Separating principles below RT^2_2

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Theorem (RT^2_2)

Let $c : [\mathbb{N}]^2 \to \{0, 1\}$ *be a 2-coloring of the two element subsets of* $\mathbb{N}$. *There is an infinite set* $H \subseteq \mathbb{N}$ *such that* c *is constant on* $[H]^2$.

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Let $(P, \leq_P)$ be a (countable) partial order.

$C \subseteq P$ is a *chain* if every pair of elements in C is comparable.

$$\forall a, b \in C\,(a \leq_P b \text{ or } b \leq_P a)$$

$A \subseteq P$ is an *antichain* if no pair of distinct elements in A is comparable.

$$\forall a, b \in A\,(a \neq b \to a \not\leq_P b \text{ and } b \not\leq_P a)$$

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Theorem (CAC)

Every infinite partial order contains either an infinite chain or an infinite antichain.

Think of an instance $c : [\mathbb{N}]^2 \to 2$ of RT^2_2 as a problem. The solution to this problem is an infinite homogeneous set.

Think of an infinite partial order as a CAC problem. The solution to this problem is an infinite chain or antichain.

RT^2_2 implies CAC

Let $(P, \leq_P)$ be a partial order with $P = \{p_0, p_1, \ldots\}$.
Define a coloring $c : [\mathbb{N}]^2 \to \{0, 1\}$ by

$$c(n, m) = 1 \Leftrightarrow p_n \text{ and } p_m \text{ are comparable}$$

Fix a homogeneous set H for c and define $B = \{p_n \mid n \in H\}$.

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Fix a homogeneous set H for c and define $B = \{p_n \mid n \in H\}$.
Suppose H is homogeneous for color 1.

For all $n \neq m \in H$, $c(n, m) = 1$, so $p_n, p_m \in B$ are comparable.
Therefore B is a chain.

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For all $n \neq m \in H$, $c(n, m) = 1$, so $p_n, p_m \in B$ are comparable.
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Suppose H is homogeneous for color 0.

For all $n \neq m \in H$, $c(n, m) = 0$, so $p_n, p_m \in B$ are incomparable.
Therefore B is an antichain.

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Does CAC imply RT_2^2?

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- $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is a Turing ideal if it is closed under $\leq_T$ and $\oplus$.
  - $\mathcal{I} = \{B \subseteq \omega \mid B \text{ is computable}\}$ is a Turing ideal.
  - If $A \subseteq \omega$, then $\mathcal{I}_A = \{B \subseteq \omega \mid B \leq_T A\}$ is a Turing ideal.
- $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an ω-model of $RCA_0 \Leftrightarrow \mathcal{I}$ is a Turing ideal.

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To separating CAC and RT^2_2 on an ω-model of RCA_0

Construct a Turing ideal $\mathcal{I}$ such that

- every instance of CAC in $\mathcal{I}$ has a solution in $\mathcal{I}$
- some instance of RT^2_2 in $\mathcal{I}$ has no solution in $\mathcal{I}$.

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Theorem (ADS)

If $(L, \leq_L)$ is an infinite linear order, then L contains either an infinite ascending sequence or an infinite descending sequence.

Theorem (Hirschfeldt, Shore)

$RCA_0 + CAC \nvdash RT^2_2$.

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Let P be poset. For $p \in P$

$p \in A^*(P) \Leftrightarrow p$ is below almost every element of P
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P is stable if either $A^*(P) \cup B^*(P) = P$ or $C^*(P) \cup B^*(P) = P$.

- $SCAC$ is CAC restricted to stable posets.
- $CCAC$ says every poset has a stable suborder.

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$RCA_0 + ADS + SCAC \vdash CAC$.

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There is a Turing ideal $\mathcal{I}$:

$\mathcal{I}$ is closed under solutions to ADS and $SCAC$, so $\mathcal{I} \models CAC$
but $\mathcal{I}$ does not contain a diagonally nonrecursive function
and therefore $\mathcal{I}$ is not a model of RT^2_2 by known results

A *tournament* is a directed graph $(T, \to)$ such that for all $x \neq y$, exactly one of $x \to y$ or $y \to x$ holds.

A tournament is *transitive* if $x \to y$ and $y \to z$ implies $x \to z$.

Theorem (EM)

Every infinite tournament has an infinite transitive subtournament.

Think of ADS and EM as problems to be solved.

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$RCA_0 + CAC \vdash ADS$
$RCA_0 + RT^2_2 \vdash EM$
$RCA_0 + EM + ADS \vdash RT^2_2$ (Bovykin and Weiermann)

Questions

Does $ADS \Rightarrow CAC$? (Equivalently, does $ADS \Rightarrow SCAC$?)

