Turing degrees of orders on torsion-free abelian groups

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joint with Asher Kach and Karen Lange

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Ordered abelian groups

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The *positive cone* of this order is $P_{\leq_G} = \{g \in G \mid 0_G \leq_G g\}$. Since

$$a \leq_G b \Leftrightarrow b - a \in P_{\leq_G}$$

we can (effectively) equate orders and positive cones. Let

$$\mathbb{X}(G) = \{P \subseteq G \mid P \text{ is a positive cone on } G\} \subseteq 2^G$$

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- $\mathbb{X}(G)$ is a closed subspace of 2^G and hence is a Boolean topological space (compact, Hausdorff and has basis of clopen sets).
- If G is computable, then $\mathbb{X}(G)$ is a Π^0_1 class.

Motivating Question

Let G be computable torsion-free abelian group. What can we say about the elements of $\deg(\mathbb{X}(G)) = \{\deg(P) \mid P \in \mathbb{X}(G)\}$?

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- (Craven) For any Boolean topological space T, there is a field F such that $T \cong \mathbb{X}(F)$.
- (Metakides and Nerode) For any Π^0_1 class $\mathcal{C}$, there is a computable field F and a Turing degree preserving homeomorphism $\mathbb{X}(F) \to \mathcal{C}$.

Back to groups: classical structure of $\mathbb{X}(G)$

Let G be a (countable) torsion-free abelian group. $\{b_i \mid i \in I\} \subseteq G$ is *independent* if

$$\alpha_0 b_{i_0} + \cdots + \alpha_k b_{i_k} = 0_G \leftrightarrow \forall i \leq k(\alpha_i = 0)$$

where the coefficients are taken from $\mathbb{Z}$. A *basis* for G is a maximal independent set and the *rank* of G is the size of any basis.

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- $\text{rank}(G) = 1 \leftrightarrow G$ embeds into $\mathbb{Q}$
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- $\text{rank}(G) = 1 \Rightarrow |\mathbb{X}(G)| = 2$
- $\text{rank}(G) > 1 \Rightarrow \mathbb{X}(G) \cong 2^\omega$

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- Hence, there is a computable $H \cong G$ such that $\deg(\mathbb{X}(H))$ contains all degrees.
- (Downey and Kurtz) There is a computable copy of $\oplus_\omega \mathbb{Z}$ which has no computable order.

Question

Is $\deg(\mathbb{X}(G))$ always closed upwards in the degrees? If G has a computable order, does it have orders of every degree?

Lemma (Kach, Lange and Solomon)

Let G be a computable torsion-free abelian group with infinite rank. If G has a computable basis, then G has a basis of each degree.

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Theorem (Kach, Lange, Solomon)

There is a computable copy G of $\oplus_\omega \mathbb{Q}$ and a non computable c.e. set C such that

- *G has exactly two computable orders, and*
- *every C-computable order on G is computable.*

In particular, deg$(\mathbb{X}(G))$ is not closed upwards.

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In particular, $deg(\mathbb{X}(G))$ is not closed upwards.

Question

Does the conclusion of this theorem hold for computable groups other than $\oplus_\omega \mathbb{Q}$?

Effectively completely decomposable groups

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Unlike vector spaces, torsion-free abelian groups do not necessarily decompose into direct sums of smaller rank components.

A computable infinite rank torsion-free abelian group G is *effectively completely decomposable* if there is a uniformly computable sequence of rank 1 subgroups G_i of G such that G is computably isomorphic to $\bigoplus_i G_i$ (with the standard presentation).

Main Theorem

Theorem (Kach, Lange and Solomon)

Let G be an effectively completely decomposable torsion-free abelian group. There is a computable copy H of G and a noncomputable c.e. set C such that

- *H has exactly two computable orders, and*
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- *H has exactly two computable orders, and*
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Open Question

Does every infinite rank torsion-free abelian group have a computable copy which admits a computable order but does not have orders of every degree?

Other strange properties?

Theorem (Kach, Lange, Solomon and Turetsky)

For any infinite rank computable torsion-free abelian group G, $\mathbb{X}(G)$ contains infinitely many low degrees and infinitely many hyperimmune-free degrees.

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Theorem (Kach, Lange, Solomon and Turetsky)

For any infinite rank computable torsion-free abelian group G, $\mathbb{X}(G)$ contains infinitely many low degrees and infinitely many hyperimmune-free degrees.

It is possible to have an uncountable Π^0_1 class with isolated elements whose only low members are computable (or whose only hyperimmune-free members are computable).

Thank you!