  $\sigma = \langle x, y, i, j, \epsilon, \delta \rangle$  such that  $i \neq j \in \mathbb{N}$ ,  $x \neq y \in X$  and  $B_\epsilon(x) \cap B_\delta(y) = \emptyset$ , where  $B_\epsilon(x)$  and  $B_\delta(y)$  are calculated relative to the original complete metric corresponding to  $\tau$ . Since each  $\tau_i$  is a refinement of  $\tau$ , the balls  $B_\epsilon(x)$  and  $B_\delta(y)$  are open in all of the topologies  $(X, \tau_i)$ . For each such tuple  $\sigma$ , the subset  $V_\sigma$  of  $\prod (X, \tau_i)$  given by  $\pi_i^{-1}(B_\epsilon(x)) \cap \pi_j^{-1}(B_\delta(y))$  is open. Furthermore, for any  $f \in V_\sigma$ ,  $f(i) \neq f(j)$  since  $f(i) \in B_\epsilon(x)$ ,  $f(j) \in B_\delta(y)$  and  $B_\epsilon(x) \cap B_\delta(y) = \emptyset$ . The union  $\bigcup V_\sigma$  taken over all such tuples  $\sigma$  is equal to  $\overline{\phi(X)}$ . **Check** that this equality is true.

Second, we consider the topology  $\tau^*$  on  $X$  defined by  $U \in \tau^*$  if and only if there is an open set  $V \in \prod(X, \tau_i)$  such that  $U = \phi^{-1}(V)$ . We claim that  $\tau^*$  is a Polish topology on  $X$ . To see why this is true, notice the following facts. First,  $\phi$  is a bijection from  $X$  onto  $\phi(X)$ . Second,  $\phi(X)$  is closed and hence is a Polish space with the subspace topology from  $\prod(X, \tau_i)$ . Third, by our definition of  $\tau^*$ , both  $\phi$  and  $\phi^{-1}$  are continuous. Therefore,  $(X, \tau^*)$  is homeomorphic to  $\phi(X)$  with the subspace topology, and hence  $(X, \tau^*)$  is a Polish topology.

Third, recall that the sets of the form  $\pi_i^{-1}(U)$  for  $i \in \mathbb{N}$  and  $U$  open in  $(X, \tau_i)$  form a subbasis for  $\prod(X, \tau_i)$ . By intersection these sets with  $\phi(X)$ , we get that sets of the form  $\pi_i^{-1}(U) \cap \phi(X)$  form a subbasis for  $\phi(X)$  in the subspace topology, and hence sets of the form

$$\phi^{-1}(\pi_i^{-1}(U) \cap \phi(X))$$

form a subbasis for  $X$  in the topology  $\tau^*$ . However, because  $\phi$  is the diagonal map, we have that

$$\phi^{-1}(\pi_i^{-1}(U) \cap \phi(X)) = U.$$

Hence, the collection of all subsets  $U \subseteq X$  which are  $\tau_i$  open for some  $i \in \mathbb{N}$  forms a subbasis for  $\tau^*$ . Since each  $\tau_i$  is a refinement of  $\tau$ , every  $\tau$ -open set is  $\tau_i$ -open, and hence is  $\tau^*$ -open. Furthermore, because  $\tau_i$  and  $\tau$  have the same Borel sets, each such set  $U$  is  $\tau$ -Borel. Therefore,  $\tau^*$  has a subbasis of sets which are all  $\tau$ -Borel. It follows that  $\tau$  and  $\tau^*$  have the same Borel sets as required.

Fourth, remember that we started all of this process with a sequence of sets  $A_0, A_1, \dots \in \Omega$  such that each  $A_i$  is  $\tau_i$ -clopen. By the last paragraph, each  $A_i$  is also  $\tau^*$ -clopen, so  $B = \bigcap A_i$  is  $\tau^*$ -closed. Therefore, we have shown that there is a Polish topology  $\tau^*$  on  $X$  extending  $\tau$  which has the same Borel sets as  $\tau$  and which makes  $B$  closed. Applying Lemma 5.1 to  $(X, \tau^{ast})$  and  $B \subseteq X$  (which is  $\tau^*$ -closed) gives us a Polish topology on  $X$  extending  $\tau^*$  (and hence extending  $\tau$ ) which has the same Borel sets as  $\tau^{ast}$  (and hence the same Borel sets as  $\tau$ ) and which makes  $B$  clopen. Therefore,  $\Omega$  is closed under taking countable intersections which complete this proof.  $\square$