Does $EM \Rightarrow RT^2_2$? (Equivalently, does $EM \Rightarrow ADS$?)

Theorem (Lerman, Solomon and Towsner)

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Focus on $RCA_0 + ADS \nvdash SCAC$

Build Turing ideal $\mathcal{I}$ such that $\mathcal{I} \models ADS$ and $\mathcal{I} \not\models SCAC$.

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Build Turing ideal $\mathcal{I}$ such that $\mathcal{I} \models ADS$ and $\mathcal{I} \not\models SCAC$.

To satisfy $\mathcal{I} \not\models SCAC$, build poset $(M, A^*(M), B^*(M)) \in \mathcal{I}$

- $a \in A^*(M) \Leftrightarrow a$ is below almost every element of M
- $b \in B^*(M) \Leftrightarrow b$ is incomparable with almost every element of M
- $A^*(M) \cup B^*(M) = M = \omega$, so M is stable
- If $X \in \mathcal{I}$ is infinite, then $X \cap A^*(M) \neq \emptyset$ and $X \cap B^*(M) \neq \emptyset$

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To satisfy $\mathcal{I} \models ADS$

- For $X \in \mathcal{I}$, $e \in \omega$ s.t. Φ^X_e is an infinite linear order $\preceq^X_e$ on ω there is $f \in \mathcal{I}$ such that f is ascending or descending sequence in $\preceq^X_e$

Fix a linear order $(\omega, \prec)$. Let V be the initial segment consisting of the elements with finitely many predecessors. $(\omega, \prec)$ is called *stable-ish* if V and $\omega \setminus V$ are nonempty, V does not have a greatest element and $\omega \setminus V$ does not have a least element.

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Fact

If $(\omega, \prec)$ is not stable-ish, then there is a solution to $(\omega, \prec)$ computable from $\prec$.

Therefore, when building $\mathcal{I}$, we only need to add solutions to linear orders which are stable-ish.

Ground forcing: Build $(M, A^*(M), B^*(M))$

(1) M does not compute infinite chain or antichain in M.
If Φ_e^M is infinite, then $\Phi_e^M \cap A^*(M) \neq \emptyset$ and $\Phi_e^M \cap B^*(M) \neq \emptyset$

(2) Requirements for first level of iteration forcing are appropriately dense.

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Iteration forcing - context is fixed set X and index e

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Iteration forcing - action is to build generic solution G to $\prec_e^X$

(1) $X \oplus G$ does not compute solution to M

(2) Each requirement indexed by $X \oplus G$ is uniformly dense.

Full construction

Use ground forcing to build stable poset M.
Each step of iteration forcing adds solution f to $\prec_e^X$ to form $X \oplus f$.

Ground forcing

$\mathcal{F}$ is the set of conditions (F, A^*, B^*) where

- F is finite partial order, domain is an initial segment of ω
- if $x \prec_F y$, then $x <_{\mathbb{N}} y$
- $A^* \cup B^* \subseteq F$ with $A^* \cap B^* = \emptyset$
- A^* closed downward and B^* closed upward under $\prec_F$

$(F, A^*, B^*) \leq (F_0, A_0^*, B_0^*)$ if and only if

- F extends F_0 as partial order
- $A_0^* \subseteq A^*$ and $B_0^* \subseteq B^*$
- $(a \in A_0^*$ and $x \in F \setminus F_0) \Rightarrow a \prec_F x$
- $(b \in B_0^*$ and $x \in F \setminus F_0) \Rightarrow b$ is incomparable with x

We define generic sequence of conditions

$$(F_0, A_0^*, B_0^*) > (F_1, A_1^*, B_1^*) > \cdots$$

and set $M = \cup_n F_n$ to satisfy

(C1) M is stable: $\forall i \, \exists n \, (i \in A_n^* \cup B_n^*)$

(C2) M does not compute solution to itself: If Φ_e^M is infinite, then

$$\exists a \in A^*(M) \, \exists b \in B^*(M) \, (\Phi_e^M(a) = \Phi_e^M(b) = 1)$$

(C3) If $\prec_e^M$ is stable-ish, then each requirement for the iterated forcing is uniformly dense.

