Partial results on the complexity of finding roots in Puiseux and Hahn fields

Reed Solomon (with Julia Knight and Karen Lange)

January 11, 2018

Background

Let K be an ACF_0 (or a RCF).

- $K[t]$ is ring of formal power series.
- Expand $K[t]$ to the field of Laurent series $K(t)$.
- Expand $K(t)$ to the algebraically closed field $K\{t\}$ of Puiseux series.

Definition

A *Puiseux series* (*over* K) is a formal sum $s = \sum_{i \geq k} a_i t^{\frac{i}{n}}$, where $n \in \mathbb{N}^+$, $k \in \mathbb{Z}$ and each $a_i \in K$. The *weight* $w(s)$ of s is the exponent of the first non-zero term in s, and is ∞ if $s = 0$.

That is, a Puiseux series is a Laurent series in $t^{\frac{1}{n}}$ for some positive n. When K is RCF, think of t as an infinitesimal.

Theorem (Newton, Puiseux)

If K is ACF_0 (or RCF), the $K\{t\}$ is ACF_0 (or RCF).

Main question

- Fix a countable (computable) ACF_0 K.
- A Puiseux series is a formal sum $s = \sum_{i \geq k} a_i t^{\frac{i}{n}}$, where $n \in \mathbb{N}^+$, $k \in \mathbb{Z}$ and each $a_i \in K$.

For a polynomial $p(x) = A_n x^n + \cdots + A_1 x + A_0$ with $A_i \in K\{t\}$, how hard is it to find a root $r \in K\{t\}$?

Represent Puiseux series by $s : \omega \to K \times \mathbb{Q}$ such that if $s(m) = \langle a_m, q_m \rangle$, then s represents the series $\sum_{m \in \omega} a_m t^{q_m}$. We require that q_m increases with m and there is a uniform bound on the size of the denominators of the q_m terms. Note that the q_m terms are unbounded.

Represent Puiseux series by $s : \omega \to K \times \mathbb{Q}$ such that if $s(m) = \langle a_m, q_m \rangle$, then q_m increases with m and there is a uniform bound on the size of the denominators of the q_m terms.

- Addition and multiplication of Puiseux series are computable.
- Equality is Π^0_1.
- Determining $w(s)$ is in general Δ^0_2, but there is a uniform computable procedure to find $w(s)$ for any $s \neq 0$.
- For any $q \in \mathbb{Q}$, determining whether $w(s) \geq q$ is computable.

Q: Given $A_0, \ldots, A_n \in K\{t\}$, how difficult is it to compute a root of

$$p(x) = A_0 + A_1 x + \cdots + A_n x^n \quad ?$$

Answer 1. The classical algebraic geometry literature gives a uniform Δ^0_2 procedure that will construct a root by initial segments

$$a_0 t^{q_0}$$
$$a_0 t^{q_0} + a_1 t^{q_1}$$
$$a_0 t^{q_0} + a_1 t^{q_1} + a_2 t^{q_2}$$

with $q_0 < q_1 < q_2 < \cdots$ and each $a_i \neq 0$. Furthermore, if we reach a root at a finite stage, the procedure will terminate and declare the root complete. Any uniform procedure with this termination feature is Δ^0_2 hard, so the classical result is sharp in this sense.

Answer 2. We can do better if we drop uniformity and termination conditions.

Theorem (Knight, Lange and Solomon)

For any countable ACF_0 K and any nonconstant polynomial $p(x) = A_0 + \cdots + A_n x^n$ over $K\{t\}$, $p(x)$ has a root computable from K and the coefficients $A_0, \ldots, A_n$.

In particular, the Newton-Puiseux theorem holds in every Turing ideal.

Conjecture

There is a uniform computable procedure that will produce roots for any nonconstant polynomial $p(x)$ from K and the coefficients $A_0, \ldots, A_n$.

- This procedure will not have the termination property.
- There is no such uniform procedure which will also work on the constant polynomial $p(x) = 0$.

Hahn fields

Let K be an ACF_0 (or RCF) and let G be a divisible ordered abelian group. The *Hahn field* $K((G))$ consists of formal sum

$$s = \sum_{g \in I} a_g t^g$$

where $I \subseteq G$ is well ordered and each $a_g \in K$.

- The *support* of s is $Supp(s) = \{g \in I : a_g \neq 0\}$.
- The *length* of s is the order type of $Supp(s)$.
- The *weight* $w(s)$ of s is the least $g \in I$ such that $a_g \neq 0$, and is ∞ if $s = 0$.

Theorem (Maclane)

$K((G))$ is an ACF_0 (or an RCF if K is RCF).

Given $p(x) = A_0 + A_1 x + \cdots + A_n x^n$ over $K((G))$, how complicated are the roots of $p(x)$?

Theorem (Knight and Lange)

If each coefficient A_i has countable length α_i and γ is a limit ordinal such that $\alpha_i < \gamma$, then the roots of $p(x)$ all have length less than ω^{ω^γ}.

- This result can be extended to uncountable ordinals.
- Maclane's theorem about Hahn fields holds in any admissible set.
- We would like a finer analysis of the complexity of roots.

We represent an element of $K((G))$ by a function $s : G \to K$ such that

$$Supp(s) = \{g \in G : s(g) \neq 0\} \text{ is well ordered.}$$

- Addition is computable.
- Computing multiplication, equality and weights are all uniformly Δ^0_2.

Given $p(x) = A_0 + A_1 x + \cdots + A_n x^n$ over $K((G))$, we want to uniformly produce initial segments $r_\alpha \in K((G))$ of a root r

$$r_0 = 0$$
$$r_1 = a_1 t^{\nu_1}$$
$$r_2 = a_1 t^{\nu_1} + a_2 t^{\nu_2}$$
$$\vdots$$
$$r_\omega = \sum_{n \in \omega} a_n t^{\nu_n}$$
$$r_{\omega+1} = r_\omega + a_\omega t^{\nu_\omega}$$
$$\vdots$$

Moreover, we would like to terminate the process when we complete the root. At stage $\alpha + 1$, either we declare r_α is a root, or we produce the next non-zero term in a root.

Let f be a function such that it is uniformly $\Delta^0_{f(\alpha)}$ to produce r_α.

Lemma (Knight, Lange and Solomon)

- *For a limit ordinal α, $f(\alpha) = \sup_{\beta<\alpha} f(\beta) + 1$.*
- *For a successor β, write $\beta = \alpha + n$ with α a limit ordinal. Then $f(\beta) = f(\alpha) + 1$.*

- r_0 is uniformly Δ^0_1
- $r_1, r_2, \ldots$ are each uniformly Δ^0_2
- r_ω is uniformly Δ^0_3
- $r_{\omega+1}, r_{\omega+2}, \ldots$ are each uniformly Δ^0_4.

Up to this point, we know the complexity results are sharp. Once we get to the Δ^0_5 bound on $r_{\omega+\omega}$, we do not know whether the bound is sharp, but we do know that our current methods will not work to show sharpness.

Thank you!