**Theorem 5.3.** *Let  $(X, \tau)$  be a Polish space and let  $Y \subseteq X$  be a nonempty Borel set. There is a continuous function  $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow X$  such that  $\phi(\mathbb{N}^{\mathbb{N}}) = Y$ .*

*Proof.* By Theorem 5.2, there is a topology  $\tau^*$  on  $X$  such that  $Y$  is closed (even clopen) in  $(X, \tau^*)$ . By Exercise 3.3,  $Y$  is a Polish space with the subspace topology from  $\tau^*$ . By Theorem 2.3, there is an onto function  $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow Y$  that is continuous with respect to the  $\tau^*$ -subspace topology on  $Y$ . When  $\phi$  is viewed as a function from  $\mathbb{N}^{\mathbb{N}} \rightarrow X$ , it has image  $Y$  and is continuous with respect to  $\tau^*$ . However, every  $\tau$ -open set in  $X$  is  $\tau^*$ -open, and therefore  $\phi$  is also continuous with respect to the topology  $\tau$ .  $\square$

## 6 Analytic sets

Not all sets are Borel so it is useful to develop some definitions for larger classes of sets which are still reasonably well behaved, if not quite as nice as the Borel sets themselves. One of the natural classes extending the Borel sets are the analytic sets. These sets are the continuous images of the Borel sets.

**Definition 6.1.** Let  $X$  be a Polish space.  $A \subseteq X$  is called **analytic** if there is a Polish space  $Y$ , a continuous function  $f : Y \rightarrow X$  and a Borel set  $B \in \mathcal{B}(Y)$  such that  $f(B) = A$ . We denote the collection of analytic subsets of  $X$  by  $\Sigma_1^1(X)$ .

Not surprisingly, we also have a notation for the complements of the analytic sets. Although these sets will not be important for us, they are important in their own right as well. A set  $A \subseteq X$  is called **co-analytic** if  $X \setminus A$  is analytic. We denote the collection of co-analytic subsets of  $X$  by  $\Pi_1^1(X)$  and we denote the collection of sets which are both analytic and co-analytic by  $\Delta_1^1(X)$ .

**Exercise 6.2.** Show that every Borel subset of  $X$  is  $\Sigma_1^1(X)$ .

**Exercise 6.3.** Show that every Borel subset of  $X$  is  $\Delta_1^1(X)$ .

It is a non-trivial fact that every  $\Delta_1^1(X)$  set is a Borel subset of  $X$ . From this non-trivial fact and Exercise 6.3, it follows that  $\Delta_1^1(X) = \mathcal{B}(X)$  – that is, the class of sets which are both analytic and co-analytic is exactly the same as the class of Borel sets! The main fact that we will need about analytic subsets of  $X$  is contained in the following lemma.

**Lemma 6.4.** *Let  $X$  be a Polish space. For every nonempty subset  $A \subseteq X$ , the following are equivalent.*

1.  $A \in \Sigma_1^1(X)$ .
2. There is a continuous map  $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow X$  such that  $\phi(\mathbb{N}^{\mathbb{N}}) = A$ .

*Proof.* To see  $1 \Rightarrow 2$ , suppose  $A \in \Sigma_1^1(X)$  is nonempty. Fix the Polish space  $Y$ , the continuous map  $f : Y \rightarrow X$  and the Borel subset  $B \in \mathcal{B}(Y)$  such that  $f(B) = A$ . By Theorem 5.3, there is a continuous map  $g : \mathbb{N}^{\mathbb{N}} \rightarrow Y$  which is onto  $B$ . Let  $\phi = f \circ g$ . It is clear that  $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow X$  is continuous and onto  $A$  as required.

To see that  $2 \Rightarrow 1$ , just let  $Y = B = \mathbb{N}^{\mathbb{N}}$  in the definition of an analytic set. □

## 7 Continuum hypothesis for analytic sets

In this section, we show that the continuum hypothesis holds for analytic subsets of Polish spaces. That is, if  $A \subseteq X$  is analytic, then either  $|A| \leq \aleph_0$  or  $|A| = 2^{\aleph_0}$ . By Exercise 6.2, this result implies that for all  $B \in \mathcal{B}(X)$ , either  $|B| \leq \aleph_0$  or  $|B| = 2^{\aleph_0}$ . We begin our analysis by looking at perfect sets and the Cantor-Bendixson Theorem for closed sets.