To meet (C2)

$(F, A^*, B^*) \Vdash (\Phi_e^G \text{ is finite})$ if there is k such that

$$\forall (F_0, A_0^*, B_0^*) \leq (F, A^*, B^*)\, \forall x\, (\Phi_e^{F_0}(x) = 1 \to x \leq k)$$

$(F, A^*, B^*) \Vdash (\Phi_e^G \not\subseteq A^*(G) \wedge \Phi_e^G \not\subseteq B^*(G))$ if

$$\exists a \in A^*\, \exists b \in B^*\, (\Phi_e^F(a) = \Phi_e^F(b) = 1)$$

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$(F, A^*, B^*) \Vdash (\Phi_e^G \not\subseteq A^*(G) \wedge \Phi_e^G \not\subseteq B^*(G))$ if

$$\exists a \in A^* \, \exists b \in B^* \, (\Phi_e^F(a) = \Phi_e^F(b) = 1)$$

Lemma

The set of conditions which either force Φ_e^G is finite or force

$$\Phi_e^G \not\subseteq A^*(G) \wedge \Phi_e^G \not\subseteq B^*(G)$$

is dense in $\mathcal{F}$.

Assume (F, A^*, B^*) has no extension forcing Φ_e^G is finite.

(1) Find extension (F_1, A_1^*, B_1^*) with $a \in A_1^*$ and $\Phi_e^{F_1}(a) = 1$.

Fix $a > F$ and $(F_0, A^*, B^*) \leq (F, A^*, B^*)$ such that $\Phi_e^{F_0}(a) = 1$.
Without loss of generality, $a \in F_0$.
Since $a \notin F$, for all $b \in B^*$, a and b are incomparable.
Define (F_1, A_1^*, B_1^*) by $F_1 = F_0$, $B_1^* = B^*$ and

$$A_1^* = A^* \cup \{c \in F_0 \mid c \preceq_{F_0} a\}$$

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(2) Find extension (F_3, A_3^*, B_3^*) with $b \in B_3^*$ and $\Phi_e^{F_3}(b) = 1$.

Fix $b > F_1$ and $(F_2, A_1^*, B_1^*) \leq (F_1, A_1^*, B_1^*)$ such that $\Phi_e^{F_2}(b) = 1$.
Without loss of generality, $b \in F_2$.
Since $b \notin F_1$, for all $a \in A_1^*$, $a \prec_F b$
Define (F_3, A_3^*, B_3^*) by $F_3 = F_2$, $A_3^* = A_2^*$ and

$$B_3^* = B_2^* \cup \{c \in F_0 \mid b \preceq_{F_0} c\}$$

Ground forcing: Build $(M, A^*(M), B^*(M))$

(1) M does not compute infinite chain or antichain in M.
If Φ_e^M is infinite, then $\Phi_e^M \cap A^*(M) \neq \emptyset$ and $\Phi_e^M \cap B^*(M) \neq \emptyset$

(2) Requirements for first level of iteration forcing are appropriately dense.

Iteration forcing - context is fixed set X and index e

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Φ_e^X is stable-ish linear order $\prec_e^X$ on ω
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Iteration forcing

Conditions

$\mathbb{A}_e^X = \{\sigma \mid \sigma$ is a finite ascending sequence in $\prec_e^X\}$.
$\mathbb{D}_e^X = \{\tau \mid \tau$ is a finite descending sequence in $\prec_e^X\}$.
$\mathbb{P}_e^X = \{(\sigma, \tau) \mid \sigma \in \mathbb{A}_e^X$ and $\tau \in \mathbb{D}_e^X$ and $\sigma \prec_e^X \tau\}$.

$$q \leq p \Leftrightarrow \sigma_p \sqsubseteq \sigma_q \text{ and } \tau_p \sqsubseteq \tau_q$$

If $p \in \mathbb{P}_e^X$, we write $p = (\sigma_p, \tau_p)$.

σ_p is attempt at an ascending solution to $\prec_e^X$

τ_p is attempt at a descending solution to $\prec_e^X$

A significant concern

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- V is nonempty initial segment with no maximal element.
- $\omega \setminus V$ is nonempty with no minimum element.

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Let $\mathbb{V}_e^X = \{p \in \mathbb{P}_e^X \mid \sigma_p \subseteq V \text{ and } \tau_p \subseteq \omega \setminus V\}$.

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A split pair below p is $q_0 = (\sigma_p\frown\sigma', \tau_p)$ and $q_1 = (\sigma_p, \tau_p\frown\tau')$ with $\sigma' \prec_e^X \tau'$.

If $p \in \mathbb{V}_e^X$ and q_0, q_1 is split pair below p, then $q_0 \in \mathbb{V}_e^X$ or $q_1 \in \mathbb{V}_e^X$.

We always look for split pairs and stay inside $\mathbb{V}_e^X$.

$$\mathbb{P}_e^X = \{(\sigma,\tau) \mid \sigma \in \mathbb{A}_e^X \text{ and } \tau \in \mathbb{D}_e^X \text{ and } \sigma \prec_e^X \tau\}.$$

$$q \leq p \Leftrightarrow \sigma_p \sqsubseteq \sigma_q \text{ and } \tau_p \sqsubseteq \tau_q$$

A diagonalization requirement is specified by indices m and n.