**Definition 7.1.** Let  $X$  be a Polish space and let  $P \subseteq X$ . A point  $x \in P$  is **isolated** if there is an open set  $U$  such that  $P \cap U = \{x\}$ .  $P$  is called **perfect** if it is closed and has no isolated points.

Our first lemma is the generalization of Exercise 1.19 from Baire space to general Polish spaces.

**Lemma 7.2.** *Let  $X$  be a Polish space and  $P \subseteq X$  be a nonempty perfect set. There is a continuous one-to-one map  $\phi : 2^{\mathbb{N}} \rightarrow P$ , and hence  $|P| = 2^{\aleph_0}$ .*



*Proof.* We define a family of nonempty open subsets  $U_\sigma \subseteq X$  indexed by  $\sigma \in 2^\omega$  such that

1.  $U_\emptyset = X$ ,
2.  $\overline{U_\tau} \subseteq U_\sigma$  for  $\sigma \subset \tau$ ,
3.  $\overline{U_{\sigma*0}} \cap \overline{U_{\sigma*1}} = \emptyset$  for all  $\sigma$ ,
4.  $\text{diam}(U_\sigma) < 1/|\sigma|$ , and
5.  $U_\sigma \cap P \neq \emptyset$  for all  $\sigma$

by induction on  $\sigma$ . The base case is set already with  $U_\emptyset = X$ . For the induction case, assume we have defined  $U_\sigma$  with  $U_\sigma \cap P \neq \emptyset$ . Because  $P$  is perfect, we know that  $U_\sigma \cap P$  is not a singleton, so we can pick two points  $x_0 \neq x_1 \in P \cap U_\sigma$ . Because  $U_\sigma$  is open and  $x_0 \neq x_1 \in U_\sigma$ , there are open balls  $U_{\sigma*0}$  and  $U_{\sigma*1}$  such that  $x_i \in U_{\sigma*i}$ ,  $\overline{U_{\sigma*i}} \subseteq U_\sigma$ ,  $\overline{U_{\sigma*0}} \cap \overline{U_{\sigma*1}} = \emptyset$  and  $\text{diam}(U_{\sigma*i}) < 1/(|\sigma| + 1)$ . (Look back at Exercise 1.7 to see why this is true.) Therefore, we can continue our definition by induction.

To define the map  $\phi$ , consider any element  $f \in 2^\mathbb{N}$ . The sequence

$$\overline{U_{f|0}} \supseteq \overline{U_{f|1}} \supseteq \overline{U_{f|2}} \supseteq \cdots$$

is a nested sequence of nonempty closed sets such that  $\lim_{n \rightarrow \infty} \text{diam}(U_{f|n}) \rightarrow 0$ . By Lemma 2.1, there is a unique element in the intersection of this nested sequence. We define  $\phi(f)$  to be equal to this unique element. Furthermore, since  $\overline{U_{f|n}} \cap P \neq \emptyset$  for all  $n$  and both  $\overline{U_{f|n}}$  and  $P$  are closed, we have that

$$\phi(f) = \bigcap \overline{U_{f|n}} = \bigcap (\overline{U_{f|n}} \cap P)$$

so  $\phi(f) \in P$ . **Check:**  $\phi$  is one-to-one and continuous. □

The importance of Lemma 7.2 is that if we want to show that a subset  $A \subseteq X$  has size  $|A| = 2^{\aleph_0}$ , it suffices to show that  $A$  contains a nonempty perfect set. One way to do this in the case when  $A \subseteq X$  is closed is to use the Cantor-Bendixson derivative.

**Definition 7.3.** Let  $X$  be a Polish space and  $A \subseteq X$  be closed. The **Cantor-Bendixson derivative** of  $A$  is

$$\Gamma(A) = \{x \in A \mid x \text{ is not isolated}\}.$$

**Lemma 7.4.** Let  $X$  be a Polish space and  $A \subseteq X$  be closed.  $\Gamma(A) = A$  if and only if  $A$  is perfect.

*Proof.* By definition,  $A \setminus \Gamma(A)$  is the set of isolated points of  $A$ . Therefore  $\Gamma(A) = A \Leftrightarrow A$  has no isolated points  $\Leftrightarrow A$  is perfect. □

**Lemma 7.5.** Let  $X$  be a Polish space,  $A \subseteq X$  be closed, and  $A_0 \subseteq A$  be the set of isolated points of  $A$ .  $A_0$  is countable and the Cantor-Bendixson derivative  $\Gamma(A) = A \setminus A_0$  is closed.

*Proof.* Fix a countable basis for  $X$ . For each  $a \in A_0$ , there is a basic open set  $U_a$  from this countable basis such that  $U_a \cap A = \{a\}$ . (We say that  $U_a$  isolated  $a$ .) Since the basis is countable and each element of the basis can isolate at most one element of  $A$ , the set  $A_0$  is countable. Furthermore,

$$\Gamma(A) = A \setminus A_0 = A \setminus (\cup_{a \in A_0} U_a) = A \cap (\overline{\cup_{a \in A_0} U_a}).$$

Since  $\cup_{a \in A_0} U_a$  is open,  $\Gamma(A)$  is closed as required.  $\square$

For a closed set  $A \subseteq X$ , we define the iterated Cantor-Bendixson derivative by induction on  $\alpha < \omega_1$  as follows:  $\Gamma_0(A) = A$ ,  $\Gamma_{\alpha+1}(A) = \Gamma(\Gamma_\alpha(A))$  and  $\Gamma_\beta(A) = \bigcap_{\alpha < \beta} \Gamma_\alpha(A)$  for limit ordinals  $\beta$ . By Lemma 7.5,  $\Gamma_\alpha(A)$  is closed for each  $\alpha < \omega_1$ , so this iteration is possible. Notice that these derivatives form a nested sequence of closed sets

$$A = \Gamma_0(A) \supseteq \Gamma_1(A) \supseteq \Gamma_2(A) \supseteq \cdots \supseteq \Gamma_\omega(A) \supseteq \Gamma_{\omega+1}(A) \supseteq \cdots$$

If there is an  $\alpha$  such that  $\Gamma_{\alpha+1}(A) = \Gamma_\alpha(A)$ , then by Lemma 7.4,  $\Gamma_\alpha(A)$  is perfect and  $\Gamma_\gamma(A) = \Gamma_\alpha(A)$  for all  $\gamma \geq \alpha$ . In the next lemma, we show that such a fixed point must exist.

**Lemma 7.6.** *Let  $X$  be a Polish space and  $A \subseteq X$  be closed.*

1.  $\Gamma_\alpha(A) \setminus \Gamma_{\alpha+1}(A)$  is countable for each  $\alpha < \omega_1$ .
2. There is an  $\alpha < \omega_1$  such that  $\Gamma_{\alpha+1}(A) = \Gamma_\alpha(A)$ .

*Proof.* Since  $\Gamma_\alpha(A) \setminus \Gamma_{\alpha+1}(A)$  is equal to the set of isolated points of  $\Gamma_\alpha(A)$ , 1 follows immediately from Lemma 7.5. To prove 2, suppose for a contradiction that there is no such  $\alpha$ . Fix an enumeration  $U_0, U_1, \dots$  of a countable basis for  $X$ . For every  $\alpha < \omega_1$ , we can fix an element  $a_\alpha \in \Gamma_\alpha(A) \setminus \Gamma_{\alpha+1}(A)$  and an index  $n_\alpha \in \mathbb{N}$  such that  $U_{n_\alpha}$  isolates  $a_\alpha$ . Because  $a_\alpha \in \Gamma_\beta(A)$  for all  $\beta \leq \alpha$ ,  $U_{n_\alpha}$  cannot also isolate another point  $a_\beta$  for  $\beta < \alpha$ . Therefore,  $n_\alpha \neq n_\beta$  for all  $\alpha \neq \beta$ . But then the map  $f : \omega_1 \rightarrow \mathbb{N}$  given by  $f(\alpha) = n_\alpha$  is one-to-one, giving the desired contradiction.  $\square$

**Definition 7.7.** The **Cantor-Bendixson rank** of a closed set  $A$  is the least  $\alpha < \omega_1$  such that  $\Gamma_{\alpha+1}(A) = \Gamma_\alpha(A)$ .

We now arrive at what is frequently called the Cantor-Bendixson Theorem.