Given $p \in \mathbb{P}_e^X$, we want to do (1) or (2).

(1) Find $\sigma \sqsupseteq \sigma_p$ with $\sigma \prec_e^X \tau_p$ such that

$$\exists a \in A^*(M)\, \exists b \in B^*(M)\, (\Phi_m^{X\oplus\sigma}(a) = \Phi_m^{X\oplus\sigma}(b) = 1)$$

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(2) Find $\tau \sqsupseteq \tau_p$ with $\sigma_p \prec_e^X \tau$ such that

$$\exists a \in A^*(M)\, \exists b \in B^*(M)\, (\Phi_n^{X \oplus \tau}(a) = \Phi_n^{X \oplus \tau}(b) = 1)$$

An $\mathbb{A}$-side half requirement is downward closed set

$$\mathcal{R}^{X,A^*(M),B^*(M)} = \{\sigma \in \mathbb{A}_e^X \mid \exists a \in A^*(M)\, \exists b \in B^*(M)\, (R^X(\sigma, a, b))\}$$

where $R^X(x, y, z)$ is computable in X.

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A $\mathbb{D}$-side half requirements is defined similarly.

A requirement is downward closed set

$$\mathcal{K}^{X,A^*(M),B^*(M)} = \{p \in \mathbb{P}_e^X \mid \exists a \in A^*(M)\, \exists b \in B^*(M)\, (K^X(p, a, b))\}$$

where $K^X(x, y, z)$ is computable in X.

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A $\mathbb{D}$-side half requirements is defined similarly.

A requirement is downward closed set

$$\mathcal{K}^{X,A^*(M),B^*(M)} = \{p \in \mathbb{P}_e^X \mid \exists a \in A^*(M)\, \exists b \in B^*(M)\, (K^X(p, a, b))\}$$

where $K^X(x, y, z)$ is computable in X.

The requirements we are concerned with have the form

$$\mathcal{K}_{\mathcal{R},\mathcal{S}}^{X,A^*(M),B^*(M)} = \left\{p \in \mathbb{P}_e^X \mid \sigma_p \in \mathcal{R}^{X,A^*(M),B^*(M)} \text{ or } \tau_p \in \mathcal{S}^{X,A^*(M),B^*(M)}\right\}$$

where $\mathcal{R}$ and $\mathcal{S}$ are $\mathbb{A}$ and $\mathbb{D}$-side half requirements.

Fix an $\mathbb{A}$-side half requirement $\mathcal{R}^{X,A^*(M),B^*(M)}$

For finite sets A and B, let

$$\mathcal{R}^{X,A,B} = \{\sigma \in \mathbb{A}_e^X \mid \exists a \in A \exists b \in B (R^X(\sigma, a, b))\}$$

Let $\mathcal{R}^X$ be operator mapping A, B to $\mathcal{R}^{X,A,B}$.

Fix an $\mathbb{A}$-side half requirement $\mathcal{R}^{X,A^*(M),B^*(M)}$

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Let $\mathcal{R}^X$ be operator mapping A, B to $\mathcal{R}^{X,A,B}$.

Fix an infinite ascending sequence Λ in $\prec_e^X$.

$\mathcal{R}^X$ is *essential in* Λ if for every n and x,

$$\exists A > x \, \forall y \, \exists B > y \, \exists m > n \, (\Lambda \restriction m \in \mathcal{R}^{X,A,B})$$

Fix an $\mathbb{A}$-side half requirement $\mathcal{R}^{X,A^*(M),B^*(M)}$

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$$\exists A > x \, \forall y \, \exists B > y \, \exists m > n \, (\Lambda \upharpoonright m \in \mathcal{R}^{X,A,B})$$

Λ *satisfies* $\mathcal{R}^{X,A^*(M),B^*(M)}$ if either

(1) $\mathcal{R}^X$ is not essential in Λ, or
(2) there is an n such that $\Lambda \upharpoonright n \in \mathcal{R}^{X,A^*(M),B^*(M)}$.

Consider the $\mathbb{A}$-side half requirement $\mathcal{A}_m^{X,A^*(M),B^*(M)}$

$$\{\sigma \in \mathbb{A}_e^X \mid \exists a \in A^*(M)\,\exists b \in B^*(M)\,(\Phi_m^{X\oplus\sigma}(a) = \Phi_m^{X\oplus\sigma}(b) = 1)\}$$

$\mathcal{A}_m^X$ is essential in Λ if and only if $\Phi_m^{X\oplus\Lambda}$ is infinite.