**Theorem 7.8.** *Let  $X$  be a Polish space and  $A \subseteq X$  be closed. Then  $A = P \cup C$  where  $P$  is perfect,  $C$  is countable and  $P \cap C = \emptyset$ . In particular, if  $A$  is uncountable, then  $A$  has size  $2^{\aleph_0}$ .*

*Proof.* Let  $\alpha$  be the Cantor-Bendixson rank of  $A$ . Since  $\Gamma_{\alpha+1}(A) = \Gamma_\alpha(A)$ ,  $\Gamma_\alpha(A)$  is perfect and

$$A \setminus \Gamma_\alpha(A) = \bigcup_{\beta < \alpha} (\Gamma_\beta(A) \setminus \Gamma_{\beta+1}(A)).$$

Since each  $\Gamma_\beta(A) \setminus \Gamma_{\beta+1}(A)$  is countable (by Lemma 7.6), this union is a countable union of countable sets, and hence is countable. Therefore, setting  $P = \Gamma_\alpha(A)$  and  $C = A \setminus P$  gives the desired decomposition. If  $A$  is uncountable, then  $P$  is a nonempty perfect set, and hence by Lemma 7.2,  $|P| = 2^{\aleph_0}$  and  $|A| = 2^{\aleph_0} + |C| = 2^{\aleph_0}$ .  $\square$

We can extend these ideas to analytic sets fairly easily, but not using such a nice method as the Cantor-Bendixson analysis.

**Lemma 7.9.** *Let  $X$  be a Polish space. If  $A \subseteq X$  is uncountable, then there are disjoint open sets  $V_0$  and  $V_1$  such that  $V_0 \cap A$  and  $V_1 \cap A$  are both uncountable.*

*Proof.* For a contradiction, assume there do not exist such open sets  $V_0$  and  $V_1$ . Applying Lemma 2.2, for each  $n > 0$ , we can find an open cover  $U_0^n, U_1^n, \dots$  of  $X$  by balls of radius  $< 1/n$ . Because  $A$  is uncountable and

$$A = (U_0^n \cap A) \cup (U_1^n \cap A) \cup (U_2^n \cap A) \cup \dots$$

one of the sets  $U_m^n \cap A$  must be uncountable. Fix  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $U_{f(n)}^n \cap A$  is uncountable. Let

$$A_n = A \setminus \overline{U_{f(n)}^n}.$$

We must have that either some  $A_n$  is uncountable or that all of the  $A_n$  are (at most) countable.

First, consider the case when some  $A_n$  is uncountable. Set  $V_0 = U_{f(n)}^n$  and  $V_1 = X \setminus \overline{U_{f(n)}^n}$ . Clearly  $V_0 \cap V_1 = \emptyset$ .  $V_0 \cap A$  is uncountable by the definition of  $f(n)$ .  $V_1$  is open since  $\overline{U_{f(n)}^n}$  is closed and  $V_1 \cap A$  is uncountable since  $V_1 \cap A = A \setminus \overline{U_{f(n)}^n} = A_n$  which is uncountable by assumption.

Second, consider the case when all of the  $A_n$  are countable. We show that this case cannot happen as it implies that  $A$  is countable. By the definition of  $A_n$  we have

$$A \setminus \left( \bigcup A_n \right) \subseteq \bigcap \overline{U_{f(n)}^n}.$$

But, since  $\text{diam}(\overline{U_{f(n)}^n}) \rightarrow 0$ , we have that  $\bigcap \overline{U_{f(n)}^n}$  has at most one element in it. Therefore,  $A$  can be decomposed in  $\bigcup A_n$  (which is countable as it is a countable union of countable sets) plus at most one more element. Therefore,  $A$  is countable contradicting our assumption that  $A$  is uncountable.  $\square$

**Theorem 7.10.** *Let  $X$  be a Polish space and  $A \subseteq X$  be analytic and uncountable.  $A$  contains a perfect set and therefore  $|A| = 2^{\aleph_0}$ .*

*Proof.* Because  $A$  is analytic, we can fix a continuous function  $\phi : \mathbb{N}^{\mathbb{N}} \rightarrow X$  such that  $\phi(\mathbb{N}^{\mathbb{N}}) = A$  by Lemma 6.4. We build a one-to-one function from  $2^{<\mathbb{N}}$  to  $\mathbb{N}^{<\mathbb{N}}$  which we denote by  $\sigma \mapsto \tau_\sigma$  that has the following properties. (You should think of this function as assigning a basic clopen subset  $N_{\tau_\sigma}$  of  $\mathbb{N}^{\mathbb{N}}$  to each string  $\sigma \in 2^{<\mathbb{N}}$ .)