$\mathcal{A}_m^{X,A^*(M),B^*(M)}$ is satisfied by Λ if and only if $\Phi_m^{X\oplus\Lambda}$ is finite or

$$\exists a \in A^*(M)\,\exists b \in B^*(M)\,(\Phi_m^{X\oplus\Lambda}(a) = \Phi_m^{X\oplus\Lambda}(b) = 1)$$

Either way, Λ is a solution to $\prec_e^X$ which doesn't compute a solution to M.

Fix a requirement $\mathcal{K}^{X,A^*(M),B^*(M)}$

$$\{p \in \mathbb{P}_e^X \mid \exists a \in A^*(M)\, \exists b \in B^*(M)\, (K^X(p,a,b))\}$$

For finite sets A and B, let

$$\mathcal{K}^{X,A,B} = \{p \in \mathbb{P}_e^X \mid \exists a \in A\, \exists b \in B\, (K^X(p,a,b))\}$$

Fix a requirement $\mathcal{K}^{X,A^*(M),B^*(M)}$

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For finite sets A and B, let

$$\mathcal{K}^{X,A,B} = \{p \in \mathbb{P}_e^X \mid \exists a \in A\, \exists b \in B\, (K^X(p,a,b))\}$$

$\mathcal{K}^X$ is *essential below* p if for every x

$$\exists A > x\, \forall y\, \exists B > y\, (q_0, q_1 \in \mathcal{K}^{X,A,B} \text{ for some split pair } q_0, q_1 \text{ below } p)$$

Fix a requirement $\mathcal{K}^{X,A^*(M),B^*(M)}$

$$\{p \in \mathbb{P}_e^X \mid \exists a \in A^*(M)\, \exists b \in B^*(M)\, (K^X(p,a,b))\}$$

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$\mathcal{K}^{X,A^*(M),B^*(M)}$ is *uniformly dense* if whenever $\mathcal{K}^X$ is essential below p, there is a split pair q_0, q_1 below p with $q_0, q_1 \in \mathcal{K}^{X,A^*(M),B^*(M)}$.

This is the notion of density from the set-up for the iteration forcing.

A sequence $p_0 > p_1 > \cdots$ from $\mathbb{P}_e^X$ *satisfies* $\mathcal{K}^{X,A^*(M),B^*(M)}$ if either

(1) for cofinitely many p_i, $\mathcal{K}^X$ is not essential below p_i, or

(2) there is a $p_n \in \mathcal{K}^{X,A^*(M),B^*(M)}$

A sequence $p_0 > p_1 > \cdots$ from $\mathbb{P}_e^X$ *satisfies* $\mathcal{K}^{X,A^*(M),B^*(M)}$ if either

(1) for cofinitely many p_i, $\mathcal{K}^X$ is not essential below p_i, or

(2) there is a $p_n \in \mathcal{K}^{X,A^*(M),B^*(M)}$

There is a sequence $p_0 > p_1 > \cdots$ from $\mathbb{V}_e^X$ which satisfies every requirement $\mathcal{K}^{X,A^*(M),B^*(M)}$.

Let $p_0 = (\emptyset, \emptyset) \in \mathbb{V}_e^X$

Given $p_n \in \mathbb{V}_e^X$, let m be least s.t. $\mathcal{K}_m^{X,A^*(M),B^*(M)}$ is essential below p_n but not satisfied yet.

By assumption, $\mathcal{K}_m^{X,A^*(M),B^*(M)}$ is uniformly dense.

So, there is split pair $q_0, q_1 < p_n$ in $\mathcal{K}_m^{X,A^*(M),B^*(M)}$

Let p_{n+1} be which of q_0, q_1 is in $\mathbb{V}_e^X$.

Check that these notions of satisfaction work together.

Let $\sigma = \cup_n \sigma_n$ and $\tau = \cup_n \tau_n$. If $\mathcal{R}^X$ is essential in σ and $\mathcal{S}^X$ is essential in τ, then $\mathcal{K}^X_{\mathcal{R},\mathcal{S}}$ is essential below every p_n.

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- Either σ satisfies every $\mathbb{A}^X_e$-side half requirement or τ satisfies every $\mathbb{D}^X_e$-side half requirement.

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- Either σ satisfies every $\mathbb{A}^X_e$-side half requirement or τ satisfies every $\mathbb{D}^X_e$-side half requirement.

Check that requirements forcing $\mathcal{K}^{X \oplus G, A^*(M), B^*(M)}$ to be uniformly dense (for the next iteration stage) can be written using X-computable relations as described here.

Thank you!