1.  $\tau_\emptyset = \emptyset$ , so  $N_{\tau_\emptyset} = N_\emptyset = X$ .
2. If  $\sigma \subseteq \mu$ , then  $\tau_\sigma \subseteq \tau_\mu$ .
3.  $\phi(N_{\tau_\sigma}) \subseteq A$  is uncountable for each  $\sigma$ .
4.  $\phi(N_{\tau_{\sigma*0}}) \cap \phi(N_{\tau_{\sigma*1}}) = \emptyset$  for all  $\sigma$ .

We define this function  $\sigma \mapsto \tau_\sigma$  by induction on  $\sigma$ . We begin with  $\emptyset \mapsto \tau_\emptyset = \emptyset$ . Notice that  $N_{\tau_\emptyset} = X$  and  $\phi(X) = A$  is uncountable.

Assume that we have defined  $\tau_\sigma$  so that  $\phi(N_{\tau_\sigma})$  is uncountable. By Lemma 7.9, there are open sets  $U_0$  and  $U_1$  in  $X$  such that  $U_0 \cap \phi(N_{\tau_\sigma})$  and  $U_1 \cap \phi(N_{\tau_\sigma})$  are uncountable. Let  $W_i = \phi^{-1}(U_i) \cap N_{\tau_\sigma}$  (for  $i = 0, 1$ ) and notice that  $W_i$  is open in  $\mathbb{N}^{\mathbb{N}}$ , that  $\phi(W_i)$  is uncountable and that  $\phi(W_0) \cap \phi(W_1) = \emptyset$ .

$W_i \subseteq N_{\tau_\sigma}$  is open, so  $W_i$  can be written as a countable union of basic clopen sets of the form  $N_\mu$  with  $\mu \supseteq \tau_\sigma$ . Because this union is countable and  $\phi(W_i)$  is uncountable, there must be strings  $\mu_i \in 2^{<\mathbb{N}}$  such that  $\tau_\sigma \subseteq \mu_i$ ,  $N_{\mu_i} \subseteq W_i$  and  $\phi(N_{\mu_i})$  is uncountable.

Fix such strings  $\mu_i$  and set  $\tau_{\sigma*i} = \mu_i$ . It is clear that this definition maintains Properties 2 and 3. To see that Property 4 holds, notice that  $\phi(N_{\sigma*i}) \subseteq \phi(W_i)$ , so since  $\phi(W_0) \cap \phi(W_1) = \emptyset$  we have that Property 4 holds. This completes the construction of our map  $\sigma \mapsto \tau_\sigma$ .

As with many earlier proofs, we use this mapping of strings to define a map  $\alpha : 2^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  by setting  $\alpha(g) = \bigcap N_{\tau_{g \upharpoonright n}} = \bigcup \tau_{g \upharpoonright n}$ . **Check:** both  $\alpha : 2^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$  and  $\phi \circ \alpha : 2^{\mathbb{N}} \rightarrow X$  are continuous and one-to-one.

To finish this proof, we use a couple of basic facts from point set topology. Because  $2^{\mathbb{N}}$  is compact and  $\alpha$  is continuous,  $\alpha(2^{\mathbb{N}})$  is a compact subset of  $\mathbb{N}^{\mathbb{N}}$ . Again, because  $\phi$  is continuous and  $\alpha(2^{\mathbb{N}})$  is compact,  $\phi(\alpha(2^{\mathbb{N}}))$  is a compact subset of  $X$  (contained in  $A$ ). Because  $X$  is Hausdorff (recall Exercise 1.2), every compact subset of  $X$  is closed. Therefore,  $\phi(\alpha(2^{\mathbb{N}}))$  is closed in  $X$  and because  $\phi \circ \alpha$  is one-to-one,  $\phi(\alpha(2^{\mathbb{N}}))$  is uncountable. It follows by Theorem 7.8 that  $\phi(\alpha(2^{\mathbb{N}}))$  (and hence  $A$ ) contains a nonempty perfect set and  $|A| = 2^{\aleph_0}$ .  $\square$

You might wonder if this type of analysis can be extended to show that every uncountable co-analytic (that is, every  $\mathbf{\Pi}_1^1(X)$ ) subset of  $X$  has size  $2^{\aleph_0}$ . Unfortunately, whether or not every  $\mathbf{\Pi}_1^1(X)$  set satisfies the continuum hypothesis is independent of ZFC!